The median function on distributive semilattices

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Abstract

A median of a \(k\)-tuple \(\pi = (x_1, \ldots, x_k)\) of elements of a finite metric space \((X, d)\) is an element \(x\) for which \(\sum_{i=1}^{k} d(x, x_i)\) is minimum. The function \(m\) with domain the set of all \(k\)-tuples with \(k \geq 0\) and defined by \(m(\pi) = \{x: x\) is a median of \(\pi\}\) is called the median function on \(X\). Continuing with the program of characterizing \(m\) on various metric spaces, this paper presents a characterization of the median function on distributive semilattices endowed with the standard lattice metric.

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1. Introduction

Let \((X, d)\) be a finite metric space and \(k\) denote a positive integer. Set \(X^* = \bigcup_{k \geq 0} X^k\), where \(X^k\) is the \(k\)-fold cartesian product. Given a profile \(\pi = (x_1, \ldots, x_k) \in X^*,\) it is of interest in studies of consensus, data aggregation and location theory to find a point in the space that is in some sense “closest” to \(\pi\). Measuring closeness can be done using a remoteness function \(r_{\pi}\) which maps \(X\) into the non-negative real numbers. There are many ways that a meaningful remoteness function can be defined, but one of the most popular is \(r_{\pi}(x) = \sum_{i=1}^{k} d(x, x_i)\). When \(\pi\) is clear from the context, we will use \(r\) instead of \(r_{\pi}\), and \(r\) will denote this particular remoteness function throughout. A closest point in \(X\) based on this remoteness function is a median for \(\pi\). That is, a point \(x \in X\) for
which \( \sum_{i=1}^{k} d(x, x_i) \) is minimum. The median function on \((X, d)\) is the function that returns the set of all medians of a profile \(\pi\). Letting \(m\) denote the median function, we then have \(m: X^* \to 2^X \setminus \{\emptyset\} \) defined by \(m(\pi) = \{x: x \text{ is a median of } \pi\}\), for all \(\pi \in X^*\).

Adding structure to \(X\) via graph or order theoretic conditions has allowed for some nice characterizations of \(m\) and the reader can consult [1–3,5,8] for more information. In the present paper, we characterize the median function on \((X, d)\) where \(X\) is a finite distributive semilattice and \(d\) is a minimum path length metric on the covering graph of \(X\). We utilize results and ideas found in [9,10], and generalize those in [10].

2. Preliminaries, definitions and axioms

In this section, we will give some basic ideas and terminology necessary for our main results in Section 3. We first need to recall some order theoretic information. A partially ordered set is a non-empty set \(X\) together with a reflexive, antisymmetric, transitive relation \(\leq\) defined on \(X\). If \(X\) is a partially ordered set and \(x, y \in X\), then \(y\) covers \(x\) if \(x < y\) and for any \(z \in X\), \(x \leq z < y\) implies \(x = z\). The covering graph of \(X\) is the graph \(G = (X, E)\) where \(xy \in E\) if and only if \(x\) covers \(y\) or \(y\) covers \(x\). The partially ordered set \((X, \leq)\) is a meet semilattice if and only if every two element set \(\{x, y\}\) has an infimum, denoted \(x \land y\), and is a join semilattice if and only if every \(\{x, y\}\) has a supremum, \(x \lor y\). An element \(j\) in the meet semilattice \(X\) is join irreducible if \(j = x \lor y\) implies that either \(j = x\) or \(j = y\). Thus \(j\) is a join irreducible if and only if \(j\) covers exactly one element, which we denote by \(p(j)\). We let \(J\) denote the set of join irreducibles of \(X\), and for \(x \in X\) let \(J(x) = \{j \in J: j \leq x\}\). A lattice is a partially ordered set \(X\) for which \(x \land y\) and \(x \lor y\) exist for all \(x, y \in X\). The lattice \((X, \leq)\) is distributive when \((x \lor y) \land z = (x \land z) \lor (y \land z)\) for all \(x, y, z \in X\).

A semilattice \(X\) is a distributive semilattice if, for every \(x \in X\), the principal ideal \(\{y: y \leq x\}\) is a distributive lattice. A well known fact, (see [6]), that will be used in proofs is that a meet semilattice is distributive if and only if for any \(j \in J\) and \(y_1, \ldots, y_m \in X\), the inequality \(j \leq y_1 \lor \cdots \lor y_m\) implies that \(j \leq y_i\) for some \(i\). In addition, each \(x \in X\) has a unique irredundant representation [4] where \(x = j_1 \lor \cdots \lor j_n\) for \(j_1, \ldots, j_n \in J\) and where equality is destroyed if any \(j_i\) is removed from the join.

Following [6] we now define a particular type of minimum path length metric on the meet semilattice \(X\). Let \(w\) be an arbitrary mapping from \(J\) into the positive real numbers. Define \(v: X \to \mathbb{R}\) by setting \(v(0) = 0\), and for all \(x \neq 0\),

\[
v(x) = \sum_{j \in J(x)} w(j).
\]

Now, weight each edge \(xy\) of the covering graph \(X\) by \(|v(x) - v(y)|\), and let \(d_v\) be the minimum path length metric on this edge-weighted graph. Then from results found in [5,6] we have

\[
d_v(x, y) = v(x) + v(y) - 2v(x \land y) = \sum_{j \in J(x) \triangle J(y)} w(j),
\]
where $\Delta$ is the symmetric difference operator. It is not hard to show that if $w$ is defined by $w(j) = 1$ for all $j \in J$ and $X$ is distributive, then $|v(x) - v(y)| = 1$ for every edge $xy$ in the covering graph of $X$. Thus, in this case, $d_v$ is the usual graph metric on the covering graph.

A consensus function on $X$ is a function $c : X^* \to 2^X \setminus \{\emptyset\}$. Our goal is to characterize the median function among the consensus functions on the metric space $(X, d_v)$, where $X$ is a distributive semilattice. For the remainder of this paper, $X$ will denote a distributive semilattice. We now consider some properties, or axioms, that are useful for studying consensus functions. Call $c$ faithful if $c((x)) = \{x\}$ for all $x \in X$. Next, $c$ is consistent if $c(\pi \rho) = c(\pi) \cap c(\rho)$ whenever $c(\pi) \cap c(\rho) \neq \emptyset$, for profiles $\pi$ and $\rho$ on $X$, where $\pi \rho$ is the concatenation of $\pi$ and $\rho$. It is not hard to show that the median function $m$ is both faithful and consistent on any metric space. Now let $\pi = (x_1, \ldots, x_k)$ be a profile on $X$. For $x \in X$, we define the index of $x$ with respect to $\pi$ to be

$$\gamma(x, \pi) = \frac{|\{i \mid x \leq x_i\}|}{k}$$

and the score of $x$ to be

$$s(x) = k\gamma(x, \pi).$$

A consensus function $c : X^* \to 2^X \setminus \{\emptyset\}$ on $(X, \leq)$ is $\frac{1}{2}$-condorcet if the following holds for any profile $\pi$ on $X$: for $j \in J$ and $\gamma(j, \pi) = \frac{1}{2}$, then $x \lor j \in c(\pi)$ if and only if $x \lor p(j) \in c(\pi)$, provided $x \lor j$ exists.

When $X$ is a median semilattice (a distributive semilattice where any three elements have an upper bound whenever each pair of them has an upper bound) the above axioms suffice to characterize $m$. Specifically in [7] we show that if $c$ is a consensus function on a median semilattice, then $c = m$ if and only if $c$ is faithful, consistent, and $\frac{1}{2}$-condorcet.

Unfortunately, when $X$ is not a median semilattice there may exist consensus functions that are faithful, consistent, and $\frac{1}{2}$-condorcet yet are not the median function [9]. The next axiom will be needed for our characterization of $m$. A consensus function on $X$ is $\frac{1}{2}$-optimal if, for every profile $\pi \in X^k$,

$$c(\pi) \subseteq \{y \in X : \sigma(y) \text{ is minimum}\},$$

where

$$\sigma(y) = \sum_{j \in J_{1/2} \setminus \{y\}} (2w(j) - k)w(j)$$

with $J_{1/2} = \{r \in J : \gamma(r, \pi) \geq \frac{1}{2}\}$.

3. The results

The following lemmas generalize results proved in [10] for the special case where $X$ is the semilattice of all weak hierarchies on a set.
Lemma 1. The median function is $\frac{1}{2}$-condorcet on the metric space $(X,d_x)$.

Proof. Let $j \in J$ and $\pi \in X^k$ be a profile such that $\gamma(j,\pi) = \frac{1}{2}$. Let $x \in X$ and suppose $x \lor j$ exists. As observed in [6]

$$r(x \lor p(j)) = \rho(\pi) - \sum_{j' \in J(x \lor p(j))} (2s(j') - k)w(j')$$

and

$$r(x \lor j) = \rho(\pi) - \sum_{j' \in J(x \lor j)} (2s(j') - k)w(j'),$$

where

$$\rho(\pi) = \sum_{j' \in J} s(j')w(j').$$

If $j \leq x$, then $x \lor p(j) = x \lor j = x$ and we are done. Otherwise, using the fact that $X$ is a distributive semilattice, we have $J(x \lor j) \setminus J(x \lor p(j)) = \{j\}$. Thus

$$r(x \lor j) = r(x \lor p(j)) - (2s(j) - k)w(j).$$

Since $s(j) = k/2$ it follows that $r(x \lor j) = r(x \lor p(j))$. Hence $x \lor p(j) \in m(\pi)$ if and only if $x \lor j \in m(\pi)$. □

Before stating the next lemma, we need an alternative description of the remoteness function $r$. We start with Leclerc’s formula (4) in [6, p. 286]:

$$r(x) = \sum_{j \in J(x)} (k - s(j))w(j) + \sum_{j \in J \setminus J(x)} s(j)w(j).$$

Through a series of grouping terms and rewriting the expressions we get

$$r(x) = \rho'(\pi) + \sum_{j \in (J \setminus J_1) \setminus J(x)} (2s(j) - k)w(j) + \sigma(x),$$

where $\sigma(x)$ is as defined previously, and

$$\rho'(\pi) = \sum_{j \in J} (k - s(j))w(j).$$

Lemma 2. The median function is $\frac{1}{2}$-optimal on the metric space $(X,d_x)$.

Proof. For each $x \in X$ and $\pi \in X^k$, using the notation in [6], let

$$x_b = \bigvee \left\{ j \in J(x): s(j) \geq \frac{k}{2} \right\}.$$

Since $J_{1/2} \setminus J(x) = J_{1/2} \setminus J(x_b)$ it follows that $\sigma(x) = \sigma(x_b)$. Let $j \in J(x_b)$, so that $j \leq x_b$. Because $X$ is distributive, this implies that $j \leq j_i$ for some $j_i \in J(x)$ with $s(j_i) \geq \frac{1}{2}$,
and thus we have \( J(x_b) \subseteq J_{1/2} \). Hence \([(J \setminus J_{1/2}) \setminus J(x_b)] = J \setminus J_{1/2} \). Therefore, using the alternative formula for \( r \),
\[
r(x_b) = \rho'(\pi) + \rho''(\pi) + \sigma(x_b),
\]
where
\[
\rho''(\pi) = \sum_{j \in J \setminus J_{1/2}} (2s(j) - k)w(j).
\]

If \( x \in m(\pi) \), then, by Proposition 4.2 in [6], \( x = x_b \) and so \( \sigma(x) = r(x) - \rho'(\pi) - \rho''(\pi) \).

So, for every \( y \in X \), \( \sigma(x) \leq r(y_b) - \rho'(\pi) - \rho''(\pi) = \sigma(y_b) = \sigma(y) \). Therefore, \( m \) is \( \frac{1}{2} \)-optimal. \( \square \)

We state the next result without proof since it follows from an easy modification of the proof of Theorem 2 in [9].

**Lemma 3.** Let \( c \) be a consensus function on \( X \) that is faithful, consistent, and \( \frac{1}{2} \)-condorcet. For any \( \pi \in X^* \) and \( j \in J \), if \( x = y \lor j \in c(\pi) \) for some \( y \neq x \), then \( \gamma(j, \pi) \geq \frac{1}{2} \).

We can now state and prove our characterization theorem for the median function.

**Theorem 4.** The median function \( m \) on the metric space \((X, d_x)\) is faithful, consistent, \( \frac{1}{2} \)-condorcet, and \( \frac{1}{2} \)-optimal. Moreover, if \( c \) is a consensus function on \((X, d_x)\) that is faithful, consistent, \( \frac{1}{2} \)-condorcet, and \( \frac{1}{2} \)-optimal, then \( c(\pi) \subseteq m(\pi) \) for every profile \( \pi \).

**Proof.** The first part follows from Lemmas 1 and 2 so assume that \( c \) is a consensus function that is faithful, consistent, \( \frac{1}{2} \)-condorcet, and \( \frac{1}{2} \)-optimal. Let \( \pi \) be a profile and \( x \in c(\pi) \). If \( x \neq 0 \), then \( x \) can be expressed in its irredundant representation as \( x = j_1 \lor \cdots \lor j_s \) where each \( j_s \) is a join irreducible. Thus, for each \( i, x = y \lor j_i \) where \( y \neq x \). For example, if \( s = 1 \) then \( x = 0 \lor j_1 \). It follows from Lemma 3 that \( \gamma(j_1, \pi) \geq \frac{1}{2} \).

Using the argument in the proof of Lemma 2 we then have \( J(x) \subseteq J_{1/2} \) so that, as before,
\[
r(x) = \rho'(\pi) + \rho''(\pi) + \sigma(x).
\]

Since \( c \) is \( \frac{1}{2} \)-optimal, \( \sigma(x) \leq \sigma(y_b) \) so it follows that \( r(x) \leq r(y_b) \) for all \( y \in X \). Next observe that
\[
\rho''(\pi) = \sum_{j \notin J_{1/2}} (2s(j) - k)w(j) \leq \sum_{j' \in (J \setminus J_{1/2}) \cup (y)} (2s(j') - k)w(j')
\]
using the facts that \( s(j), s(j') \leq \frac{k}{2} \) and \((J \setminus J_{1/2}) \setminus J(y) \subseteq J \setminus J_{1/2} \). From the comments before Lemma 2, this implies that \( r(y_b) \leq r(y) \) for all \( y \in X \) and therefore \( x \in m(\pi) \).

For the case \( 0 \in c(\pi) \), since \([(J \setminus J_{1/2}) \setminus J(0)] = J \setminus J_{1/2} \) we have, using the argument above, \( 0 \in m(\pi) \). Hence \( c(\pi) \subseteq m(\pi) \). \( \square \)

**Corollary 5.** The median function \( m \) on \( X \) is the maximum element of the set of all consensus functions on \( X \) that are faithful, consistent, \( \frac{1}{2} \)-condorcet, and \( \frac{1}{2} \)-optimal.
References