On a new system of nonlinear $A$-monotone multivalued variational inclusions

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Abstract

In this paper, we introduce and study a new system of nonlinear $A$-monotone multivalued variational inclusions in Hilbert spaces. By using the concept and properties of $A$-monotone mappings, and the resolvent operator technique associated with $A$-monotone mappings due to Verma, we construct a new iterative algorithm for solving this system of nonlinear multivalued variational inclusions associated with $A$-monotone mappings in Hilbert spaces. We also prove the existence of solutions for the nonlinear multivalued variational inclusions and the convergence of iterative sequences generated by the algorithm. Our results improve and generalize many known corresponding results.

Keywords: $A$-monotone mapping; Resolvent operator technique; Nonlinear multivalued variational inclusion system; Existence; Convergence

1. Introduction

In 2001, Verma [13] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of system of variational

inequalities. Very recently, Cho et al. [1] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness theorems of solutions for the system of nonlinear variational inequalities. They also constructed an iterative algorithm for approximating the solution of the system of nonlinear variational inequalities. Some related works, we refer to [2,6–8,14].

Furthermore, Fang and Huang [4] introduced a new class of $H$-monotone mappings in the context of solving a system of variational inclusions involving a combination of $H$-monotone and strongly monotone mappings based on the resolvent operator technique. The notion of the $H$-monotonicity has revitalized the theory of maximal monotone mappings in several directions, especially in the domain of applications. Later, Yan et al. [20] introduce and study a new system of set-valued variational inclusions with $H$-monotone operators in Hilbert spaces. By using the resolvent operator method associated with $H$-monotone operator due to Fang and Huang, the authors constructed a new iterative algorithm for solving this kind of system of set-valued variational inclusions and proved the existence of solutions for the system of set-valued variational inclusions and the convergence of iterative sequences generated by the algorithm.

On the other hand, Verma [16] introduced the notion of the $A$-monotone mappings and its applications to the solvability of systems of nonlinear variational inclusions. As Verma pointed out, “the class of the $A$-monotone mappings generalizes the $H$-monotonicity. On the top of that, $A$-monotonicity originates from hemivariational inequalities, and emerges as a major contributor to the solvability of nonlinear variational problems on nonconvex settings.” As a matter of fact, some nice examples on $A$-monotone (or generalized maximal monotone) mappings can be found in Naniewicz and Panagiotopoulos [10] and Verma [15]. Hemivariational inequalities—initiated and developed by Panagiotopoulos [11]—are connected with nonconvex energy functions and turned out to be useful tools proving the existence of solutions of nonconvex constrained problems. We note that the $A$-monotonicity is defined in terms of relaxed monotone mappings—a more general notion than the monotonicity or strong monotonicity—which gives a significant edge over the $H$-monotonicity. Very recently, Verma [18] explored the role of $A$-monotonicity in constructing a general framework for $A$-resolvent operator technique, and considered the existence and uniqueness of the solution and convergence analysis for approximate solution of a new class of nonlinear variational inclusion problems involving relaxed cocoercive mappings using $A$-resolvent operator technique.

Inspired and motivated by the recent works [5,12,17–20], in this paper, we introduce and study a new system of nonlinear $A$-monotone multivalued variational inclusions in Hilbert spaces. By using the concept and properties of $A$-monotone mappings, and the resolvent operator technique associated with $A$-monotone mappings due to Verma, we construct a new iterative algorithm for solving this system of nonlinear multivalued variational inclusions associated with $A$-monotone mappings in Hilbert spaces. We also prove the existence of solutions for the nonlinear multivalued variational inclusions and the convergence of iterative sequences generated by the algorithm. Our results improve and generalize many known corresponding results of [1,2,4,14,16,20].

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle \cdot, \cdot \rangle$, respectively, $2^\mathcal{H}$ denote the family of all the nonempty subsets of $\mathcal{H}$. If $M : \mathcal{H} \to 2^\mathcal{H}$ is a multivalued mapping, then we denote the effective domain $D(M)$ of $M$ as follows:

$$D(M) = \{x \in \mathcal{H} : M(x) \neq \emptyset \}.$$
In the sequel, let us recall some concepts and lemmas.

**Definition 2.1.** Let $T, A : \mathcal{H} \to \mathcal{H}$ be two single-valued mappings. Then $T$ is said to be

(i) monotone if
\[ \langle T(x) - T(y), x - y \rangle \geq 0, \quad x, y \in \mathcal{H}; \]

(ii) strictly monotone, if $T$ is monotone and
\[ \langle T(x) - T(y), x - y \rangle = 0 \]
if and only if $x = y$;

(iii) $r$-strongly monotone, if there exists a constant $r > 0$ such that
\[ \langle T(x) - T(y), x - y \rangle \geq r \| x - y \|^2, \quad x, y \in \mathcal{H}; \]

(iv) $\gamma$-strongly monotone with respect to $A$, if there exists a constant $\gamma > 0$ such that
\[ \langle T(x) - T(y), A(x) - A(y) \rangle \geq \gamma \| x - y \|^2, \quad x, y \in \mathcal{H}; \]

(v) $m$-relaxed cocoercive with respect to $A$, if there exists a constant $m > 0$ such that
\[ \langle T(x) - T(y), A(x) - A(y) \rangle \geq m \| T(x) - T(y) \|^2, \quad x, y \in \mathcal{H}; \]

(vi) $(\alpha, \xi)$-relaxed cocoercive with respect to $A$, if there exist constants $\alpha, \xi > 0$ such that
\[ \langle T(x) - T(y), A(x) - A(y) \rangle \geq -\alpha \| x - y \|^2 - \xi \| x - y \|^2, \quad x, y \in \mathcal{H}; \]

(vii) $s$-Lipschitz continuous, if there exists a constant $s > 0$ such that
\[ \| T(x) - T(y) \| \leq s \| x - y \|, \quad x, y \in \mathcal{H}. \]

**Example 2.1.** [19] Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. Then $I - T$ is $\frac{1}{2}$-cocoercive and $\gamma$-relaxed cocoercive for $\frac{1}{2} > -\gamma$, where $\gamma > 0$.

**Example 2.2.** [18] Let $T : \mathcal{H} \to \mathcal{H}$ be an $r$-strongly monotone (and hence $r$-expanding) mapping. Then $T$ is $(1, r + r^2)$-relaxed cocoercive.

**Definition 2.2.** Let $H, A : \mathcal{H} \to \mathcal{H}$ be two single-valued mappings. Then multivalued mapping $M : \mathcal{H} \to 2^\mathcal{H}$ is said to be

(i) monotone if
\[ \langle u - v, x - y \rangle \geq 0, \quad x, y \in D(M), \ u \in M(x), \ v \in M(y); \]

(ii) $r$-strongly monotone, if there exists a constant $r > 0$ such that
\[ \langle u - v, x - y \rangle \geq r \| x - y \|^2, \quad x, y \in D(M), \ u \in M(x), \ v \in M(y); \]

(iii) $m$-relaxed monotone, if there exists a constant $m > 0$ such that
\[ \langle u - v, x - y \rangle \geq -m \| x - y \|^2, \quad x, y \in D(M), \ u \in M(x), \ v \in M(y); \]

(iv) maximal monotone if and only if
(a) $M$ is monotone,
(b) and for every $x \in D(M)$ and $u \in E$ such that
\[
\langle u - v, x - y \rangle \geq 0, \quad \forall y \in D(M), \ v \in M(y)
\]
implies $u \in M(x)$;

(v) $H$-monotone if $M$ is monotone and $(H + \lambda M)(E) = E$ for all $\lambda > 0$.

This is equivalent to stating that $M$ is $H$-monotone if $M$ is monotone and $(H + \rho M)$ is
maximal monotone;

(vi) $A$-monotone with constant $m$ if
(a) $M$ is $m$-relaxed monotone,
(b) and $A + \lambda M$ is maximal monotone for all $\lambda > 0$.

Remark 2.1. $H$-monotone operator was first introduced by Fang and Huang [3]. Obviously, if $m = 0$, that is, $M$ is 0-relaxed monotone, then the $A$-monotone mappings reduce to an
$H$-monotone operator. Therefore, the class of $A$-monotone mappings provides a unifying frame-
works for classes of maximal monotone operators and $H$-monotone operators. For details about
these operators, we refer the reader to [3,16,17,19] and the references therein.

Example 2.3. [9, Lemma 7.11] Let $X$ be a reflexive Banach space with $X^*$ its dual, and
$A : X \rightarrow X^*$ be $m$-strongly monotone and $f : X \rightarrow \mathbb{R}$ be locally Lipschitz such that $\partial f$ is
$\alpha$-relaxed monotone. Then $\partial f$ is $A$-monotone, that is, $A + \partial f$ is maximal monotone for
$m - \alpha > 0$, where $m, \alpha > 0$.

Example 2.4. [15, Theorem 5.1] Let $X$ be a reflexive Banach space with $X^*$ its dual, and
$A : X \rightarrow X^*$ be $m$-strongly monotone and $B : X \rightarrow X^*$ be $c$-Lipschitz continuous. Let $f : X \rightarrow \mathbb{R}$
be locally Lipschitz such that $\partial f$ is $\alpha$-relaxed monotone. Then $\partial f$ is $(A - B)$-monotone.

Definition 2.3. Let $F : \mathcal{H} \rightarrow 2^{2^\mathcal{H}}$ be a multivalued mapping. For all $x, y \in \mathcal{H}$, the mapping
$T(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ is called to be

(i) $\tau$-Lipschitz continuous with respect to the first argument, if there exists a constant $\tau > 0$
such that
\[
\| T(x, \cdot) - T(y, \cdot) \| \leq \tau \| x - y \| \quad \forall x, y \in \mathcal{H};
\]

(ii) $F$ is said to be $\xi$-$\hat{H}$-Lipschitz continuous, if there exists a constant $\xi > 0$ such that
\[
\hat{H}(F(x), F(y)) \leq \xi \| x - y \|, \quad \forall x, y \in \mathcal{H},
\]

where $\hat{H} : 2^{2^\mathcal{H}} \times 2^{2^\mathcal{H}} \rightarrow (-\infty, +\infty) \cup \{ +\infty \}$ is the Hausdorff pseudo-metric, i.e.,
\[
\hat{H}(A, B) = \max\left\{ \sup_{x \in A} \inf_{y \in B} \| x - y \|, \sup_{y \in A} \inf_{x \in B} \| x - y \| \right\}, \quad \forall A, B \in 2^{2^\mathcal{H}}.
\]

Note that if the domain of $\hat{H}$ is restricted to closed bounded subsets, then $\hat{H}$ is the Hausdorff metric.

In a similar way, we can define Lipschitz continuity of the mapping $F(\cdot, \cdot)$ with respect to the
second argument.
**Lemma 2.1.** [16,17] Let $A : \mathcal{H} \to \mathcal{H}$ be $r$-strongly monotone and $M : \mathcal{H} \to 2^\mathcal{H}$ be $A$-monotone with constant $m$. Then $M$ is maximal monotone and the $A$-resolvent operator $J_{A,M}^\rho : \mathcal{H} \to \mathcal{H}$ associated with $M$ and defined by

$$J_{A,M}^\rho(x) = (A + \rho M)^{-1}(x), \quad \forall x \in \mathcal{H},$$

is $(\frac{1}{r-\rho m})$-Lipschitz continuous for $0 < \rho < \frac{r}{m}$, i.e.,

$$\|J_{A,M}^\rho(x) - J_{A,M}^\rho(y)\| \leq \frac{1}{r-\rho m}\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

If $m = 0$, that is, $M$ is 0-relaxed monotone, then we have the following result.

**Lemma 2.2.** [3] Let $H : \mathcal{H} \to \mathcal{H}$ be a $r$-strongly monotone operator and $M : \mathcal{H} \to 2^\mathcal{H}$ be an $H$-monotone operator. Then the resolvent operator $J_{H,M}^\rho$ is $\frac{1}{r}$-Lipschitz continuous, i.e.,

$$\|J_{H,M}^\rho(x) - J_{H,M}^\rho(y)\| \leq \frac{1}{r}\|x - y\|, \quad \forall x, y \in \mathcal{H},$$

where $J_{H,M}^\rho(x) = (H + \rho M)^{-1}(x)$ for all $x \in \mathcal{H}$.

### 3. Variational inclusion systems and iterative algorithms

In this section, we shall introduce a new system of nonlinear $A$-monotone multivalued variational inclusions and construct a new iterative algorithm for solving this kind of system of nonlinear variational inclusions in Hilbert spaces.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces, $S : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$, $T : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$, $p : \mathcal{H}_1 \to \mathcal{H}_1$ and $q : \mathcal{H}_2 \to \mathcal{H}_2$ be single-valued mappings, $E : \mathcal{H}_1 \to 2^\mathcal{H}_1$, $F : \mathcal{H}_2 \to 2^\mathcal{H}_2$ be any two multivalued mappings. Let $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$, $A_2 : \mathcal{H}_2 \to \mathcal{H}_2$, $M : \mathcal{H}_1 \to 2^\mathcal{H}_1$ and $N : \mathcal{H}_2 \to 2^\mathcal{H}_2$ be any nonlinear mappings, $f : \mathcal{H}_1 \to \mathcal{H}_1$, $g : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear mappings with $f(\mathcal{H}_1) \cap D(M) \neq \emptyset$ and $g(\mathcal{H}_2) \cap D(N) \neq \emptyset$, respectively. Then the problem of finding $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in E(x)$, $v \in F(y)$ such that

$$\begin{align*}
0 &\in S(p(x),v) + M(f(x)), \\
0 &\in T(u, q(y)) + N(g(y)),
\end{align*}$$

(3.1)

is called the system of generalized nonlinear multivalued variational inclusion problems.

If $p = q = f = g \equiv I$, the identity mapping, then the problem (3.1) is equivalent to the following system of set-valued variational inclusion problem: find $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in E(x)$, $v \in F(y)$ such that

$$\begin{align*}
0 &\in S(x,v) + M(x), \\
0 &\in T(u, y) + N(y),
\end{align*}$$

(3.2)

which is considered by Huang and Fang [6].

If $E : \mathcal{H}_1 \to \mathcal{H}_1$ and $F : \mathcal{H}_2 \to \mathcal{H}_2$ are two single-valued mappings, then the problem (3.2) collapses to the following system of nonlinear variational inclusion problem: find $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$\begin{align*}
0 &\in S(x,F(y)) + M(x), \\
0 &\in T(E(x), y) + N(y),
\end{align*}$$

(3.3)

which is studied by Verma [16] with $E = F = I$. 

If \( M(x) = \partial \phi(x) \) and \( N(y) = \partial \varphi(y) \) for all \( x \in H_1 \) and \( y \in H_2 \), where \( \phi : H_1 \to \mathbb{R} \cup \{ +\infty \} \) and \( \varphi : H_2 \to \mathbb{R} \cup \{ +\infty \} \) are two proper, convex and lower semi-continuous functionals, and \( \partial \phi \) and \( \partial \varphi \) denote the subdifferential operators of \( \phi \) and \( \varphi \), respectively, then the problem (3.3) reduces to the following problem: find \((x, y) \in H_1 \times H_2 \) such that

\[
\begin{align*}
\langle S(x, F(y)), a - x \rangle + \varphi(a) - \varphi(x) \geq 0, & \quad \forall a \in H_1, \\
\langle T(E(x), y), b - y \rangle + \varphi(b) - \varphi(y) \geq 0, & \quad \forall b \in H_2,
\end{align*}
\]

which is called a system of nonlinear mixed variational inequalities. Some special cases of the problem (3.4) can be found in [14]. Further, if \( E = F \equiv I \), then the problem (3.4) reduces to the system of nonlinear variational inequalities problem considered by Cho et al. [1].

If \( M(x) = \partial \delta K_1(x) \) and \( N(y) = \partial \delta K_2(y) \) for all \( x \in K_1 \) and \( y \in K_2 \), where \( K_1 \) and \( K_2 \), respectively, are nonempty closed convex subsets of \( H_1 \) and \( H_2 \), and \( \partial \delta K_1 \) and \( \partial \delta K_2 \) denote indicator functions of \( K_1 \) and \( K_2 \), respectively, then the problem (3.4) becomes to determining an element \((x, y) \in K_1 \times K_2 \) such that

\[
\begin{align*}
\langle S(x, F(y)), a - x \rangle \geq 0, & \quad \forall a \in K_1, \\
\langle T(E(x), y), b - y \rangle \geq 0, & \quad \forall b \in K_2,
\end{align*}
\]

which is just the problem in [7] when \( E \) and \( F \) are single-valued and \( E = F \equiv I \).

If \( H_1 = H_2 = H \), \( K_1 = K_2 = K \), \( S(x, F(y)) = \rho F(y) + x - y \) and \( T(E(x), y) = \lambda E(x) + y - x \) for all \( x, y \in H \), where \( \rho > 0 \) and \( \lambda > 0 \) are two constants, then the problem (3.5) is equivalent to finding an element \((x, y) \in K \times K \) such that

\[
\begin{align*}
\langle \rho F(y) + x - y, a - x \rangle \geq 0, & \quad \forall a \in K, \\
\langle \lambda E(x) + y - x, b - y \rangle \geq 0, & \quad \forall b \in K,
\end{align*}
\]

which is the system of nonlinear variational inequalities considered by Verma [14] with \( E = F \).

If \( x = y, E = F \) and \( \rho = \lambda \), then the problem (3.6) reduces to the following classical nonlinear variational inequality problem: find an element \( x \in K \) such that

\[
\langle F(x), z - x \rangle \geq 0, \quad \forall z \in K.
\]

**Lemma 3.1.** Let \( H_1 \) and \( H_2 \) be two real Hilbert spaces. Suppose that \( A_1 : H_1 \to H_1 \) and \( A_2 : H_2 \to H_2 \) are strictly monotone, \( M : H_1 \to 2^{H_1} \) is \( A_1 \)-monotone and \( N : H_2 \to 2^{H_2} \) is \( A_2 \)-monotone. Let \( S : H_1 \times H_2 \to H_1 \) and \( T : H_1 \times H_2 \to H_2 \) be any two single-valued mappings, \( p, f : H_1 \to H_1 \) and \( q, g : H_2 \to H_2 \) be any nonlinear mappings with \( f(H_1) \cap D(M) \neq \emptyset \) and \( g(H_2) \cap D(N) \neq \emptyset \), respectively, and \( E : H_1 \to 2^{H_1}, F : H_2 \to 2^{H_2} \) be any two multivalued mappings. Then a given element \((x, y, u, v) \in H_1 \times H_2 \) is a solution to the problem (3.1) if and only if \((x, y, u, v)\) satisfies

\[
\begin{align*}
f(x) &= J^\rho_{A_1, M}(A_1(f(x)) - \rho S(p(x), v)), \\
g(y) &= J^\lambda_{A_2, N}(A_2(g(y)) - \lambda T(u, q(y))),
\end{align*}
\]

where \( \rho > 0 \) and \( \lambda > 0 \) are two constants.

**Remark 3.1.** The equality (3.7) can be written as

\[
\begin{align*}
x &= (1 - \mu)x + \mu[x - f(x) + J^\rho_{A_1, M}(A_1(f(x)) - \rho S(p(x), v))], \\
y &= (1 - v)y + v[y - g(y) + J^\lambda_{A_2, N}(A_2(g(y)) - \lambda T(u, q(y)))],
\end{align*}
\]

where \( 0 < \mu, v \leq 1 \) are two parameters and \( \rho, \lambda > 0 \) are constants. This fixed point formulation enables us to suggest the following iterative algorithm.
Algorithm 3.1. Assume that \( \mathcal{H}_1, \mathcal{H}_2, A_1, A_2, M, N, S, T, p, f, q, g, E \) and \( F \) are the same as in the problem (2.1). For any given \( (x_0, y_0) \in \mathcal{H}_1 \times \mathcal{H}_2 \), we choose \( u_0 \in E(x_0) \), \( v_0 \in F(y_0) \) and let

\[
\begin{align*}
  x_1 &= (1 - \mu)x_0 + \mu \left[ x_0 - f(x_0) + J_{A_1,M}^\rho (A_1(f(x_0)) - \rho S(p(x_0), v_0)) \right] + \mu d_0, \\
  y_1 &= (1 - v)y_0 + v \left[ y_0 - g(y_0) + J_{A_2,N}^\lambda (A_2(g(y_0)) - \lambda T(u_0, q(y_0))) \right] + ve_0.
\end{align*}
\]

Since \( u_0 \in E(x_0) \) and \( v_0 \in F(y_0) \), for any \( x_1 \in \mathcal{H}_1 \), \( y_1 \in \mathcal{H}_2 \), by Nadler [9], there exist \( u_1 \in E(x_1) \), \( v_1 \in F(y_1) \) such that

\[
\begin{align*}
  \|u_0 - u_1\| &\leq (1 + 1)\hat{H}_1(E(x_0), E(x_1)), \\
  \|v_0 - v_1\| &\leq (1 + 1)\hat{H}_2(F(y_0), F(y_1)).
\end{align*}
\]

Let

\[
\begin{align*}
  x_2 &= (1 - \mu)x_1 + \mu \left[ x_1 - f(x_1) + J_{A_1,M}^\rho (A_1(f(x_1)) - \rho S(p(x_1), v_1)) \right] + \mu d_1, \\
  y_2 &= (1 - v)y_1 + v \left[ y_1 - g(y_1) + J_{A_2,N}^\lambda (A_2(g(y_1)) - \lambda T(u_1, q(y_1))) \right] + ve_1.
\end{align*}
\]

Continuing this way, we can obtain the sequences \( \{x_n\}, \{y_n\} \) satisfying

\[
\begin{align*}
  x_{n+1} &= (1 - \mu)x_n + \mu \left[ x_n - f(x_n) + J_{A_1,M}^\rho (A_1(f(x_n)) - \rho S(p(x_n), v_n)) \right] + \mu d_n, \\
  y_{n+1} &= (1 - v)y_n + v \left[ y_n - g(y_n) + J_{A_2,N}^\lambda (A_2(g(y_n)) - \lambda T(u_n, q(y_n))) \right] + ve_n
\end{align*}
\]

and choose \( u_{n+1} \in E(x_{n+1}) \) and \( v_{n+1} \in F(y_{n+1}) \) such that

\[
\begin{align*}
  \|u_n - u_{n+1}\| &\leq (1 + (n + 1)^{-1})\hat{H}_1(E(x_n), E(x_{n+1})), \\
  \|v_n - v_{n+1}\| &\leq (1 + (n + 1)^{-1})\hat{H}_2(F(y_n), F(y_{n+1})),
\end{align*}
\]

where \( 0 < \mu, v \leq 1 \) are two parameters and \( \rho, \lambda > 0 \) are constants, \( d_n \in \mathcal{H}_1, e_n \in \mathcal{H}_2 (n \geq 0) \) are errors to take into account a possible inexact computation of the resolvent operator point, and \( \hat{H}_i(., .) \) is the Hausdorff pseudo-metric on \( 2^{\mathcal{H}_i} \) for \( i = 1, 2 \).

Remark 3.2. If we choose suitable \( \mu, v, d_n, e_n, A_1, A_2, M, N, S, T, p, f, q, g, E, F \) and \( \mathcal{H}_1, \mathcal{H}_2 \), then Algorithm 3.1 can be degenerated to a number of algorithms involving many known algorithms which due to classes of variational inequalities, complementarity problems, and variational inclusions (see, for example, [1–8,12–17,19,20]).

4. Existence and convergence theorems

In this section, we will prove the existence of solutions for problem (3.1) and the convergence of iterative sequences generated by Algorithm 3.1.

Theorem 4.1. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two real Hilbert spaces. Suppose that \( A_1 : \mathcal{H}_1 \to \mathcal{H}_1 \) is \( r_1 \)-strongly monotone and \( \alpha_1 \)-Lipschitz continuous, and \( A_2 : \mathcal{H}_2 \to \mathcal{H}_2 \) is \( r_2 \)-strongly monotone and \( \alpha_2 \)-Lipschitz continuous, \( M : \mathcal{H}_1 \to 2^{\mathcal{H}_1} \) is \( A_1 \)-monotone with constant \( m_1 \) and \( N : \mathcal{H}_2 \to 2^{\mathcal{H}_2} \) is \( A_2 \)-monotone with constant \( m_2 \), \( S : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \) is a single-valued mapping such that \( S(\cdot, y) \) is \( (\gamma, r) \)-relaxed cocoercive with respect to \( A_1 \) and \( \sigma \)-Lipschitz continuous in the first variable and \( S(x, \cdot) \) is \( q \)-Lipschitz continuous in the second variable for all \( (x, y) \in \mathcal{H}_1 \times \mathcal{H}_2 \), \( T : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2 \) is a nonlinear mapping such that \( T(x, \cdot) \) is \( (\delta, s) \)-relaxed cocoercive with respect to
A2 and β-Lipschitz continuous in the second variable and \( T(\cdot, y) \) is τ-Lipschitz continuous in the first variable for all \((x, y) \in \mathcal{H}_1 \times \mathcal{H}_2\). Let \( p: \mathcal{H}_1 \to \mathcal{H}_1 \) be k-Lipschitz continuous, \( q: \mathcal{H}_2 \to \mathcal{H}_2 \) be \( \zeta \)-Lipschitz continuous, \( f: \mathcal{H}_1 \to \mathcal{H}_1 \) be \( \pi \)-strongly monotone and \( \epsilon \)-Lipschitz continuous, \( g: \mathcal{H}_2 \to \mathcal{H}_2 \) be \( \sigma \)-monotone and \( \epsilon \)-Lipschitz continuous, \( E: \mathcal{H}_1 \to C(\mathcal{H}_1) \) be \( \xi \)-H1-Lipschitz continuous, \( F: \mathcal{H}_2 \to C(\mathcal{H}_2) \) be \( \eta \)-H2-Lipschitz continuous, where \( C(\mathcal{H}_i) \) denotes the collection of all closed subsets of \( \mathcal{H}_i \) for \( i = 1, 2 \). If, in addition, for every parameters \( \mu, \nu \in (0, 1) \), there exist positive constants \( \rho \) and \( \lambda \) such that

\[
\begin{align*}
k_1 &= \sqrt{1 - 2\pi + \epsilon^2} + \frac{\sqrt{\alpha_1^2k^2 + \alpha_2^2k^2}}{r_1 - \rho m_1} < 1, \\
k_2 &= \sqrt{1 - 2\sigma + \epsilon^2} + \frac{\sqrt{\alpha_2^2k^2 + 2\lambda k^2 + \lambda^2k^2}}{r_2 - \lambda m_2} < 1, \\
\lambda\nu \xi &\leq (1 - k_1)(r_2 - \lambda m_2), \quad \rho \mu \nu \xi < \nu(1 - k_2)(r_1 - \rho m_1), \\
\sum_{i=1}^{\infty} \|d_i - d_{i-1}\|k^{-i} &< \infty, \quad \sum_{i=1}^{\infty} \|e_i - e_{i-1}\|k^{-i} < \infty, \quad \forall k \in (0, 1), \\
\lim_{n \to \infty} d_n &= \lim_{n \to \infty} e_n = 0,
\end{align*}
\]

then the iterative sequences \( \{x_n\}, \{y_n\}, \{u_n\} \) and \( \{v_n\} \) generated by Algorithm 3.1 converge strongly to \( x^*, y^*, u^* \) and \( v^* \), respectively, and \( (x^*, y^*, u^*, v^*) \) is a solution of the system of nonlinear multivalued variational inclusion problem (3.1).

**Proof.** From (3.9) and Lemma 2.1, we have

\[
\begin{align*}
\|x_{n+1} - x_n\| &\leq \|(1 - \mu)x_n + \mu [x_n - f(x_n) + J_{A_1,M}^\rho (A_1(f(x_n)) - \rho S(p(x_n), v_n))]
+ \mu d_n - \{(1 - \mu)x_{n-1} + \mu [x_{n-1} - f(x_{n-1})]
+ J_{A_1,M}^\rho (A_1(f(x_{n-1})) - \rho S(p(x_{n-1}), v_{n-1}))\} + \mu d_{n-1}\|
\leq (1 - \mu)\|x_n - x_{n-1}\| + \mu \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|
+ \mu \|J_{A_1,M}^\rho (A_1(f(x_n)) - \rho S(p(x_n), v_n))
- J_{A_1,M}^\rho (A_1(f(x_{n-1})) - \rho S(p(x_{n-1}), v_{n-1}))\| + \mu \|d_n - d_{n-1}\|
\leq (1 - \mu)\|x_n - x_{n-1}\| + \mu \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|
+ \frac{\mu}{r_1 - \rho m_1} \|A_1(f(x_n)) - A_1(f(x_{n-1})) - \rho [S(p(x_n), v_n) - S(p(x_{n-1}), v_n)]\|
+ \frac{\rho \mu}{r_1 - \rho m_1} \|S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1})\| + \mu \|d_n - d_{n-1}\|.
\end{align*}
\]

Since \( f \) is \( \pi \)-strongly monotone and \( \epsilon \)-Lipschitz continuous, \( F \) is \( \eta \)-H2-Lipschitz continuous, \( A_1 \) is \( \alpha_1 \)-Lipschitz continuous, \( \rho \) is \( \kappa \)-Lipschitz continuous and \( S(\cdot, y) \) is \( (y, r) \)-relaxed cocoercive with respect to \( A_1 \) and \( \sigma \)-Lipschitz continuous in the first variable and \( S(x, \cdot) \) is \( \varrho \)-Lipschitz continuous in the second variable for all \((x, y) \in \mathcal{H}_1 \times \mathcal{H}_2\), from (3.10) we get

\[
\begin{align*}
\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|^2 &\leq \|x_n - x_{n-1}\|^2 - 2\|x_n - x_{n-1}, f(x_n) - f(x_{n-1})\| + \|f(x_n) - f(x_{n-1})\|^2 \\
&\leq (1 - 2\pi + \epsilon^2)\|x_n - x_{n-1}\|^2, \\
\|S(p(x_{n-1}), v_n) - S(p(x_{n-1}), v_{n-1})\| &\leq \varrho \|v_n - v_{n-1}\| \leq \varrho (1 + n^{-1})\hat{H}(F(y_n), F(y_{n-1})) \leq \varrho (1 + n^{-1})\|y_n - y_{n-1}\|.
\end{align*}
\]
and
\[
\| A_1(f(x_n)) - A_1(f(x_{n-1})) - \rho [S(p(x_n), v_n) - S(p(x_{n-1}), v_n)] \|^2 \\
= \| A_1(f(x_n)) - A_1(f(x_{n-1})) \|^2 + \rho^2 \| S(p(x_n), v_n) - S(p(x_{n-1}), v_n) \|^2 \\
- 2\rho [S(p(x_n), v_n) - S(p(x_{n-1}), v_n), A_1(f(x_n)) - A_1(f(x_{n-1}))] \\
\leq \alpha^2 \| f(x_n) - f(x_{n-1}) \|^2 + \rho^2 \sigma^2 \| p(x_n) - p(x_{n-1}) \|^2 \\
- 2\rho [\| p(x_n) - p(x_{n-1}) \|^2 + r \| f(x_n) - f(x_{n-1}) \|^2] \\
= (\alpha^2 - 2\rho r) \| f(x_n) - f(x_{n-1}) \|^2 + (\rho^2 \sigma^2 + 2\rho \gamma) \| p(x_n) - p(x_{n-1}) \|^2 \\
\leq (\alpha^2 \epsilon^2 - 2\rho r \epsilon^2 + 2\rho \gamma \kappa^2 + \rho^2 \sigma^2 \kappa^2) \| x_n - x_{n-1} \|^2. \tag{4.5}
\]

From (4.2)–(4.5), we have
\[
\| x_{n+1} - x_n \| \leq \left[ (1 - \mu) + \mu \sqrt{1 - 2\pi + \epsilon^2} \\
+ \frac{\mu}{r_1 - \rho m_1} \sqrt{\alpha^2 \epsilon^2 - 2\rho r \epsilon^2 + 2\rho \gamma \kappa^2 + \rho^2 \sigma^2 \kappa^2} \right] \| x_n - x_{n-1} \| \\
+ \frac{\rho \mu \eta \rho}{r_1 - \rho m_1} (1 + n^{-1}) \| y_n - y_{n-1} \| + \mu \| d_n - d_{n-1} \| \\
= \theta \| x_n - x_{n-1} \| + b_n \| y_n - y_{n-1} \| + \mu \| d_n - d_{n-1} \|, \tag{4.6}
\]

where
\[
\theta = 1 - \mu (1 - k_1), \quad k_1 = \sqrt{1 - 2\pi + \epsilon^2} + \frac{\sqrt{\alpha^2 \epsilon^2 - 2\rho r \epsilon^2 + 2\rho \gamma \kappa^2 + \rho^2 \sigma^2 \kappa^2}}{r_1 - \rho m_1}, \\
b_n = \frac{\rho \mu \eta \rho}{r_1 - \rho m_1} (1 + n^{-1}).
\]

Similarly, by the assumptions of \( g, E, A_2, q, T (\cdot, \cdot) \), we can obtain
\[
\| y_n - y_{n-1} - (g(y_n) - g(y_{n-1})) \|^2 \leq (1 - 2\sigma + \epsilon^2) \| y_n - y_{n-1} \|^2, \\
\| T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1})) \| \leq \tau \| u_n - u_{n-1} \| \leq \tau (1 + n^{-1}) \| \hat{H}(E(x_n), E(x_{n-1})) \| \\
\leq \xi \tau (1 + n^{-1}) \| x_n - x_{n-1} \|, \tag{4.7}
\]

and
\[
\| y_{n+1} - y_n \| \\
\leq (1 - \nu) \| y_n - y_{n-1} \| + \nu \| y_n - y_{n-1} - (g(y_n) - g(y_{n-1})) \| \\
+ \frac{\nu}{r_2 - \lambda m_2} \| A_2(g(y_n)) - A_2(g(y_{n-1})) - \lambda \| T(u_{n-1}, q(y_{n-1})) - T(u_{n-1}, q(y_{n-1})) \| \\
+ \frac{\lambda \nu}{r_2 - \lambda m_2} \| T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1})) \| + \nu \| e_n - e_{n-1} \|.
\]
then (4.6) and (4.8) imply that

Letting

\[ t_n \quad \text{hence there exist} \quad (4.9) \quad \text{that} \]

where

\[ \vartheta = 1 - \nu(1 - k_2), \quad k_2 = \sqrt{1 - 2\sigma + \varepsilon^2} + \frac{\alpha_2^2 \xi^2}{r_2 - \lambda m_2} \]

\[ a_n = \frac{\lambda \nu \xi \tau}{r_2 - \lambda m_2} (1 + n^{-1}). \]

Now (4.6) and (4.8) imply that

\[
\begin{aligned}
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|
\leq (a_n + \theta)\|x_n - x_{n-1}\| + (b_n + \vartheta)\|y_n - y_{n-1}\|
+ \mu\|d_n - d_{n-1}\| + \nu\|e_n - e_{n-1}\|
\leq t_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) + \mu\|d_n - d_{n-1}\| + \nu\|e_n - e_{n-1}\|,
\end{aligned}
\]

(4.9)

where

\[ t_n = \max\{a_n + \theta, b_n + \vartheta\}, \quad \forall n = 1, 2, \ldots. \]

Letting \( t = \max\{a + \theta, b + \vartheta\} \), where

\[ a = \nu \cdot \frac{\lambda \xi \tau}{r_2 - \lambda m_2}, \quad b = \mu \cdot \frac{\rho \eta \phi}{r_1 - \rho m_1}, \]

then \( t_n \to t, a_n \to a \) and \( b_n \to b \) as \( n \to \infty \). From condition (4.1), we know that \( 0 < t < 1 \) and hence there exist \( n_0 > 0 \) and \( t_0 \in (t, 1) \) such that \( t_n \leq t_0 \) for all \( n \geq n_0 \). Therefore, it follows from (4.9) that

\[
\begin{aligned}
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|
\leq t_0(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) + \mu\|d_n - d_{n-1}\| + \nu\|e_n - e_{n-1}\|, \quad \forall n \geq n_0.
\end{aligned}
\]

This implies that

\[
\begin{aligned}
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\leq t_0^{n-n_0}(\|x_{n_0+1} - x_{n_0}\| + \|y_{n_0+1} - y_{n_0}\|)
+ \mu \sum_{j=1}^{n-n_0} t_0^{j-1} l_{n-(j-1)} + \nu \sum_{j=1}^{n-n_0} t_0^{j-1} l_{n-(j-1)}, \quad \forall n \geq n_0,
\end{aligned}
\]

where \( t = \|d_n - d_{n-1}\|, \quad l_n = \|e_n - e_{n-1}\| \) for all \( n > n_0 \). Hence, for any \( m \geq n > n_0 \), we have

\[
\begin{aligned}
\|x_m - x_n\| + \|y_m - y_n\|
&\leq \sum_{i=n}^{m-1}(\|x_{i+1} - x_i\| + \|y_{i+1} - y_i\|)
\end{aligned}
\]
\begin{equation}
\begin{aligned}
&\leq \sum_{i=n}^{m-1} t_0^{-i-n_0} \left(\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|\right) \\
&\quad + \mu \sum_{i=n}^{m-1} \left[ \sum_{j=1}^{i-n_0} i^{-j-1} t_i^{-1-j} \right] + \nu \sum_{i=n}^{m-1} \left[ \sum_{j=1}^{i-n_0} i^{-j-1} t_i^{-1-j} \right] \\
&\leq \sum_{i=n}^{m-1} t_0^{-i-n_0} \left(\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|\right) \\
&\quad + \mu \sum_{i=n}^{m-1} \left[ \sum_{j=1}^{i-n_0} i^{-j-1} t_i^{-1-j} \right] + \nu \sum_{i=n}^{m-1} \left[ \sum_{j=1}^{i-n_0} i^{-j-1} t_i^{-1-j} \right].
\end{aligned}
\end{equation}

Since \(\sum_{i=1}^{\infty} t_i k^{-i} < \infty\) and \(\sum_{i=1}^{\infty} l_i k^{-i} < \infty\) for all \(k \in (0, 1)\), and \(t_0 < 1\), it follows from (4.10) that \(\|x_m - x_n\| \to 0\) and \(\|y_m - y_n\| \to 0\), as \(n \to \infty\), and so \(\{x_n\}\) and \(\{y_n\}\) are both Cauchy sequences in \(H_1\) and \(H_2\), respectively. Thus, there exist \(x^* \in H_1\) and \(y^* \in H_2\) such that \(x_n \to x^*\) and \(y_n \to y^*\) as \(n \to \infty\).

Now we prove that \(u_n \to u^* \in E(x^*)\) and \(v_n \to v^* \in F(y^*)\). In fact, it follows from (4.4) and (4.7) that \(\{u_n\}\) and \(\{v_n\}\) are also Cauchy sequences. Let \(u_n \to u^*\) and \(v_n \to v^*\), respectively. In the sequel, we will show that \(u^* \in E(x^*)\) and \(v^* \in F(y^*)\). Noting \(u_n \in E(x_n)\), we have

\[ d(u^*, E(x^*)) = \inf\{\|u_n - z\| : z \in E(x^*)\} \leq \|u^* - u_n\| + \hat{H}(E(x_n), E(x^*)) \leq \|u^* - u_n\| + \hat{\mu} \to 0. \]

Hence \(d(u^*, E(x^*)) = 0\) and therefore \(u^* \in E(x^*)\). Similarly, we can prove that \(v^* \in F(y^*)\).

By continuity, (3.9) and \(\lim_{n \to \infty} d_n = \lim_{n \to \infty} e_n = 0\), it is easy to see that \(x^*, y^*, u^*, v^*\) satisfy the following relation:

\[
\begin{cases}
    f(x^*) = J^0_{\alpha_1, M}(A_1(f(x^*)) - \rho S(p(x^*), v^*)), \\
    g(y^*) = J^\alpha_{\beta_2, N}(A_2(g(y^*)) - \lambda T(u^*, q(y^*)�).
\end{cases}
\]

It follows from Lemma 3.1 that \((x^*, y^*, u^*, v^*)\) is a solution of the system of generalized multivalued variational inclusion problem (3.1). This completes the proof. \(\Box\)

From Theorem 4.1, we have the following result associated with \(H\)-monotone operators.

**Theorem 4.2.** Suppose that \(H_1, H_2, p, q, f, g, E\) and \(F\) are the same in Theorem 4.1. Let \(H_1 : H_1 \to H_1\) be \(r_1\)-strongly monotone and \(\alpha_1\)-Lipschitz continuous, and \(H_2 : H_2 \to H_2\) be \(r_2\)-strongly monotone and \(\alpha_2\)-Lipschitz continuous, \(M : H_1 \to 2^{H_1}\) be \(H_1\)-monotone and \(N : H_2 \to 2^{H_2}\) be \(H_2\)-monotone, \(S : H_1 \times H_2 \to H_1\) be a single-valued mapping such that \(S(\cdot, y)\) is \((\gamma, r)\)-relaxed cocoercive with respect to \(H_1\) and \(\sigma\)-Lipschitz continuous in the first variable and \(S(x, \cdot)\) is \(\rho\)-Lipschitz continuous in the second variable for all \((x, y) \in H_1 \times H_2\), \(T : H_1 \times H_2 \to H_2\) be a nonlinear mapping such that \(T(x, \cdot)\) is \((\delta, \kappa)\)-relaxed cocoercive with respect to \(H_2\) and \(\eta\)-Lipschitz continuous in the second variable and \(T(\cdot, y)\) is \(\tau\)-Lipschitz continuous in the first variable for all \((x, y) \in H_1 \times H_2\). If for every parameters \(\mu, \nu \in (0, 1)\), there exist positive constants \(\rho\) and \(\lambda\) such that
\[
\begin{aligned}
&k_1 = \sqrt{1 - 2\pi + \varepsilon^2 + r_1^{-1} \sqrt{\alpha_1^2 \varepsilon^2 - 2 \rho \gamma k^2 + \rho^2 \sigma^2 k^2}} < 1, \\
&k_2 = \sqrt{1 - 2\omega + \varepsilon^2 + r_2^{-1} \sqrt{\alpha_2^2 \varepsilon^2 - 2 \lambda \delta \xi^2 + \lambda^2 \beta^2 \xi^2}} < 1, \\
&\lambda, \gamma, \omega, \varepsilon, \delta, \lambda, \mu, \rho, \xi, \beta, \sigma, k, r_1, r_2 \text{ are the same as in Lemma 2.2,}
\end{aligned}
\]
and the resolvent operators \( J_{\tilde{H}_1, M}^\rho \) and \( J_{\tilde{H}_2, N}^\gamma \) are the same as in Lemma 2.2. \( \mu, v, \rho, \lambda, d_n, e_n \) are the same as in (3.9), then the iterative sequences \( \{x_n\}, \{y_n\}, \{u_n\} \) and \( \{v_n\} \) generated by the following iteration:

\[
\begin{aligned}
x_{n+1} &= (1 - \mu)x_n + \mu [x_n - f(x_n) + J_{\tilde{H}_1, M}^\rho (A_1(f(x_n)) - \rho S(p(x_n), v_n))] + \mu d_n, \\
y_{n+1} &= (1 - v)y_n + v [y_n - g(y_n) + J_{\tilde{H}_2, N}^\gamma (A_2(g(y_n)) - \lambda T(u_n, q(y_n)))] + \nu e_n, \\
u_{n+1} &\in E(x_{n+1}), \quad \|u_{n+1} - u_n\| \leq (1 + (n + 1)^{-1}) \tilde{H}_1(E(x_n), E(x_{n+1})), \\
v_{n+1} &\in F(y_{n+1}), \quad \|v_{n+1} - v_n\| \leq (1 + (n + 1)^{-1}) \tilde{H}_2(F(y_n), F(y_{n+1})), \\
&n = 0, 1, 2, \ldots,
\end{aligned}
\]

converge strongly to \( x^*, y^*, u^* \) and \( v^* \), respectively, and \( (x^*, y^*, u^*, v^*) \) is a solution of the problem (3.1).

**Remark 4.1.** From Theorems 4.1 and 4.2, we can get the existence and convergence results of solutions for the problems (3.2)–(3.6). Our results improve and generalize many known corresponding results of [1, 2, 4, 14, 16, 20].

**References**


