Intersection Graphs on Trees with a Tolerance

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An intriguing theme in graph theory is that of the intersection graph of a family of subsets of a set: the vertices of the graph are represented by the subsets of the family and adjacency is defined by a non-empty intersection of the corresponding subsets. Prime examples are interval graphs and chordal graphs. An interval graph is the intersection graph of a family of closed intervals on the real line. A classical result is the characterization of interval graphs by forbidden subgraphs by Lekkerkerker and Boland [7]. A chordal graph is a graph without induced cycles of length four or more. They were proven to be the intersection graphs of a family of subtrees of a host tree by Gavril [1] and Walter [9]. In [8] McMorris and Scheinerman observed, without adding a proof, that this result may be sharpened in the following way: a graph $G$ is chordal if and only if it is the intersection graph of a family of leaf-generated subtrees of a full binary tree such that intersecting subtrees share a leaf. In this very special type of representation ‘intersection’ may be even replaced by ‘edge-intersection’. Note that, if there are no restrictions on the host tree, then any graph is the edge-intersection graphs of some family of subtrees of a tree. Note that one may even take the tree and all subtrees to be stars (of high enough degree).

In this paper we study a class of generalizations of chordal graphs using the idea of tolerance intersection graph, cf. [2, 3]. In [2] the idea of tolerance was introduced. And in [3] a very broad Master Plan, based on this idea from [2], was formulated: the study of $\varphi$-tolerance $(S, \mu)$-graphs.

Let $T$ be a tree of maximum degree $\Delta$, called the *host tree*, and let \{$S_v\}$ $v \in V$ be a (finite) family of non-empty subtrees of $T$ of maximum degree $d$, and let $t$ be a positive integer, called the *tolerance*. Let $G = (V, E)$ be the graph with vertex set $V$, where $uv$ is an edge if and only if $|S_u \cap S_v| \geq t$. Then $(T, \{S_v\}_{v \in V}, t)$ is a $\{\Delta, d, t\}$-representation of $G$. A graph $G$ is said to be $\{\Delta, d, t\}$-representable if it has a $\{\Delta, d, t\}$-representation. We will denote the class of $\{\Delta, d, t\}$-representable graphs simply by $\{\Delta, d, t\}$. This idea of a representation by subtrees of a host tree with a tolerance on the intersections opens up a whole new area of graph theory. On the one hand, known classes of graphs are subsumed under this approach. On the other hand, many new questions and problems present themselves. We will survey the state of the art at this moment and present some challenging open problems.
The classical result of Gavril [1] and Walter [9] in this terminology reads as follows: a graph is chordal if and only if it is \( \{\Delta, d, 1\} \)-representable for some \( \Delta \) and \( d \). And the result by McMorris and Scheinerman [8] then reads that a graph is chordal if and only if it is in \( \{3, 3, 1\} \) or, equivalently, if and only if it is in \( \{3, 3, 2\} \). From this point of view, the first step away from chordal graphs is the class \( \{3, 3, 3\} \). This is the class of graphs \( G = (V, E) \) having a \( \{3, 3, 3\} \)-representation, that is, there is a binary host tree \( T \) and a family of binary subtrees \( \{S_v\}_{v \in V} \) of \( T \) such that \( u \) and \( v \) are adjacent in \( G \) if and only if \( |S_u \cap S_v| \geq 3 \). It is a simple exercise to show that cycles of any length are in this class.

In [4] a study of the class \( \{3, 3, 3\} \) is initiated. In the generalization of chordal graphs, this case may be considered as the first step away from chordal graphs from the viewpoint of tolerance intersection graphs. But it is already a giant step: the relatively easy results mentioned above on chordal graphs are not available here. For example, it turns out that it makes a difference, whether we allow the representing subtrees to be arbitrary subtrees of the host tree or that we require all the representing subtrees to be leaf-generated. We will show that, basically, there are three different classes, depending on such conditions as the subtrees being allowed to be arbitrary subtrees or necessarily leaf-generated. We present results on representability and non-representability of graphs, but we are still far from a characterization by forbidden subgraphs of the class \( \{3, 3, 3\} \) and its pertinent subclasses.

A vertex \( v \) of a graph \( G \) is simplicial if the set of its neighbors induces a complete graph. A classical result is the following: a graph \( G \) is chordal if and only if it admits a simplicial elimination scheme. Such a scheme is an ordering of the vertices \( v_1, v_2, \ldots, v_n \) of \( G \) such that \( v_i \) is simplicial in the subgraph of \( G \) induced by \( \{v_i, v_{i+1}, \ldots, v_n\} \), for \( i = 1, 2, \ldots, n \). The question arises whether \( \{\Delta, d, t\} \)-representable graphs, being a generalization of chordal graphs, admit some elimination scheme, or are even characterized by it. We present a first result in this direction. A \( k \)-simplex is a graph consisting of \( k \), possibly intersecting, complete subgraphs, in other words, a \( k \)-simplex can be vertex covered by at most \( k \) complete graphs. Note that a \( k \)-simplex is an \( n \)-simplex, for any \( n \geq k \). Let us call a vertex \( v \) of a graph \( G \) \( k \)-simplicial if its neighbors induce a \( k \)-simplex in \( G \). A \( k \)-simplicial elimination scheme is an ordering of the vertices \( v_1, v_2, \ldots, v_n \) of \( G \) such that \( v_i \) is \( k \)-simplicial in the subgraph of \( G \) induced by \( \{v_i, v_{i+1}, \ldots, v_n\} \), for \( i = 1, 2, \ldots, n \). In [4] it is proved that a \( \{3, 3, 3\} \)-representable graph admits a 3-simplicial elimination scheme. A nice subclass of the class \( \{3, 3, 3\} \) even admits a 2-simplicial elimination scheme. We call these the 3-orthodox graphs. This idea of ‘orthodoxy’ may be employed to construct larger representable graphs from smaller ones. Or, conversely, to break up large graphs into smaller pieces and just test the small pieces on representability to decide whether the large graph is representable or not.
Another special subclass of the class \( \{3, 3, 3\} \) is the class \( \{3, 2, 3\} \). Here the host tree is a binary tree, but all subtrees must be paths. An example of a result here are nice new characterizations of outerplanar graphs and maximal outerplanar graphs, cf. [6]. The results on outerplanar graphs may be generalized to arbitrary planar graphs. But then the host graph need not be a tree, and we must allow also cycles as representing subgraphs. Thus these results do not fit into the approach taken here (but they still fit into the general scheme of tolerance intersection graphs of [3]).

In [5] the study of the general class \( \{\Delta, d, t\} \) is initiated. We mention here some of the main results. We denote by \( N(u) \) the neighborhood of \( u \), i.e. the set of neighbors of \( u \). Let \( G \) be a connected graph with bandwidth \( b \), maximum degree \( D \) and \( D_2 = \max_{u \neq v} |N(u) \cap N(v)| \). Then \( G \) is in class \( \{3, 3, t\} \), for any \( t > b + D_2 \log D \). Let us denote by \( \lambda(u) \) the independence number of the subgraph of \( G \) induced by \( N(u) \). The \( \lambda(G) \) the local independence number of \( G \) is the maximum of the independence numbers of \( \lambda(u) \). Let \( F_k \) be the rooted \( k \)-ary tree, that is, the directed tree in which all nodes, except the leaves, have outdegree \( k \). And let \( \gamma(k, t) \) be the number of subtrees of \( F_k \) of order \( t \) containing the root of \( F_k \). Let \( G \) be a graph such that \( \lambda(G) > \gamma(k, t) \), then \( G \) is not in class \( \{\Delta, k + 1, t\} \), for any \( \Delta \geq k + 1 \). Loosely speaking, if \( \lambda(G) \) is too large with respect to \( t \), the graph \( G \) is not in class \( \{\Delta, d, t\} \), for any \( \Delta \) and \( d \). So, if we fix the tolerance, then we get a restricted class of graphs. In [5] it is even proved that, for any fixed \( t \), there are infinitely many minimal non-\( \{\Delta, d, t\} \) graphs.

References


