Tolerance intersection graphs on binary trees with constant tolerance 3

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Abstract

A chordal graph is the intersection graph of a family of subtrees of a tree, or, equivalently, it is the (edge-)intersection graph of leaf-generated subtrees of a full binary tree. In this paper, a generalization of chordal graphs from this viewpoint is studied: a graph $G=(V,E)$ is representable if there is a family of subtrees $\{S_v\}_{v \in V}$ of a binary tree, such that $uv \in E$ if and only if $|S_u \cap S_v| \geq 3$. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

An intriguing theme in graph theory is that of the intersection graph of a family of subsets of a set: the vertices of the graph are represented by the subsets of the family and adjacency is defined by a non-empty intersection of the corresponding subsets. Prime examples are interval graphs and chordal graphs. An interval graph is the intersection graph of a family of closed intervals on the real line. A classical result is the characterization of interval graphs by forbidden subgraphs by Lekkerkerker and Boland [10]. A chordal graph is a graph without induced cycles of length four or more. They were proven to be the intersection graphs of a family of subtrees of a tree by Gavril [3] and Walter [13]. In [11] McMorris and Scheinerman observed, without adding a proof, that this result may be sharpened in the following way: a graph $G$ is chordal if and only if it is the intersection graph of a family of leaf-generated subtrees of a full binary tree such that intersecting subtrees share a leaf. In this very special type of representation ‘intersection’ may be even replaced by ‘edge-intersection’. Note that

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all terms used here will be properly introduced in Section 2. Special classes of chordal graphs are the vertex-intersection graphs or edge-intersection graphs of subpaths of a tree, see [4,5,12].

Golumbic and Monma [6] introduced a generalization of interval graphs using tolerances: each representing interval is assigned a positive real number, its tolerance, and two vertices are adjacent if the length of the intersection of their corresponding intervals exceeds the minimum of the two tolerances, cf. [7]. This idea of tolerance was used in [8] to formulate a broad Master plan on tolerance intersection graphs, which reads as follows. Let \( Z \) be a set and let \( \mu : \mathcal{S} \to \mathbb{R}_+ \) be a measure on \( Z \) assigning to each non-empty subset \( S \) of \( Z \) a positive real number \( \mu(S) \). Let \( \mathcal{S} = \{ S_v \}_{v \in V} \) be a (finite) family of non-empty subsets of \( Z \), and let \( \tau : \mathcal{S} \to \mathbb{R}_+ \) be a mapping, which assigns to each subset \( S_v \) a tolerance \( \tau_v \). Finally, let \( \phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) be a symmetric function assigning a positive real number to each pair of positive reals. Then the tolerance graph of \((\mathcal{S}, \mu)\) and \((\tau, \phi)\) is the graph \( G = (V, E) \) with vertex-set \( V \) and
\[
uv \in E \text{ if and only if } \mu(S_u \cap S_v) \geq \phi(\tau_u, \tau_v).
\]

By specifying conditions on \( \mathcal{S}, \mu, \tau, \) and \( \phi \), one gets special classes of tolerance graphs, amongst which are the tolerance interval graphs in the sense of Golumbic and Monma. Such a class is a hereditary class since each induced subgraph of such a graph is a tolerance graph of the same type. Hence, a prototype problem one asks here is to characterize classes of tolerance intersection graphs by forbidden subgraphs. Some classes of tolerance intersection graphs were studied in [8].

In [9] a special type of tolerance graph, which generalizes chordal graphs, was studied: the set \( Z \) is a `host’ tree \( T \), the representing subsets are subtrees of \( T \), the measure \( \mu \) is just the order of the subtree, that is, its number of vertices, and the tolerances are a constant \( c \). To see that chordal graphs are in this class, just take a subtree representation of a chordal graph, add to each vertex \( v \) of the host tree \( c - 1 \) pendant vertices, and add these pendant vertices also to all subtrees containing \( v \). A number of results on representability and non-representability are obtained in [9]. Parameters in these results are the maximum degree in the host tree \( T \), the maximum degree in the subtrees, and the constant tolerance \( c \). A chordal graph is then the tolerance intersection graph of leaf-generated subtrees of a full binary tree with constant tolerance 1 (or 2, in the case of edge-intersection). Because this fact was observed without proof in [11], we present a sketch of proof here. One may proceed as follows. Let \( G = (V, E) \) be a chordal graph, and let \( T \) with subtrees \( \{ S_v \}_{v \in V} \) be a representation of \( G \). If a subtree \( S_v \) contains a leaf \( x \), which is not a leaf in \( T \), then we add a new vertex \( y \) to \( T \) pending at \( x \) and add this vertex to all subtrees having \( x \) as leaf. We proceed until we obtain a representation in which all subtrees are leaf-generated. If necessary, we add some extra leaves to \( T \) and the subtrees such that intersecting subtrees of the representation share a leaf of \( T \). If \( T \) contains an internal vertex \( x \) with neighbors \( z_1, z_2, \ldots, z_k \), for some \( k > 3 \), then replace \( x \) by an edge \( xy \), join \( z_1 \) and \( z_2 \) to \( y \) and \( z_3, \ldots, z_k \) to \( x \). In each subtree \( S_v \) containing \( x \), we replace \( x \) by the edge \( xy \) and make the corresponding adjacencies. Note that leaf-generated subtrees remain leaf-generated.
in this process. Continuing in this way, we end with a binary host tree, which is easily transformed into a full binary tree, without changing the required characteristics of the representation. To make the representation edge-intersecting, we just add an extra layer to the full binary host tree, and extend all subtrees where necessary to this new layer.

In this paper we continue the research in [9], but we impose even further restrictions: the host tree \( T \) is a binary tree, that is, the maximum degree is 3, and the constant tolerance is also 3. In other words, a graph \( G=(V,E) \) is representable if there is a family of subtrees \( \{S_v\}_{v \in V} \) of a binary tree, such that \( uv \in E \) if and only if \( |S_u \cap S_v| \geq 3 \). In the generalization of chordal graphs, this case may be considered as the first step away from chordal graphs from this viewpoint. But it is already a giant step: the relatively easy results mentioned above on chordal graphs are not available here. For example, it turns out that it makes a difference, whether we allow the representing subtrees to be arbitrary subtrees of the host tree or that we require all the representing subtrees to be leaf-generated. We present some results on representability and non-representability of graphs, but we are still far from a characterization by forbidden subgraphs, for example. The presentation in this paper is independent from that in [9].

In Section 2, we present the necessary preliminaries. In Section 3, we distinguish between various possibilities for generalizing chordal graphs along these lines. In Section 4, we study the critical case of \( K_{n,m} \). In Section 5, we present a first, infinite, class of forbidden subgraphs: the \( \Theta \)-graphs with at least three non-trivial paths or at least five paths. A classical result on chordal graphs is that they are characterized as being the graphs admitting a simplicial elimination scheme. In Section 6, we give a first generalization of this result for the graphs under study. We conclude the paper with some further non-representability results.

2. Preliminaries

In this paper \( G=(V,E) \) is a connected, finite, simple graph with vertex set \( V \) and edge set \( E \). The order \( |G|=|V| \) of \( G \) is the number of vertices in \( G \). For two graphs \( G_1 \) and \( G_2 \), the intersection \( G_1 \cap G_2 \) is the graph with vertex set \( V_1 \cap V_2 \) and edge set \( E_1 \cap E_2 \). The neighborhood \( N(u) \) of a vertex \( u \) is the set consisting of all neighbors of \( u \), i.e. vertices adjacent to \( u \).

A complete binary tree is a tree in which each vertex has degree 1 (i.e. is a leaf) or 3 (i.e. is an internal vertex). A binary tree is a subtree of a complete binary tree. A subtree \( S \) of a tree \( T \) is leaf-generated if each leaf in \( S \) is also a leaf in \( T \). A full binary tree \( F \) is a rooted tree, in which the root has degree 2, all other internal vertices have degree 3, and all leaves have the same distance \( h=h(F) \) to the root, called the height of the tree \( F \). A rooted tree can be viewed as a partially ordered set (poset) with its root as universal lower bound. Thus a full binary tree \( F \) is a poset, in which the maximal elements are the leaves of \( F \). Any other element of \( F \) is covered by exactly two elements: its children. The children of the children of \( v \) are called its grandchildren. Each element of \( F \) distinct from the root covers exactly one element:
its parent. Any subtree $S$ of $F$ has a minimal element, its root, which is the element of $S$ closest to the root of $F$. Note that a leaf-generated subtree of $F$ contains both children of its root.

A $3$-subtree-representation $(T, \mathcal{S})$ of a connected graph $G = (V, E)$ consists of a binary host tree $T$ and a family of subtrees $\mathcal{S} = \{S_u \mid u \in V\}$ of $T$ such that

$$uv \in E \text{ if and only if } |S_u \cap S_v| \geq 3,$$

i.e., $S_u$ and $S_v$ are $3$-intersecting. In the sequel we will call a $3$-subtree-representation just a representation. The subtree $S_u$ represents vertex $u$ of $G$. Note that, if $G$ is a non-trivial graph, then, by connectedness, each vertex is represented by a subtree of size at least $3$. The graph $G$ is said to be representable if it has a representation $(T, \mathcal{S})$. Note that a representation in the above sense is just a $(3,3,3)$-representation in the sense of Jamison and Mulder [9]. If all subtrees in $\mathcal{S}$ are paths, then we say that $(T, \mathcal{S})$ is a path-representation of $G$. A representation $(T, \mathcal{S})$ of $G$ is faithful if all subtrees in $\mathcal{S}$ are leaf-generated subtrees of $T$ and subtrees in $\mathcal{S}$ sharing a leaf (of $T$) represent adjacent vertices of $G$. A graph that admits a faithful representation will be called a faithful graph, for short. A representation $(T, \mathcal{S})$ is orthodox if all subtrees in $\mathcal{S}$ are leaf-generated subtrees of $T$ and subtrees in $\mathcal{S}$ share a leaf (of $T$) if and only if they represent adjacent vertices of $G$. A graph that admits an orthodox representation will be called an orthodox graph, for short. In [9], we use orthodoxy to amalgamate orthodox graphs along certain cliques to obtain larger orthodox graphs. In [9] we also show that, if a graph is orthodox for some tolerance $c$, then it is orthodox for any tolerance $t \geq c$. These facts do not hold in the non-orthodox case.

Evidently, representability is a hereditary property: every induced subgraph of a representable graph is itself representable, and the characteristics of the representations, like being faithful or orthodox, are inherited as well. So an obvious problem is finding a list of forbidden subgraphs characterizing each type of representable graph.

Note that we can add a pendant vertex to any vertex of degree $2$ in the host tree without changing the representation or its type (e.g. faithful remains faithful, and orthodox remains orthodox). Therefore, without loss of generality, we will assume in the sequel that the host tree $T$ is a complete binary tree, unless stated otherwise.

3. Three classes only

There are various ways to distinguish between subclasses of the class of representable graphs by the properties of their representations. For example, we may require the host tree $T$ to be either an arbitrary (complete) binary tree or a full binary tree. Second, we may require $\mathcal{S}$ to consist of either arbitrary subtrees or leaf-generated subtrees of the host tree. Finally, we may place extra conditions on (non-)adjacency of vertices (by definition, two subtrees represent adjacent vertices if and only if they are $3$-intersecting).
Thus, we end up with four possibilities, the first of which is just the arbitrary case with no extra conditions on adjacency:

(i) no extra conditions,
(ii) \( S_u \) and \( S_v \) share a leaf of \( T \Rightarrow uv \in E \),
(iii) \( S_u \) and \( S_v \) share a leaf of \( T \Leftarrow uv \in E \),
(iv) \( S_u \) and \( S_v \) share a leaf of \( T \Leftrightarrow uv \in E \).

At first sight we may thus distinguish between 16 different possibilities. The aim of this section is to show that, with the above distinctions, there are only three different classes.

Lemma 1. Let \( G \) be a representable graph. Then \( G \) has a representation in which non-adjacent vertices are represented by subtrees having no leaf of the host tree in common.

Proof. Take any representation \((T, \mathcal{S})\) of \( G \). Without loss of generality, let \( T \) be a complete binary tree. Let \( p \) be a leaf of \( T \) adjacent to \( q \), with \( x \) and \( y \) being the other neighbors of \( q \) in \( T \). Let \( \mathcal{S}_x \) consist of the subtrees in \( \mathcal{S} \) containing \( p, q \) and \( x \), and \( \mathcal{S}_y \) of those containing \( p, q \) and \( y \). We extend \( T \) to a complete binary tree \( T^* \) by adding two new vertices \( p_x \) and \( p_y \) adjacent to \( p \). We extend each subtree \( S_u \) in \( \mathcal{S}_x \) to \( p_x \) and each subtree \( S_v \) in \( \mathcal{S}_y \) to \( p_y \), thus obtaining a subtree \( S_u^* \) of \( T^* \). Note that any subtree in \( \mathcal{S}_x \cap \mathcal{S}_y \) is thus extended to \( p_x \) as well as \( p_y \). For any \( S_u \) in \( \mathcal{S} - (\mathcal{S}_x \cup \mathcal{S}_y) \), we set \( S_u^* = S_u \). Thus we get a representation \((T^*, \mathcal{S}^*)\) of \( G \). If \( S_u \) and \( S_v \) from \( \mathcal{S} \) represent non-adjacent vertices of \( G \) sharing leaf \( p \) of \( T \), then \( S_u^* \) and \( S_v^* \) do not share leaves \( p_x \) or \( p_y \) of \( T^* \).

We apply the above construction to all leaves of \( T \) shared by subtrees in \( \mathcal{S} \) representing non-adjacent vertices of \( G \). Thus we get the required representation of \( G \).

From Lemma 1 we can deduce that the above conditions (i) and (ii) do not define distinct classes, and the same holds for conditions (iii) and (iv). If we leave the host tree aside for the moment, then four possibly different subclasses remain, see Fig. 1.

The next Lemma deals with the question mark in Fig. 1.

Lemma 2. Let \( G = (V, E) \) be a graph with a representation \((T, \mathcal{S})\) such that vertices of \( G \) are adjacent if and only if they are represented by subtrees sharing a leaf of \( T \). Then \( G \) has an orthodox representation.

Proof. For each leaf \( p \) of \( T \), we proceed as follows: we add two new vertices \( p_x \) and \( p_y \) adjacent to \( p \), and we add these vertices also to the subtrees in \( \mathcal{S} \) containing \( p \). Thus we get a representation \((T^*, \mathcal{S}^*)\) of \( G \), where \( S_u^* \) is obtained from \( S_u \) in \( \mathcal{S} \). This representation has the following property: if \( S_u^* \) and \( S_v^* \) share a leaf \( p_x \) of \( T^* \), then
they also share $p$ and $p_y$, where $p$ is a leaf in $T$. For each $S_u^*$, let $R_u$ be the subtree of $T^*$ generated by the leaves of $T^*$ in $S_u^*$. Set $\mathcal{R} = \{R_u\}_{u \in V}$. Now $R_u$ and $R_v$ share a leaf $p_x$ of $T^*$ if and only if they share $p$ and $p_y$ as well (whence are 3-intersecting) if and only if $S_u$ and $S_v$ share the leaf $p$ of $T$ if and only if $u$ and $v$ are adjacent in $G$. So $(T^*, \mathcal{R})$ is an orthodox representation of $G$. $\square$

This shows that the subclass of the question mark is just the class of orthodox graphs. The next Lemma shows that, in the remaining cases, we may always take the host tree to be a full binary tree.

**Lemma 3.** Let $G = (V,E)$ be a graph with a representation $(T,\mathcal{S})$. Then there is a representation $(F,\mathcal{R})$ with a full binary tree $F$ as host tree. If $(T,\mathcal{S})$ is faithful (respectively, orthodox), then there exists a faithful (respectively, orthodox) representation $(F,\mathcal{R})$.

**Proof.** First let $(T,\mathcal{S})$ be an arbitrary representation of $G$. Choose any leaf $r$ of $T$, and let $h = \max\{d_T(r,v) \mid v \text{ vertex of } T\}$. Extend $T$ from $r$ as root to a full binary tree $F$ of height $h$. Then $(F,\mathcal{S})$ is a representation of $G$.

Now let $(T,\mathcal{S})$ be faithful (respectively, orthodox). Take any leaf $p$ of $T$. Then the subtrees of $\mathcal{S}$ containing $p$ represent mutually adjacent vertices of $G$. Let $P$ be a path in $F$ from $p$ to a leaf $p^*$ of $F$ of length $h - d_T(r,p)$, i.e. $p^*$ is a leaf of $F$ ‘above $p$’. We add the path $P$ to every subtree in $\mathcal{S}$ containing $p$. We do this for each leaf $p$ of $T$. Thus, we obtain a family of leaf-generated subtrees $\mathcal{R}$ of $F$ such that $(F,\mathcal{R})$ is a faithful (respectively, orthodox) representation of $G$. $\square$

We may display the three different classes in a diagram as follows:

<table>
<thead>
<tr>
<th></th>
<th>Arbitrary subtrees</th>
<th>Leaf-generated subtrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) &amp; (ii)</td>
<td>REPRESENTABLE</td>
<td>FAITHFUL</td>
</tr>
<tr>
<td>(iii) &amp; (iv)</td>
<td></td>
<td>ORTHODOX</td>
</tr>
</tbody>
</table>

Fig. 1. Four possible subclasses left.

In the next section we will exhibit examples that show that the inclusions in the diagram are indeed proper inclusions.

Another possible distinction is the following. A representation $(T,\mathcal{S})$ of a graph $G$ is a **strict representation** if all subtrees in $\mathcal{S}$ are distinct. In a non-strict representation,
we allow two distinct vertices \( u \) and \( v \) to be represented by the same subtree of \( T \). Such vertices \( u \) and \( v \) are an example of true twins: they are adjacent and have the property that \( N(u) - v = N(v) - u \). From the constructions above, we easily deduce that a non-strict representation of a faithful graph is easily transformed into a strict one by extending the host tree at the leaves, and extend identical subtrees of the representation differently. In the case of non-faithful graphs, it turns out that there is a difference. There are representable graphs having true twins, which do not allow a strict representation as we will see below.

In this paper we will not pursue the theme of representable graphs having no strict representation. Other problems that we leave aside here are the following. If we define the representation number of a representable graph \( G \) to be the minimum order of a host tree for \( G \), then determine the representation number of \( G \). In this case there may be a difference between strict and arbitrary representations of faithful or orthodox graphs. A proper representation is a representation in which no representing subtree is contained in another one, cf. the idea of proper interval graph, where no interval is contained in another, versus arbitrary interval graphs (see [7] for the case of proper interval tolerance graphs). The same example below having no strict representation does not have a proper representation. Hence also here there are differences. We leave the characterization of these differences as an open problem. In the sequel we do not put restrictions on the order of the host tree and we allow non-strict and non-proper representations. Hence true twins will not be an obstruction.

4. The case \( K_{n,m} \)

Non-adjacent vertices \( u \) and \( v \) with \( N(u) = N(v) \) are called false twins. Twins play an essential role in distance-hereditary graphs, see [1,2]. In that case, true and false twins play a similar role. Here, in the case of representable graphs, their roles are in a way opposite, as we will see in this section. First we present a simple lemma. By \( P_4 \) we denote the path with four vertices, and by \( 3K_2 \) we denote the disjoint union of three copies of \( K_2 \).

**Lemma 4.** Let \((T, \mathcal{S})\) be a representation of a graph \( G = (V,E) \), and let \( S_u \) and \( S_v \) be subtrees in \( \mathcal{S} \) with \( S_u \cap S_v = \emptyset \). Then the complement of the graph induced by \( N(u) \cap N(v) \) does not contain \( K_3, P_4 \) or \( 3K_2 \) as induced subgraphs.

**Proof.** Let \( P = p \rightarrow \cdots \rightarrow q \) be the path in \( T \) between \( S_u \) and \( S_v \) with \( S_u \cap P = \{p\} \) and \( S_v \cap P = \{q\} \). If \( P \) is of length at least 2, then \( N(u) \cap N(v) \) induces a clique. So assume that \( P \) consists of the single edge \( pq \). Let \( p_x \) and \( p_y \) be the other neighbors of \( p \), and \( q_x \) and \( q_y \) those of \( q \). For any \( w \) in \( N(u) \cap N(v) \), the subtree \( S_w \) contains \( p, q \), at least one of \( p_x, p_y \), and at least one of \( q_x, q_y \). If \( S_w \) contains both \( p_x, p_y \) or both \( q_x, q_y \), then \( w \) is a central vertex in \( N(u) \cap N(v) \). The remaining vertices in \( N(u) \cap N(v) \) are of
four types: the representing subtree contains

1. only $p_x$ and $q_x$, and not $p_y$ or $q_y$,
2. only $p_x$ and $q_y$, and not $p_y$ or $q_x$,
3. only $p_y$ and $q_y$, and not $p_x$ or $q_x$,
4. only $p_y$ and $q_x$, and not $p_x$ or $q_y$.

Vertices of the same type form a clique. Between vertices of different types we have the following adjacencies: (1)–(2), (2)–(3), (3)–(4), (4)–(1). On the other hand, there is no adjacency between vertices of types (1) and (3), or between types (2) and (4). Loosely speaking, there is a 4-cycle on the cliques of the four types. Clearly, the complement of the graph induced by $N(u) \cap N(v)$ consists of a number of isolates and two disjoint complete bipartite graphs.

Recall that the independence number of a graph is the order of a maximum independent set (an independent set is a set of mutually non-adjacent vertices).

**Corollary 5.** Let $(T, \mathcal{S})$ be a representation of a graph $G = (V, E)$, and let $u$ and $v$ be non-adjacent vertices of $G$, for which $S_u \cap S_v = \emptyset$. Then the independence number of the subgraph of $G$ induced by $N(u) \cap N(v)$ is at most 2.

This corollary suggests that there is a restriction on the order of induced complete bipartite subgraphs in a representable graph.

For any subset of vertices $U$ of a tree $T$, let

$$B_k(U) = \{ q \mid d_T(p, q) \leq k, \text{ for some } p \in U \}$$

be the $k$-ball in $T$ at $U$.

Recall that any family of subtrees of a tree satisfies the Helly property, that is, if any two subtrees of the family intersect, then the whole family has a non-empty intersection.

**Theorem 6.** Let $(T, \mathcal{S})$ be a representation of $K_{3,3}$. Then there is a unique vertex $p$ of $T$ such that $\cap \mathcal{S} = \{ p \}$, and $(T, \mathcal{S})$ is unique when restricted to $B_2(p)$, up to the labeling of the vertices in $B_2(p)$, as given in Fig. 2.

**Proof.** We may assume that $T$ is a complete binary tree. Let $A, B, C$ and 1, 2, 3 denote the subtrees in $\mathcal{S}$ representing the two color classes (i.e. sets of three independent vertices) of $K_{3,3}$, respectively. By Corollary 5, all six subtrees in $\mathcal{S}$ mutually intersect. So, by the Helly property, we have $\cap \mathcal{S} \neq \emptyset$, say $p$ is a common vertex. Note that $p$ is not a leaf, being contained in the subtrees representing three independent vertices. Let $x, y, z$ be the neighbors of $p$ in $T$.

If $p$ were a leaf in one of the subtrees in $\mathcal{S}$, say $x$ is in $A$ but $y$ and $z$ are not, then 1, 2 and 3 all contain $p, x$ and another neighbor of $x$. Since 1, 2, 3 are mutually non-adjacent, this is impossible, so that $p$ is an internal vertex of all six subtrees in $\mathcal{S}$.
If, besides $p$, also $x$ were in $A \cap B \cap C$, then at least two of $A, B$ and $C$ would contain also, say, $y$. Hence, each $x, y, z$ is in precisely two of $A, B, C$, and in precisely two of 1, 2, and 3. Without loss of generality, we may assume that we have

\[
\begin{align*}
x &\in B \cap C \cap 2 \cap 3, \\
y &\in A \cap C \cap 1 \cap 3, \\
z &\in A \cap B \cap 1 \cap 2, \\
x &\notin A \cup 1, \\
y &\notin B \cup 2, \\
z &\notin C \cup 3.
\end{align*}
\]

Let $x^-, x^+$ be the other two neighbors of $x$, and $y^-, y^+$ those of $y$, and $z^-, z^+$ those of $z$, see Fig. 2. In order that $A$ and 2 are 3-intersecting, they must have one of $z^-, z^+$ in common, say $z^-$. Then 1 does not contain $z^-$. In order that $B$ and 1 are 3-intersecting, they must both contain $z^+$, whence $A$ does not contain $z^+$. Proceeding in this way, we obtain the structure in Fig. 2, where subtree $S$ in $\mathcal{P}$ is the path between the vertices labeled with $S$. Up to the labeling of the vertices, this structure is unique. \(\square\)

We call the representation in Fig. 2 the **canonical representation** of $K_{3,3}$. Note that it is a faithful path representation. Furthermore, $A$ and 1 do not have a leaf in common. So $K_{3,3}$ does not have an orthodox representation. If we delete one edge from $K_{3,3}$, say between $A$ and 1, then an orthodox representation is possible, see Fig. 3. If we add one edge to $K_{3,3}$, say between $A$ and $B$, then we have true twins and, in Fig. 2, we may delete $x$ and $x^-$ from $B$ and add $y$, $y^+$ and $z^-$ to $B$, add $z^+$ to $A$, and finally add $x^-$ to $C$, thus obtaining an orthodox representation of $K_{3,3}$ plus an edge. Hence we could say that $K_{3,3}$ is critically non-orthodox.

In Fig. 4, we depict a representable graph that is not faithful. We can see this as follows. In order to represent the graph, start with the canonical representation of the
Fig. 3. An orthodox representation of $K_{3,3}$ minus an edge.

Fig. 4. The apple and its core.

$K_{3,3}$ on $A, B, C, 1, 2, 3$ of Fig. 2. Consider the $K_{3,3}$ on $A, B, D, 1, 2, 4$. Since $D$ and 4 are not adjacent to $C$ or 3, the common vertex of the canonical representation of this $K_{3,3}$ must be $z$. Hence this representation also involves $z^{-}$ and $z^{+}$ and their other neighbors. Similarly, $y$ is the common vertex of the $K_{3,3}$ on $A, C, E, 1, 3, 5$, and $x$ is the common vertex in the representation of the $K_{3,3}$ on $B, C, F, 2, 3, 6$. This is all unique (up to the labeling of the vertices). Now we must find a subtree to represent $Q$. Since $Q$ is adjacent to all of $A, B, C, 1, 2, 3$, it follows, by the Helly property, that also $Q$ contains $p$. Furthermore, $Q$ may contain $z$, $z^{-}$, or $z^{+}$, but not any other neighbor of $z^{-}$ or $z^{+}$, for, otherwise, $Q$ would be 3-intersecting with $D$ or 4. The same argument
applied to \( x \) and \( y \) and their neighbors shows that \( Q \) is contained in \( B_2(p) \). We may take \( Q \) to consists of \( B_1(p) \). So we can represent \( Q \), but \( Q \) cannot contain any leaf of the host tree. Therefore, the graph is representable but not faithful.

Thus, we have examples showing that

\[ \text{ORTHODOX} \neq \text{FAITHFUL} \neq \text{REPRESENTABLE}. \]

Let us call the graph of Fig. 4 an \textit{apple} and the \( K_{3,3} \) on \( A, B, C, 1, 2, 3 \) its \textit{core}. Now we extend the graph as follows: loosely speaking, we let each of the other three \( K_{3,3} \)'s of the graph be a core of its own apple. Then it follows that the vertex \( Q \) can only be represented by \( B_1(p) \). Hence, if we turn \( Q \) into a pair of true twins, then only a non-strict representation is available.

Using Theorem 6, we can easily show that \( K_{3,4} \) is not representable. What about \( K_{2,n} \)? In the next section, we will construct the ‘canonical’ representations of \( K_{2,4} \). These are all faithful, and one of them is orthodox. Using these, it is simple to show that \( K_{2,5} \) is not representable.

We note here that also in the case of tolerances larger than 3, mentioned in the introduction, the graphs \( K_{2,n} \) and \( K_{n,m} \) are critical in the above sense, see [9].

5. \( \Theta \)-graphs

It is easily seen that cycles have orthodox path representations. As was mentioned above, \( K_{2,4} \) is orthodox, but \( K_{2,5} \) is not representable. Cycles of length at least 4 and \( K_{2,n} \) can be viewed as instances of the following structure: two non-adjacent vertices \( u \) and \( v \) and internally disjoint paths between \( u \) and \( v \). In the case of cycles the paths can be of any length, in the case of \( K_{2,n} \) they are all of length 2. The next proposition covers the representability of structures of this type. For convenience, we exclude cycles in the following definition. Let \( t_1, t_2, \ldots, t_k \) be integers with \( k \geq 3 \) and \( t_1 \geq t_2 \geq \cdots \geq t_k \geq 1 \). The \textit{\( \Theta \)-graph} \( \Theta(t_1, t_2, \ldots, t_k) \) consists of two non-adjacent vertices \( u \) and \( v \) and mutually internally disjoint paths \( P_1, P_2, \ldots, P_k \) between \( u \) and \( v \), where path \( P_i \) has \( t_i \) internal vertices, for \( i = 1, 2, \ldots, k \). Clearly \( \Theta(1, 1, \ldots, 1) \) with \( n \) ones is just \( K_{2,n} \).

\textbf{Theorem 7.} The only representable \( \Theta \)-graphs are \( \Theta(s, t, 1) \) and \( \Theta(s, t, 1, 1) \), with \( s \) and \( t \) integers with \( s \geq t \geq 1 \).

\textbf{Proof.} Let \( G \) be a representable \( \Theta \)-graph with representation \( (T, \mathcal{D}) \). Let \( A \) and \( B \) represent the two endpoints \( u \) and \( v \) of the paths in \( G \), respectively. For the arguments below, we have to bear in mind that two consecutive vertices on a path are represented by 3-intersecting subtrees, whereas two non-consecutive vertices or vertices not on the same path are represented by subtrees that are at most 2-intersecting.

First assume that \( A \cap B = \emptyset \). Let \( x \) be the vertex in \( A \) closest to \( B \) in \( T \), and let \( y \) be the neighbor of \( x \) on the path from \( x \) to \( B \) in \( T \). In order to get from \( u \) to \( v \)
along a path in \( G \), the path necessarily contains an internal vertex represented by a subtree containing the edge \( xy \) and a neighbor of \( x \) in \( A \). Since these vertices form an independent set in \( G \), there are at most two of these vertices. This contradicts the fact that there are at least three internally disjoint paths between \( u \) and \( v \) in \( G \). So \( A \cap B \neq \emptyset \).

Assume that \( |A \cap B| = 1 \), and let \( x \) be the unique vertex in \( A \cap B \). Then \( x \) must be a pendant vertex in \( A \) or \( B \), say in \( A \). Let \( y \) be the neighbor of \( x \) in \( A \). Note that \( y \) is not in \( B \). Now we find that each path in \( G \) contains an internal vertex represented by a subtree containing \( xy \) and another neighbor of \( y \) in \( A \). Again this produces a contradiction.

Thus, we have \( |A \cap B| = 2 \), say \( A \cap B = \{x, y\} \), where \( xy \) is an edge. Let \( T_x \) be the component of \( T - xy \) containing \( x \), and \( T_y \) be the component containing \( y \). Let \( x^- \) and \( x^+ \) be the neighbors of \( x \) in \( T_x \), and \( y^- \) and \( y^+ \) those of \( y \) in \( T_y \). Without loss of generality, we may assume that \( x^- \) is in \( A \). Now suppose that \( A \) intersects \( T_y \) only in \( y \), i.e. \( A \) does not grow into \( T_y \). If \( x^+ \) is not in \( B \), then each path in \( G \) contains a vertex represented by a subtree containing \( xy \) and another neighbor of \( y \). As above, this is impossible. So \( x^+ \) must be in \( B \). Now each path in \( G \) must contain a vertex represented by a subtree containing \( x^- \) and another neighbor of \( x \), again impossible. So we conclude that \( A \) grows into \( T_y \). Similarly, \( B \) grows into \( T_x \) as well as \( T_y \), say \( x^- \) and \( y^- \) are in \( A \) and \( x^+ \) and \( y^+ \) are in \( B \).

Let us call a path \( P_t \) in \( G \) of type \( x \) if it contains a vertex represented by a subtree containing \( x^- \), \( x, x^+ \), and of type \( y \) if it contains a vertex represented by a subtree containing \( y^- \), \( y, y^+ \). Clearly, there is at most one path in \( G \) of type \( x \), and at most one path of type \( y \). We can construct a path of type \( x \) or \( y \) of any length, as is shown in Fig. 5.

Assume \( P_t \) is a path of \( G \) not of type \( x \) or \( y \). In order to get from \( A \) to \( B \) along this path, there must be an internal vertex \( p \) of \( P_t \) represented by a subtree \( S_p \) containing \( x \) and \( y \). Then \( S_p \) must contain a neighbor of \( x \) or \( y \), say \( x^- \). Hence \( p \) is adjacent to \( u \). Assume \( p \) is not adjacent to \( v \), so that \( S_p \) does not contain \( x^+ \) or \( y^- \). Let \( w \) be the vertex after \( p \) on \( P_t \) represented by subtree \( S_w \). Then \( S_w \) intersects \( A \), whence also \( S_p \), in at most \( x \) and \( y \), which is impossible. So \( p \) must be adjacent to \( v \) as well. We conclude that \( S_p \) contains \( y^- \) but not \( x^+ \) or \( y^- \), so that \( S_p \) is not of type \( x \) or type \( y \). Let us call such a path a path of type \( xy \). Clearly, there are at most two paths of type \( xy \) in \( G \). To get two such paths we may take \( S_p = \{x^-, x, y, y^+\} \) and \( S_q = \{x^+, x, y, y^-\} \). We can combine these paths with one of type \( x \) (with all representing subtrees contained in \( T_x \)) and one of type \( y \) (with all representing subtrees contained in \( T_y \)). So we conclude that \( \Theta(s, t, 1) \) and \( \Theta(s, t, 1) \) are the only representable \( \Theta \)-graphs, with \( s \geq t \geq 1 \). This concludes the proof. \( \square \)

In Fig. 5, representations of \( \Theta(s, t', 1, 1) \), with \( s \geq t' \geq 1 \), are given by there generating leaves. Here the path between \( x^+ \) and \( x^{++} \) is of length \( s - 1 \), and the path between \( y^+ \) and \( y^{++} \) is of length \( t' - 1 \). It is assumed that, for \( s = 1 \), the vertices \( x^+ \) and \( x^{++} \) coincide, and that \( x^+ \) has only two pendant vertices, viz. those labeled with letters
only. The same holds for $y^+$ in the case that $t' = 1$. If we do not use the leaves labeled $(A)$ or $(B)$, then we obtain a faithful path representation. To get an orthodox representation, we need to include the leaves labeled $(A)$ and $(B)$ for the subtrees $A$ and $B$, respectively. In the case $s = t' = 1$, we get all the possible representations of $K_{2,4}$, of which only the one using the leaves labeled $(A)$ and $(B)$ is orthodox. It is easily seen that we cannot add a subtree which only 3-intersects with $A$ and $B$ in this case, whence $K_{2,5}$ is not representable. Note that $\Theta(s, t, r)$, with $r > 1$, and $\Theta(s, t, 1, 1)$ are minimally non-representable in the following sense: they are non-representable, but as soon as we delete a vertex, or an edge, then the remaining graph is representable.

6. A 3-simplicial elimination scheme

A vertex $v$ of a graph $G$ is simplicial if the set of its neighbors induces a complete graph. A classical result is the following: a graph $G$ is chordal if and only if it admits a simplicial elimination scheme. Such a scheme is an ordering of the vertices $v_1, v_2, \ldots, v_n$ of $G$ such that $v_i$ is simplicial in the subgraph of $G$ induced by $\{v_i, v_{i+1}, \ldots, v_n\}$, for $i = 1, 2, \ldots, n$. The question arises whether representable graphs, being a generalization of chordal graphs, admit some elimination scheme, or are even characterized by it. We present a first result in this direction. A $k$-simplex is a graph that can be vertex covered by at most $k$ complete graphs. Note that a $k$-simplex is an $n$-simplex, for any $n \geq k$. Let us call a vertex $v$ of a graph $G$ $k$-simplicial if its neighbors induce a $k$-simplex in $G$. A $k$-simplicial elimination scheme is an ordering of the vertices $v_1, v_2, \ldots, v_n$ of
Let $G$ be a representable graph. Then $G$ admits a 3-simplicial elimination scheme. If $G$ is orthodox, then $G$ admits a 2-simplicial elimination scheme.

Proof. Let $(F, \mathcal{S})$ be an ordered representation of $G$. Let $S$ be a subtree in the representation with maximal root $r$, i.e. no other subtree in the representation has a root strictly above $r$. Say, $S$ represents $u$ in $G$. Since $S$ contains at least three vertices, $r$ has two children. If $r$ has no grandchildren, then $S$ consists precisely of $r$ and its two children, whence $N(u)$ is a clique. So we may assume that $r$ has four grandchildren.

Let $z$ be the parent of $r$ in $T$, let $p$ and $q$ be the children of $r$ in $T$, let $s$ and $t$ be the children of $p$, and let $x$ and $y$ be the children of $q$. For any two non-adjacent vertices $a$ and $b$ of $T$, let $K_{ab}$ be the complete graph of vertices in $G$ represented by subtrees of $T$ containing the $a,b$-path in $T$.

Case 1: $S$ contains only one child of $r$. Say, $p$ is the only child of $r$ in $S$. Then $N(u)$ is vertex covered by $K_{rs}$ and $K_{rt}$ only, and we are done.

Case 2: $S$ does not contain any children of, say, $q$. Note that, in this case, $G$ is not faithful. Now $N(u)$ is vertex covered by $K_{rs}$, $K_{rt}$, and $K_{pq}$, and again we are done.

Case 3: $S$ contains, for each child of $r$, only one grandchild. Say, $S$ contains $s$ and $y$ but not $t$ or $x$. Note that, in this case, $S$ can only contain leaves of $T$ above $s$ or $y$. Then $N(u)$ is vertex covered by $K_{rs}$, $K_{ry}$, and $K_{pq}$. If $G$ is orthodox, each neighbor of $u$ must be represented by a subtree sharing a leaf with $S$, whence $N(u)$ is vertex covered by $K_{rs}$ and $K_{ry}$, and we are done.

Case 4: $S$ contains exactly three grandchildren of $r$. Say, $S$ contains $s$, $t$ and $x$ but not $y$. Consider all representing subtrees having $r$ as root. If there is such a subtree containing only one child of $r$, then we can apply Case 1 on this subtree. So we may assume that all subtrees in the representation with $r$ as root contain $p$ as well as $q$. Hence all vertices represented by such subtrees are in $K_{pq}$. The other neighbors of $u$ are in $K_{ps}$ or $K_{px}$. Thus we conclude that $u$ is 3-simplicial.

Let $G$ be orthodox. We will show that $N(u)$ is vertex covered by the complete graphs $K_{zp}\cup(K_{rs}\cap K_{rt})$ and $K_{zt}$. First note that, since $G$ is orthodox, we have $K_{zp}=K_{zt}\cup K_{zt}$. Hence $K_{zp}\cup(K_{rs}\cap K_{rt})$ is a complete graph. Assume that $u$ has a neighbor $w$, which is not vertex covered by the two complete graphs, say, it is represented by the subtree $W$ of $T$. Then $W$ cannot contain $z$, whence $r$ is the root of $W$. Furthermore, $W$ cannot contain $x$, otherwise $w$ is in $K_{zt}$, and $W$ cannot contain both $s$ and $t$, otherwise $w$ is in $K_{rs}\cap K_{rt}$. So $W$ contains only two grandchildren of its root $r$, and we are done by Case 3 (the other cases do not apply, $G$ being orthodox).

Case 5: $S$ contains the four grandchildren of $r$. We may assume that each subtree in the representation with $r$ as root contains all four grandchildren of $r$. For, otherwise,
we may choose a subtree on which one of the previous cases applies. Now the subtrees in the representation with \( r \) as root form a complete graph \( K^* \) in \( G \). Note that the complete graph \( K^* \) is contained in the complete graph \( K_{pq} \), so \( N(u) \) is vertex covered by \( K_{pq}, K_{zp} \) and \( K_{zq} \). In the case of orthodoxy, all subtrees with root strictly below \( r \) and 3-intersecting with \( S \) all contain the parent \( z \) of \( r \), and at least one child and at least one grandchild of \( r \). This implies that \( K_{zp} \cup K^* \) and \( K_{zq} \cup K^* \) are complete graphs covering \( N(u) \). This completes the proof.

The converse is not true: having a 2-simplicial elimination scheme does not force a graph to be even representable by arbitrary subtrees. Take, for instance the graph \( \Theta(2, 2, 2) \). The internal vertices of the three paths are all 2-simplicial. Hence, we may peel them off first before getting to the common endpoints of the paths.

We have seen above that \( K_{1,3} \) is faithful. Trivially, it is 3-simplicial but not 2-simplicial. Another nice example is the 3-cube \( Q_3 \): it has no 2-simplicial vertex, whence it is not orthodox, but, up to the labeling of the vertices, it has a unique faithful path representation on \( B_2(x, y) \), where \( xy \) is an edge, see Fig. 6. We obtain this representation easily by using the ideas in the proof of Theorem 8 involving subtrees with maximal root.

The question remains whether we can find a nice condition \( C \) on a graph \( G \) such that \( C \) plus a 2-simplicial elimination scheme guarantees that \( G \) is orthodox, and a nice condition \( C^* \) on a graph \( G \) such that \( C^* \) plus a 3-simplicial elimination scheme guarantees that \( G \) is representable.

Note that the complement of a 2-simplex is 2-colorable and the complement of a 3-simplex is 3-colorable, but of a special type. The last fact raises the question whether the recognition problem for faithful graphs may be NP-complete.

Another relevant question, which is outside the scope of this paper, seems to be which connected graphs are characterized by having a \( k \)-simplicial elimination scheme, say, for \( k = 2 \) or \( 3 \).
7. A non-representability result

The $\Theta$-graphs $\Theta(t_1, t_2, \ldots, t_k)$ with $t_1 > 1$ or $k \geq 5$ form an infinite class of non-representable graphs. But we are still very far from a complete list of forbidden subgraphs for representable graphs. Here we present some first, simple non-representability results. The \emph{local independence number} of a graph $G$ is the minimum independence number of the subgraphs induced by the neighborhoods $N(u)$ in $G$.

**Proposition 9.** Let $G$ be a graph with local independence number at least four. Then $G$ is not representable.

**Proof.** A graph with local independence number four does not contain a 3-simplicial vertex. \qed

**Theorem 10.** Let $G = (V, E)$ be a graph with local independence number at least three such that the vertices with local independence number exactly three form an independent set. Then $G$ is not representable.

**Proof.** Assume the contrary, and let $(F, \mathcal{S})$ be an ordered representation of $G$. Let $S$ be a subtree of $F$ in the representation with maximal root $r$. Say, $S$ represents vertex $u$ of $G$. Let $p$ and $q$ be the children of $r$, and let $z$ be the parent of $r$. If $S$ would not contain both children of $r$, then $u$ would be 2-simplicial. So $S$ contains both $p$ and $q$. Let $a, b, c$ be three independent neighbors of $u$, with representing subtrees $S_a, S_b$ and $S_c$, respectively. Not all three of them can contain $z$. Say, $S_a$ does not contain $z$, so that $r$ is also the root of $S_a$. Then also $S_a$ has maximal root $r$, whence it contains both children of $r$. Then, $b$ and $c$ being not adjacent to $a$, it follows that neither $S_b$ nor $S_c$ can have $r$ as root. So both contain the parent $z$ of $r$. Now $u$ and $a$ are adjacent vertices, and, by assumption, at least one of them has four independent neighbors.

Thus, we have established the existence of a vertex represented by a subtree with maximal root which has four independent neighbors. Without loss of generality, it is $u$ with $a$, $b$, $c$, and $d$ as four independent neighbors, where also $a$ is represented by a subtree with $r$ as root. Then the subtrees representing $b$, $c$ and $d$ all contain $z$, $r$ and a child of $r$, which implies that they cannot be independent. This contradiction settles the proof. \qed

8. Concluding remarks

We have presented a number of results and ideas on the topic of the tolerance intersection graph of a family of subtrees of a binary tree with constant tolerance 3. This class is the first natural generalization of chordal graphs from the viewpoint of tolerance intersection graphs.
The main open problem for now is finding a characterizing list of forbidden subgraphs for the three classes of representable graphs. The case of orthodox graphs would be a good start. We have good candidates for the list, such as the \( \Theta \)-graphs with at least three non-trivial paths or with at least five paths. On the other hand, such classes as complements of paths or cycles are less promising: for the small ones, we have fairly straightforward representations, but we do not yet have a general construction or a non-representability result for the larger ones.

A second open problem is the complexity of recognizing the various classes. Also there is the question whether there exist good algorithms for coloring or clique covering. Another intriguing problem is to obtain characterizations involving the 2-simplicial and 3-simplicial elimination schemes. These elimination schemes seem already to be of interest in itself. Determining the representation number of a representable graph is a different possible line of research.

Finally, there are the problems of determining the representation number, and characterizing the differences between strict and non-strict representations, and between proper and non-proper representations.

References