Primitive Arcs in $PG(2, q)$

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We show that a complete arc $K$ in the projective plane $PG(2, q)$ admitting a transitive primitive group of projective transformations is either a cyclic arc of prime order or a known arc. If the completeness assumption is dropped, then $K$ has either an affine primitive group, or $K$ is contained in an explicit list. In order to find these primitive arcs, it is necessary to determine all complete $k$-arcs fixed by a projective elementary abelian group of order $k$. As a corollary to our result, we list all complete arcs fixed by a 2-transitive projective group. © 1995 Academic Press, Inc.

1. Introduction and Main Results

A $k$-arc $K$ of a projective plane $PG(2, q)$, also called a plane $k$-arc, is a set of $k$ points, no 3 of which are collinear. The best known example of an arc is the point set of a conic.

A point $p$ of $PG(2, q)$ extends a $k$-arc $K$ if and only if $K \cup \{p\}$ is a $(k + 1)$-arc. A $k$-arc $K$ of $PG(2, q)$ is called complete if and only if it is not contained in a $(k + 1)$-arc of $PG(2, q)$. In $PG(2, q)$, $q$ odd, $q > 3$, a conic is complete, but in $PG(2, q)$, $q$ even, a conic is not complete. It can be extended in a unique way to a $(q + 2)$-arc by its nucleus.

In the search for other examples of arcs, various methods have been used. The bibliographies of [7, 8, 9] contain a large number of articles in which arcs are constructed.

This paper continues the work of the authors in [11, 12] where arcs fixed by a large projective group are classified. In [11], all types of complete $k$-arcs, fixed by a cyclic projective group, i.e., a cyclic subgroup of $PGL_3(q)$, of order $k$, were determined. This led to a new class of such $k$-arcs, in $PG(2, q), q \equiv -1 \pmod{4}$, containing $k/2$ points of 2 conics which only have 2 conjugate points in $PG(2, q^2) \setminus PG(2, q)$ in common. In [12], a slight variation on [11] is treated. In this paper, all complete $(k + 1)$-arcs

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fixed by a cyclic projective group of order $k$, were described. Here, no new examples were found.

Now, the classification of all complete $k$-arcs fixed by a transitive projective group acting primitively on the points of the arc, is presented. This is achieved by applying the result on finite primitive permutation groups by O'Nan and Scott, in the version of Buekenhout [2], on the list of subgroups of $L_3(q)$, given by Bloom [1] for $q$ odd, and by Hartley [6] and Suzuki [14] for $q$ even.

In almost all cases, the completeness condition on the arc $K$ can be dropped. The completeness of $K$ is only assumed in the affine case where the complete $k$-arcs $K$ fixed by a transitive elementary abelian group of order $k$, are determined (Section 3). In Section 4, all classes of primitive $k$-arcs, $k \geq 5$, fixed by an almost simple projective group $G_K$, are found. They are the conic in $PG(2, q)$, the unique 5- and 6-arc in $PG(2, 4)$ fixed by $A_5$ and $A_6$, and a unique 6- and 10-arc in $PG(2, q)$, $q \equiv \pm 1 \pmod{10}$, fixed by $A_5$.

As an immediate corollary, all complete arcs fixed by a 2-transitive projective group, are determined.

From now on, suppose that $K$ is an arc in $PG(2, q)$ with automorphism group $\Gamma_K$. Put $G = PGL_3(q)$ and $G_K = \Gamma_K \cap G$.

2. Preliminary Lemmas

**Lemma 1.** If $|K| \geq 4$, then $G_K$ acts faithfully on $K$.

*Proof.* The group $G$ acts regularly on the set of all ordered 4-arcs of $PG(2, q)$. 

**Lemma 2.** If $|K| \geq 4$ and $K$ is complete, then $\Gamma_K$ acts faithfully on $K$.

*Proof.* If $\sigma \in \Gamma_K$ fixes every point of $K$, then $\sigma$ must be induced by a field automorphism and it fixes a subplane $\pi$ point by point. So $K \subset \pi$. Let $T$ be a line of $\pi$ skew to $K$ and let $x$ be a point on $T$ not in $\pi$. Then $x$ extends $K$ to a larger arc since every bisecant of $K$ is a line of $\pi$.

**Lemma 3.** Suppose $K$ is complete.

The socle $S$ of $\Gamma_K$ is either elementary abelian or simple, i.e., $\Gamma_K$ is either of affine type or almost simple. Moreover, if $\Gamma_K$ is almost simple, then $S \leq L_3(q)$.

*Proof.* Use the result of O'Nan and Scott in the version of Buekenhout [2]. According to that result, the group $\Gamma_K$ is of one and only one of the following types: affine type, biregular type, cartesian type, or simple type. The definition of cartesian and biregular type requires $\Gamma_K$ to have a normal subgroup $H$ isomorphic to the direct product of two or more
isomorphic copies of a non-abelian simple group $S$ [2]. Let $H \cong S_1 \times S_2 \times \cdots \times S_n$, where each $S_i$ is isomorphic to $S$, $1 \leq i \leq n$. For every $i \in \{1, 2, \ldots, n\}$, the group $S_i$ can be viewed as a subgroup of $H$, which is on its turn a subgroup of $PGL_3(q)$ by the previous lemmas, and either $S_i \cap L_3(q) = S_i$ or $S_i \cap L_3(q) = 1$. Suppose the latter happens, then

$$S_i \cong S_i/(S_i \cap L_3(q)) \cong S_i L_3(q)/L_3(q) \leq PGL_3(q)/L_3(q).$$

Using the ATLAS-notation [3], the latter is isomorphic to the group $3.h$ or $h$, where $q = p^h$, $p$ prime. This is impossible since then, either $S_i$ has a normal subgroup of order 3, or $S_i$ is cyclic and so abelian. Hence each $S_i$ is inside $L_3(q)$ and so is $H$. But by inspection of the list of subgroups of $L_3(q)$, see [1, Theorem 1.1], for $q$ odd, and [6, pp. 157-158], for $q$ even, one sees that this is impossible for $n \geq 2$. The case $n = 1$ corresponds to $H \cong S$. So $H$ is simple, $\Gamma_K$ is almost simple [2] and the above argument shows that the socle $S$ is a subgroup of $L_3(q)$.

In Section 3 we will consider the affine case and in Section 4, we will completely classify the simple case.

The following lemmas are elementary but turn out to be very useful.

**Lemma 4.** The group $\Gamma_K$ cannot contain a subgroup $H$ of central collineations with common center and common axis of order $r \geq 3$, when $|K| > 3$.

**Proof.** Every non-trivial orbit of such a group $H$ of collineations contains $r$ points on one line and so they cannot be points of an arc $K$. So $K$ is a subset of the set of points fixed by $H$, but then $|K| \leq 3$.

**Lemma 5.** If a central projective transformation $\sigma$ in $G_K$ fixes at least three points of an arc $K$, $|K| > 3$, then it is the identity.

This holds in particular for any involution $\sigma$ in $G_K$.

**Proof.** One of the fixed points, say $x$, must be the center of the central projective transformation $\sigma$. Any point $y$ of $K$, not on the axis of $\sigma$, is mapped onto a point $y'$ with the property that $x$, $y$ and $y'$ are points of $K$ on one line, but this is impossible.

This lemma is valid for the involutions in $PGL_3(q)$ since they are central [4, p. 172].

**Lemma 6.** Any projective transformation of $G_K$ fixing at least four points of $K$ is the identity.

**Proof.** The group $PGL_3(q)$ acts regularly on the ordered quadrangles of $PG(2, q)$.
3. The Affine Case

Assume that $G_K$ is of affine type. This means that $K$ bears the structure of a vector space $V$ over some prime field $GF(r)$ such that $G_K = H.G_0$, where $H$ is the group of all translations of $V$ and where $G_0$, the stabilizer of the origin $o$, is a subgroup of $GL(V)$ [2].

Using the fact that $H$ acts regularly on $K$, the following proposition is obtained. This is in reality the classification of all complete $k$-arcs, $k = r^n$, $r$ prime, fixed by a projective elementary abelian group of order $k$.

**Proposition 1.** Let $K$ be a complete $k$-arc, $k = r^n$ with $r$ prime, in $PG(2, q)$. Suppose $H < G_K$ is an elementary abelian group of order $r^n$, acting regularly on $K$. Then $n = 1$ and $K$ is an orbit of an element of order $r$ of a Singer group of $PGL_3(q)$, or $k = 2^2$ and $K$ is a conic in $PG(2, 3)$ or a hyperoval in $PG(2, 2)$, or $k = 9$ and $K$ is one of 2 distinct 9-arcs in $PG(2, 13)$.

**Proof.** Let $r = 2$. If $q$ is odd, then $H$ contains $2^n - 1$ involutory homologies [4, p. 172] which commute with each other. Two homologies $h_1$ and $h_2$ commute if and only if they have common center and axis or the center of one homology $h_i$ belongs to the axis of the other homology $h_j$, $\{i,j\} \neq \{1,2\}$. The first possibility cannot occur since there is a unique involutory homology with given center and given axis. The second possibility clearly implies that $|H| \leq 4$. Hence, by the completeness of $K$, $|H| = 4$, $q = 3$, and $K$ is a conic in $PG(2, 3)$.

If $q$ is even, then all involutions in $H$ are elations with either common center or common axis. If they have common center, then every non-trivial orbit of $H$ is contained in a line through the common center contradicting the fact that $K$ is an arc. In fact, this shows that no two elations of $H$ have common center. Suppose all elations have common axis $L$. Assume that a line $T$ is tangent to $K$. Then all elements of $T^H$ are tangent to $K$ and hence the point $T \cap L$ extends the arc $K$, so $K$ is not complete, a contradiction. There are no lines tangent to $K$. This implies $|K| = q + 2$ and this is a power of 2 only if $q = 2$. So $K$ is a hyperoval, the points of an affine plane, in $PG(2, 2)$.

Assume now $r$ odd. Let $O$ be an arbitrary orbit in $PG(2, q)$ under $H$. Since $H$ is an $r$-group, $|O| = r^m$ for $0 \leq m \leq n$. If $m = 0$, then $O = \{x\}$ and there is at least one line $T$ through $x$ tangent to $K$ since $|K|$ is odd. Applying $H$ to $T$, every line through $x$ meeting $K$ is a tangent line, hence $x$ extends $K$ and $K$ is not complete. If $0 < m < n$, then the kernel of $H$ on $O$ is non-trivial and so there is an element of order $r$ fixing $O$ point by point. If at least three points of $O$ are collinear, then $\sigma$ is a central projective transformation, contradicting Lemma 4. So $O$ is an arc and hence $|O| = r = 3$ by Lemma 6. We can take coordinates such that
$O = \{(1,0,0),(0,1,0),(0,0,1)\}$. A projective transformation $\varphi$ of order 3 which is not central, and which fixes $O$ point by point, has necessarily a matrix of the form

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}, \quad \omega^3 = 1, \quad \omega \neq 1.
$$

Hence $r^n = 3^2$ and $q \equiv 1 \pmod{3}$.

A projective transformation $\varphi$ of order 3 permuting cyclically the points of $O$ has, without loss of generality, matrix

$$
\begin{pmatrix}
0 & 0 & d \\
1 & 0 & 0 \\
0 & b & 0
\end{pmatrix}, \quad b, d \in GF(q)^* = GF(q) \setminus \{0\}.
$$

These matrices $A$ and $B$ define an elementary abelian group of order 9. Assume that $(1,1,1) \in K$, then we can write that

$$
K = \{(1,1,1), (1,\omega,\omega^2), (1,\omega^2,\omega), (d,1,b), (d,\omega,\omega^2b), (d,\omega^2,\omega b),
$$

$$(bd,d,b), (bd,\omega d,\omega^2b), (bd,\omega^2d,\omega b)\} = \{p_1, \ldots, p_9\}.$$

Without loss of generality, it can be assumed that $(1,0,0)$ belongs to tangent lines to $K$ through $p_1, p_2, p_3$, then $(0,1,0)$ belongs to tangent lines through $p_4, p_5, p_6$ and $(0,0,1)$ to tangent lines through $p_7, p_8, p_9$. So one of the lines $p_1p_4, p_1p_5, p_1p_6$ must pass through $(0,0,1)$ since $(0,0,1)$ must belong to a bisecant of $K$. This implies $d \in \{1, \omega, \omega^2\}$. Once again, without loss of generality, it can be assumed that $d = 1$. Then $K$ is a 9-arc if and only if $b \notin \{0,1,-2,\omega,\omega^2\}$, $\omega b^2 + (\omega - 3)b + \omega \neq 0$, $b^2 + (1 - 3\omega)b + 1 \neq 0$ and also $b \neq -1/2$ when $q$ is odd.

A complete 9-arc in $PG(2,q)$, $q \equiv 1 \pmod{3}$, can only exist if $q \in \{13,16,19,25\}$ [7,p. 206]. Using AXIOM [13], the authors checked that $K$ is complete if and only if $q = 13$ and $b \in \{2,4,7,10\}$. Since for these values of $b$, the set $O$ is the only orbit of size 3, if two 9-arcs $K_1$ and $K_2$, with different parameters $b_1$ and $b_2$, are projectively equivalent, the set $O$ must be fixed by a projective transformation $\alpha$ mapping $K_1$ onto $K_2$. Since $K_1$ and $K_2$ are transitive arcs, it can be assumed that $p_1$ is fixed by $\alpha$. Then $(1,0,0)$ must also be fixed since it belongs to a tangent line to $K_1$ and $K_2$ in $p_1$. The only possible transformation $\alpha$ which can map $K_1$ onto $K_2$ is the involution $\alpha$: $(x_0,x_1,x_2) \mapsto (x_0,x_2,x_1)$. Since this involution interchanges the arcs $K$ with parameters $b$ and $1/b$, precisely 2 of these 4 arcs are projectively distinct.
From now on, assume that every orbit of $H$ has $r^n$ points, so $r^n$ must divide $q^2 + q + 1$. If $r \neq 3$, then every Sylow $r$-subgroup of $PGL_3(q)$ must be contained in some Singer group. Indeed, $r$ does not divide $|PGL_3(q)|/(q^2 + q + 1)$, which is shown in [11, Theorem 3.1]. This implies $n = 1$ since $H$ must be cyclic and elementary abelian. The result follows. If $r = 3$, then $k = 3$ since $9 \div (q^2 + q + 1)$, but then $K$ is not complete, so this case need not be considered.

**Proposition 2.** Let $K$ be a complete primitive $k$-arc, $k = r^n$ with $r$ prime, in $PG(2, q)$. Suppose $H \leq G_K$ is an elementary abelian group of order $r^n$, acting regularly on $K$. Then $n = 1$ and $K$ is an orbit of an element of order $r$ of a Singer group of $PGL_3(q)$, or $k = 2^2$ and $K$ is a conic in $PG(2, 3)$ or a hyperoval in $PG(2, 2)$.

**Proof.** This follows from the preceding proposition. The $9$-arcs, mentioned in the preceding proposition, are not primitive since the sets \{P_1, P_2, P_3\}, \{P_4, P_5, P_6\}, \{P_7, P_8, P_9\} form a system of imprimitivity of $K$.

Every triangle, 3 non-collinear points, and every quadrilateral, 4 points no 3 of which are collinear, constitutes a primitive arc in a plane. From now on, assume $|K| \geq 5$.

**4. The Simple Case**

In this section, assume that $G_K$ acts primitively on $K$, $|K| \geq 5$, and that $G_K$ is an almost simple group with socle $S$, i.e., $S$ is a non-abelian normal simple group and $G_K \leq Aut S$ [2]. By the classification of subgroups of $L_3(q)$ by [1, Theorem 1.1], see also [10, pp. 239–242], for $q$ odd, and [6, pp. 157–158], for $q$ even, there are three infinite series for $S$, namely $L_3(q')$, $U_3(q')$, and $L_2(q')$, for suitable $q'$ dividing $q$. We first deal with them and afterwards with the sporadic cases.

Set $q = p^h, p$ a prime number and let $K$ be a $k$-arc in $PG(2, q)$.

**4.1. Infinite Classes**

4.1.1. *The $L_3$-Case.* Here, $PGL_3(q') \leq PGL_3(q)$ for every prime power $q' = p^{h'}$ such that $h'$ divides $h$.

**Proposition 3.** No arc $K$, $|K| \geq 5$, exists such that

$$L_3(q') \leq G_K \leq PGL_3(q') = Aut(L_3(q')) \cap PGL_3(q)$$

and such that $G_K$ acts primitively on $K$. 

Proof. The group $L_3(q')$ contains a subgroup of elations with common center and common axis of order $q'$, hence by Lemma 4, $q' = 2$. So there is a subplane $PG(2, 2)$ in $PG(2, q)$ stabilized by $G_K$. Clearly $K \cap PG(2, 2) = \emptyset$. If a point $x \in K$ lies on a line $L$ of $PG(2, 2)$, by applying an element of order 2 in $L_3(2)$ contained in the stabilizer of $L$, one sees that $L$ contains at least two points of $K$, but the lines of $PG(2, 2)$ partition in this way the points of $K$ in blocks of imprimitivity, a contradiction. Now let $x \in K$ and $u \in PG(2, 2)$, then $xu$ is a line of $PG(2, q)$ not in $PG(2, 2)$. The set of elations in $L_3(2)$ with center $u$ forms a subgroup of order 4 acting semi-regularly on the points of $xu \setminus \{u\}$. So $xu$ contains four points of $K$, a contradiction.

4.1.2. The $U_3$-Case. Here, $PGU_3(q') \leq PGL_3(q)$, $q' = p^{h'}$, whenever $2h'$ divides $h$. This group stabilizes a Hermitian curve in a subplane $PG(2, q')$ of $PG(2, q)$.

**Proposition 4.** No arc $K$, $|K| \geq 5$, exists such that

$$U_3(q') \leq G_K \leq PGU_3(q') = Aut(U_3(q')) \cap PGL_3(q)$$

and such that $G_K$ acts primitively on $K$.

**Proof.** The group $U_3(q')$ acts 2-transitively on a Hermitian curve $\mathcal{H}$ in some subplane $PG(2, q')$. Consider an element $\sigma$ of $U_3(q')$ fixing some point $x$ of $\mathcal{H}$ and mapping another point $y$ to some point $z$ on the line $xy$, $y, z \in \mathcal{H}$. Then $\sigma$ fixes $xy$ and its pole $u$ w.r.t. $\mathcal{H}$. Hence $\sigma$ fixes the lines $xu$ and $xy$. The order of $\sigma$ can be chosen to be $p$. So $\sigma$ fixes all lines through $x$ and it is easily seen that $xu$ is the axis. By Lemma 4, $p = 2$. But $z$ can be varied to obtain a group of elations with common center $x$ and common axis $xu$ of order $q'$. Hence $q' = 2$ by Lemma 4. But $U_3(2) \cong 3^2 \cdot Q_8$ is not simple and has no non-abelian simple socle. $\square$

4.1.3. The $L_2$-Case. Here, $PGL_2(q') \leq PGL_3(q)$, $q' = p^{h'}$, whenever $h'$ divides $h$.

**Proposition 5.** If $K$ is an arc in $PG(2, q)$ such that $G_K$, with

$$L_2(q') \leq G_K \leq PGL_2(q') = Aut(L_2(q')) \cap PGL_3(q),$$

acts primitively on $K$, then $K$ is a conic in some subplane $PG(2, q')$ of $PG(2, q)$.

**Proof.** Let $C$ be the conic on which $G_K$ acts naturally inside some subplane $PG(2, q')$. Note that we can assume $q' > 3$ since $PGL_2(2)$ and $PGL_2(3)$ have no non-abelian simple socle. Clearly if the arc $K$ has a point
in common with $\text{PG}(2, q')$, then it consists of either all internal points of $C$ ($p$ odd), all external points of $C$ ($p$ odd), the nucleus of $C$ ($p = 2$), all points not on $C$ and distinct from the nucleus of $C$ ($p = 2$) or the conic $C$ itself. Only the last set of points constitutes an arc. So we can assume that all points of $K$ lie outside $\text{PG}(2, q')$. If one point of $K$ lies on a line $L$ of $\text{PG}(2, q')$, then all points of $K$ do and the lines in the orbit of $L$ under $G_K$ define a partition $\mathcal{P}$ of $K$ invariant under $G_K$. Let $x \in K \cap L$.

If $L$ is a bisecant of $C$, then the cyclic subgroup of $L_2(q')$ fixing $L$ has at least order $(q' - 1)/2$ and acts on $L \setminus C$ in orbits of at least size $(q' - 1)/4$, if $L$ is the tangent to $C$ in $a$, the cyclic subgroup of $L_2(q')$ fixing $a$ and a second point $b$ of $C$ has again at least order $(q' - 1)/2$ and partitions $L \setminus \{a\}$ into orbits of at least size $(q' - 1)/2$. If $q'$ is odd, the $q' + 1$ transformations of $\text{PGL}_2(q')$ fixing $t = i$, with $i^2 = d_1$ non-square in $GF(q')$, are $t \mapsto (at + cd_1)/\alpha(t + a), a^2 - c^2d_1 \neq 0$, and exactly $(q' + 1)/2$ of these mappings belong to $L_2(q')$. So if $L$ is skew to $C$ in $\text{PG}(2, q')$, $L_2(q')$ contains a cyclic subgroup of order $(q' + 1)/2$, fixing $L$ and partitioning $L \setminus C$ into orbits of at least size $(q' + 1)/4$.

Assume that $K$ does not contain a point of $C$ in $\text{PG}(2, q^2)$, then the partition $\mathcal{P}$ is non-trivial if $q' > 5$. If $q' = 4$, then $L_2(4) = \text{PGL}_2(4)$ and all lower bounds on the sizes of the orbits can be doubled, so once again, the partition is non-trivial. If $q' = 5$, the previous reasoning is not valid when $G_K \cong L_2(5)$ and $L$ is a bisecant of $C$ in $\text{PG}(2, q')$ containing one point of $K$. If this occurs, then all bisecants of $C$ in $\text{PG}(2, q')$ contain one point of $K$. Hence $|K| = 15$, but this is impossible since $G_K \cong L_2(5) \cong A_5$ does not act primitively on 15 points [3].

If $K$ contains one point of $C$ in $\text{PG}(2, q^2) \setminus \text{PG}(2, q')$, since this point is fixed by a cyclic subgroup of order $(q' + 1)/2$ when $G_K \cong L_2(q')$, $q'$ odd, and of order $q' + 1$ when $G_K \cong \text{PGL}_2(q')$, $K$ contains all points of $C$ in $\text{PG}(2, q^2) \setminus \text{PG}(2, q')$. Furthermore, $K$ only consists of these points since $G_K$ acts transitively on $K$. This is however impossible since the external lines to $C$ in $\text{PG}(2, q')$ define a system of imprimitivity on $K$.

So we may assume that no point of $K$ lies on a line of $\text{PG}(2, q')$. Let $x \in K, \sigma \in G_K$ and suppose that $x^\sigma = x$. If $\sigma$ fixes two points $a, b$ of $C$, then $\sigma$ fixes four points, namely $a, b, x$ and the pole of the line $ab$ w.r.t. $C$ or the nucleus of $C$. No three of these points are collinear, otherwise $x$ lies on a line of $\text{PG}(2, q')$, contradicting our assumption, hence $\sigma$ is the identity. Suppose now $\sigma$ acts semi-regularly on $C$. Then $\sigma$ fixes two points $a, b$ of $C$ in a quadratic extension of $\text{PG}(2, q')$ and as above, this leads to $\sigma$ being the identity. Finally, suppose $\sigma$ fixes exactly one point $u$ of $C$, then it fixes the tangent line $T$ to $C$ through $u$ and it also fixes the line $xu$. Since $\sigma$ necessarily has order $p$, it readily follows that it fixes all lines through $u$. So $\sigma$ is central, $p = 2$ (Lemma 4), and the axis is $T$. But $x$ does not lie on $T$ and is fixed, hence $\sigma$ is the identity.
We have shown that no non-trivial element of $G_K$ fixes a point of $K$. So $G_K$ acts regularly on $K$ and such an action can never be primitive for groups of non-prime order. 

This completes the investigation of the infinite classes.

4.2. The Sporadic Classes

The list of these classes is given by Bloom [1, Theorem 1.1] for $q$ odd, and by Hartley [6, pp. 157–158] and Suzuki [14, Introduction] for $q$ even.

4.2.1. Case $L_2(7) \leq G_K \leq PGL_2(7)$. In this case, $q^3 \equiv 1$ (mod 7), $q$ odd, see [1, Theorem 1.1]. By the ATLAS [3], $L_2(7)$ can only act primitively on either 7 or 8 elements. If $G_K \equiv PGL_2(7)$ and $L_2(7)$, as a subgroup of $G_K$, does not act primitively on $K$, then $|K| = 21$ or 28 [3]. If $|K| = 28$, then $K$ can be identified with the pairs of points of $PG(1,7)$ and every involution fixes 4 pairs, contradicting Lemma 6. If $|K| = 21$, then $K$ can be identified with the pairs of conjugated points in $PG(1,49) \setminus PG(1,7)$. The involution sending $x$ to $-x$ fixes three such pairs, contradicting Lemma 5. We now deal with $G_K \equiv L_2(7)$.

PROPOSITION 6. The group $L_2(7)$ does not act primitively on any arc in $PG(2, q)$, $q^3 \equiv 1$ (mod 7).

Proof. Suppose $|K| = 7$. Since $L_2(7) \equiv L_3(2)$, the Klein fourgroup $K_4$ is inside $G_K$, it fixes three points $x, y, z \in K$ and acts regularly on the remaining four points of $K$. This contradicts Lemma 5.

Suppose now $|K| = 8$. Drop the restrictions on $q$. We will show that every orbit of $L_2(7)$ of length 8 which constitutes an arc in any finite projective plane must be a conic in a subplane of order 7.

We can identify the points of $K$ with the elements of $GF(7) \cup \{\infty\}$ in the natural action of $L_2(7)$. We establish this identification via the indices. So $K = \{x_0, x_1, \ldots, x_6, x_\infty\}$. We coordinatize $PG(2, q)$ and take $x_0 = (1, 0, 0)$, $x_\infty = (0, 1, 0)$, and $x_1 = (1, 1, 1)$. An element $\sigma$ in $G_K$ of order 3 fixing $x_0$ and $x_\infty$ exists. It is multiplication by 2 or 4 in the natural action, let us assume multiplication by 2. Since $1 + q + q^2 \not\equiv 2$ mod 3, $\sigma$ has to fix at least one other point $y$ of $PG(2, q)$. By Lemma 4, $\sigma$ cannot be central, hence $q \equiv 1$ (mod 3) and $y$ is not incident with the line $x_0x_\infty$. Neither lies $y$ on any other bisecant of $K$ containing $x_0$ or $x_\infty$. It would imply that $\sigma$ has to fix that bisecant point by point and so $\sigma$ would be central. Hence we can take $y = (0, 0, 1)$. The matrix of $\sigma$ looks like

$$
\begin{pmatrix}
  a & 0 & 0 \\
  0 & b & 0 \\
  0 & 0 & 1
\end{pmatrix}, \quad a, b \in GF(q)^*.
$$
Clearly \( a = b \) or \( 1 \in \{a, b\} \) implies that \( x_1, x_1^a \), and \( x_1^{a^2} \) are collinear, hence \( a \neq b \) and \( a, b \neq 1 \). Since \( \sigma \) has order 3, both \( a \) and \( b \) are non-trivial third roots of unity, say \( a = \omega \) and \( b = \omega^2 \), \( \omega^2 + \omega + 1 = 0 \). Hence \( x_2 = (\omega, \omega^2, 1) \) and \( x_4 = (\omega^2, \omega, 1) \). If we set \( x_3 = (u, v, 1) \), then \( x_6 = (\omega u, \omega^2 v, 1) \) and \( x_5 = (\omega^2 u, \omega v, 1) \). Let \( \theta \) be the element of \( L_2(7) \) mapping \( x_i \) to \( x_{i+1} \) and fixing \( x_i \). Knowing the action of \( \theta \) on 8 points of \( PG(2, q) \), we can find its matrix, namely

\[
\begin{pmatrix}
1 & 0 & b \\
1 & a & c \\
1 & 0 & d
\end{pmatrix}, \quad a, b, c, d \in GF(q).
\]

Expressing \( x_1^\theta = x_2 \), \( x_0 = x_3 \), and \( x_3^\theta = x_4 \), the elements \( a, b, c, d \) must satisfy,

(A) \( \omega + (1 + d)\omega - 1 = (\omega + \omega^2)u \),
(B) \( \omega + \omega^2 a + (1 + d)\omega^2 - 1 - a = (\omega + \omega^2)v \),
(C) \( u + (1 + d)\omega - 1 = (u + \omega)\omega^2 \),
(D) \( u + av + (1 + d)\omega^2 - 1 - a = (u + \omega)\omega \),
(E) \( b = (1 + d)\omega - 1 \),
(F) \( c = (1 + d)\omega^2 - 1 - a \).

From (A) and (C), \((u - 1)(u + 2) = 0\). If \( u = 1 \), then \( d = -1 \) by (A), so \( a(v - 1) = 0 \) by (D). Clearly \( a \neq 0 \), so \( v = 1 \), and \( x_1 = x_3 \), a contradiction. So \( u = -2 \). Noting \( \omega \neq -2 \) since \( p \neq 3 \), we deduce from (A) that \( d = -3\omega - 1 \) since \( \omega^2 + \omega + 1 = 0 \). Combining (B) and (D), gives

\[ v^2(1 - \omega) + v(5\omega + 4) + 14\omega + 4 = 0. \]

This implies \( v = -2 \) or \( v = -3\omega + 1 \). If \( v = -2 \), then \( a = -3 \) by (B). But \( x_3^\theta = x_5 \) implies

\[ (2\omega + 1, -4\omega - 2, -4\omega - 2) = k \cdot (2\omega + 2, -2\omega, 1), \]

for some \( k \in GF(q)^* \). This implies \(-2 = 1\), hence \( p = 3 \) and \( a = 0 \) which is false. So \( v = -3\omega + 1 \). Then (B) implies \( a = 3\omega + 3 \) and (E) and (F) imply that \( \theta \) has matrix

\[
\begin{pmatrix}
1 & 0 & 3\omega + 2 \\
1 & 3\omega + 3 & -3\omega - 7 \\
1 & 0 & -3\omega - 1
\end{pmatrix}.
\]

Expressing \( x_4^\theta = x_5 \), we obtain \( 7 = 0 \) and \( \omega = 4 \). So \( p = 7 \) and all points of \( K \) satisfy \( X_0X_1 = X_2^2 \), showing our assertion. \( \square \)
4.2.2. Case $A_6 \leq G_K \leq \text{Aut}(A_6)$. First, assume $G_K \cong A_6$. This can only happen for $q$ an even power of 2, see [6, p. 158] and [14, Introduction], or 5 and for $q \equiv 1$ or 19 (mod 30) [1, Theorem 1.1 (8) and (9)].

**Proposition 7.** Under the above assumptions, if $A_6 \leq \text{PGL}_3(q)$ acts primitively on an arc $K$, then $q$ is even and $K$ is the unique hyperoval consisting of 6 points in a subplane of order 4.

**Proof.** By [3], there are three distinct possibilities for $|K|$. First, suppose $|K| = 6$. Select 4 points of $K$ and give them coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$. There is an element $\sigma$ of order 3 fixing the first three points and acting regularly on the remaining three points of $K$. As in the proof of Proposition 6, $\sigma$ has matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}, \quad \omega \in GF(q), \omega \neq 1, \omega^3 = 1,
$$

and so $(1,1,1), (1,\omega,\omega^2)$, and $(1,\omega^2,\omega)$ are the remaining points of $K$.

The group element with matrix

$$
\begin{pmatrix}
-1 & 0 & 1 \\
0 & -\omega & \omega \\
0 & 0 & \omega^2
\end{pmatrix}
$$

fixes $(1,0,0)$ and $(0,1,0)$, maps $(0,0,1)$ to $(1,\omega,\omega^2)$ and $(1,1,1)$ to $(0,0,1)$. Hence, it should preserve $K$ since $A_6$ acts 4-transitively on 6 points. So the image $(-1 + \omega^2, -\omega^2 + 1, \omega)$ of $(1,\omega,\omega^2)$ must belong to $K$. This only happens if $p = 2$, in which case all points of $K$ except $(1,0,0)$ lie on the conic $X_1X_2 = X_0^2$ in $\text{PG}(2,4)$. So $K$ is the hyperoval mentioned in the statement of the proposition and this 6-arc is fixed by $A_6$ [7, p. 396].

Next, suppose $|K| = 10$. Then we can think of $A_6$ as being $L_2(9)$ acting on the elements of $GF(9) \cup \{\infty\}$. Hence, we can label the points of $K$ as $x_i, i \in GF(9) \cup \{\infty\}$, and the action of $L_2(9)$ goes via its natural action on the indices. An entirely similar argument as for the case $G_K \cong L_2(7), k = 8$, in Proposition 6, shows here that $q$ must be an even power of 3 and that $K$ is a conic in some subplane of order 9 of $\text{PG}(2,q)$. This case was treated in Section 4.1.3.

Finally, suppose $|K| = 15$. Here, $K$ can be identified with the pairs of the set $\{1,2,3,4,5,6\}$ with the natural action of $A_6$ [3]. The involution $(1\ 2)(3\ 4) \in A_6$ fixes the pairs $\{1,2\}, \{3,4\}, \{5,6\}$ and acts semi-regularly on the remaining ones. So it induces an involution in $\text{PG}(2,q)$ fixing three points of $K$, contradicting Lemma 5.
The next case deals with all groups having $A_6$ as a socle.

**Proposition 8.** Under the above assumptions, if $A_6 \leq G_K \leq \text{Aut}(A_6)$ acts primitively on an arc $K$ in $PG(2, q)$, then $A_6$ as a subgroup of $G_K$ acts primitively on $K$, $q = 2^{2h}$, $h \geq 1$, and $K$ is a hyperoval in some subplane of order 4.

**Proof.** By the previous result, we may assume that $A_6 \not\leq G_K$ and that $A_6$ does not act primitively on $K$. By the information in [3], there are two possibilities: the action of $G_K$ on $K$ is equivalent to the action of $PGL_2(9)$ on pairs of points, $O_2^+(9)$'s, of $PG(1, 9)$ or the action on $K$ is equivalent to the action of $PGL_2(9)$ on pairs of conjugated points in a quadratic extension, $O_2^+(9)$'s, of $PG(1, 9)$.

In the first case, any involution of $L_2(9)$ fixes five pairs of $PG(1, 9)$ and acts semi-regularly on the remaining 40, contradicting Lemma 6.

In the second case, the involution $x \mapsto -x$, $x \in GF(9)$, fixes four pairs of conjugated points in a quadratic extension of $GF(9)$, again contradicting Lemma 6. 

4.2.3. Case $A_7 \leq G_K \leq S_7$. This occurs when $p = 5$ and $h$ is even [1, Theorem 1.1 (8)].

**Proposition 9.** The group $A_7$ does not act primitively on any arc in $PG(2, 5^{2h})$, $h \geq 1$.

**Proof.** The group $A_7$ has a primitive action on 7, 15, 21 and 35 points [3]. Let $S := \{1, 2, 3, 4, 5, 6, 7\}$. The action of $A_7$ on 7 points is the natural one on $S$ and is 5-transitive which is impossible by Lemma 6. The action on 21 points is the action of $A_7$ on the unordered pairs of $S$. The permutation $(1 \ 2 \ 3)$ fixes 6 pairs and hence should be, by Lemma 6, the identity. The action on 35 points is the action on the triads of $S$. The permutation $(1 \ 2 \ 3)$ fixes five triads and hence should be the identity again.

The action on 15 points is the action of $A_7$ on the points of $PG(3, 2)$. Here, there is an involution fixing three points on a line of $PG(3, 2)$, contradicting Lemma 5.

To conclude, we deal with $G_K \cong S_7$.

**Proposition 10.** The group $S_7$ does not act primitively on any arc in $PG(2, 5^{2h})$, $h \geq 1$.

**Proof.** By the previous proposition, we may assume that $A_7$, as a subgroup of $S_7$, does not act primitively on $K$. This leaves only one possibility [3]: an action of $S_7$ on 120 points. The group $A_7$ acts on these points imprimitively in blocks of size 8, the stabilizer of a block being
$L_2(7)$. The 15 blocks can be identified with the points of $PG(3, 2)$. The stabilizer of a point of $PG(3, 2)$ in $A_7$ is $L_3(2)$. This contains an element $\sigma$ of order 3 and this element $\sigma$ has to fix at least 2 other points of $PG(3, 2)$. In other words, $\sigma$ stabilizes 3 blocks, and in each one of them, it must fix 2 points. So $\sigma$ fixes in total six points, contradicting Lemma 6.

4.2.4. Case $A_5 \leq G_K \leq S_5$. Since $S_5$ acts primitively on $K$ if and only if $A_5$ acts primitively on $K$ [3], assume $G_K \equiv A_5$. In this case, $q \equiv \pm 1 \pmod{10}$ [1, Theorem 1.1 (6)] or $q = 2^{2h}$, $h \geq 1$ [6, pp. 157-158]. By [3], $G_K$ can only act primitively on 5, 6, or 10 points. The action of $A_5$ in $PG(2, q)$, $q \equiv \pm 1 \pmod{10}$, is uniquely determined by the following two matrices

$$T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} - t & -t \\ t - \frac{1}{2} & t & -\frac{1}{2} \\ t & -\frac{1}{2} & -t \end{pmatrix},$$

where $4t^2 - 2t - 1 = 0$ [1, Lemma 6.4].

**Proposition 11.** Suppose $A_5$ fixes a 5-arc $K$ in $PG(2, q)$. Then $q = 2^{2h}$, $h \geq 1$, and $K$ is a conic in a subplane $PG(2, 4)$.

**Proof.** Let $K = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (1, x, y)\} = \{p_1, \ldots, p_5\}$, where $A_5$ acts naturally on the indices $i$, $1 \leq i \leq 5$.

The mapping $(1 2)(3 4)$ of $A_5$ is defined by the matrix

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and fixes $(1, x, y)$ if and only if $y = x + 1$.

The mapping $(1 2 3)$ is defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and fixes $(1, x, y)$ if and only if $1 + x = \rho$, $1 = \rho x$ and $x = \rho(1 + x)$ for some $\rho \neq 0$. This implies $x^2 + x - 1 = 0$ and $x^2 - x - 1 = 0$. So $2 = 0$ and $x^2 + x + 1 = 0$. This shows that $q = 2^{2h}$, $h \geq 1$, and $K$ is a conic in a subplane $PG(2, 4)$. 

Remark 1. This conic $K$ is contained in a unique hyperoval of $PG(2, 4)$ fixed by $A_6$ (Proposition 7).

Proposition 12. When $q \equiv \pm 1 \pmod{10}$, then the sets $K_1 = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1), (0, 4t^2, 1), (0, -4t^2, 1), (-4t^2, 1, 0), (4t^2, 1, 0), (1, 0, 4t^2), (1, 0, -4t^2)\}$, and $K_2 = \{(1, 0, 1 - 2t), (1, 0, 2t - 1), (1, 2t, 0), (1, -2t, 0), (0, 1, 2t), (0, 1, -2t)\}$, $4t^2 - 2t - 1 = 0$, constitute a 10- and 6-arc fixed by $A_5$. The points of $K_1$ are the 10 points of $PG(2, q)$ on 3 bisecants of $K_2$.

Proof. This can be verified by using the matrices $T$ and $B$ of [1, Lemma 6.4].

Proposition 13. The 10-arc $K_1$ and 6-arc $K_2$ in $PG(2, q)$, $q \equiv \pm 1 \pmod{10}$, are projectively unique.

Proof: (a) Suppose there is a second orbit $O$ of size 10. Let $p \in O$, then $p$ is fixed by a subgroup $H$ of order 6 of $A_5$. The unique subgroup of order 3 in $H$ must fix a unique point $r_1$ of $K_1$. Then $r_1$ is fixed by $H$. This implies that $pr_1$ is a tangent to $K_1$.

Since 10 is even, $p$ belongs to a second tangent $pr_2$, $r_2 \in K_1$, to $K_1$. If an element of order 3 in $H$ fixes $r_2$, it fixes 4 points of $K_1$, which is false (Lemma 6), so $p$ belongs to at least four tangents $pr_i$, $1 \leq i \leq 4$, to $K_1$. Any involution $\gamma$ in $H$ must fix two tangents through $p$ since it fixes $r_1$ and since it cannot fix four tangents (Lemma 6). Assume $\gamma(r_2) = r_2$ and $\gamma(r_3) = r_4$, then $\{p, r_1, r_2, r_3, r_4\}$ is a 5-arc fixed by $\gamma$. This contradicts Lemma 5.

(b) Suppose there is a second orbit $O$ of size 6.

If $p \in O$, then $p$ is fixed by a subgroup $H$, of order 10, of $A_5$. Since $H$ has a unique subgroup of order 5 and since $|K_2| = 6$, $H$ must fix one point $r_1$ of $K_2$ and, as in (a), $pr_1$ is tangent to $K_2$. An element of order 5 in $H$ acts transitively on $K_2 \setminus \{r_1\}$ and fixes $p$, so $p$ extends $K_2$ to a 7-arc.

An involution $\gamma$ of $H$ fixes $p, r_1$ and a second point $r_2$ of $K_2$. Let $\gamma(r_3) = r_4$, $r_3, r_4 \in K_2$, then $\{p, r_1, r_2, r_3, r_4\}$ is a 5-arc fixed by $\gamma$. This again contradicts Lemma 5.

Proposition 14. In $PG(2, 2^{2h})$, $h \geq 1$, no 10-arc $K$ is fixed by a projective group $G_K \cong A_5$ acting primitively on the points of $K$.

A 6-arc $K$ in $PG(2, 2^{2h})$, $h \geq 1$, fixed by a projective group $A_5$ acting primitively on the points of $K$ is a 6-arc in a subplane $PG(2, 4)$ and is fixed by $A_6$.

Proof. (a) Let $K$ be a 6-arc in $PG(2, 2^{2h})$, fixed by $A_5$. Since $A_5 \cong L_2(5)$ [3], the action of $A_5$ on the points of $K$ can be identified with the natural action of $L_2(5)$ on $GF(5)^+ = \{0, 1, 2, 3, 4, \infty\}$. Establish this action via the
indices, let $K = \{p_0, p_1, p_2, p_3, p_4, p_5\}$ with $p_0 = (1, 0, 0)$, $p_1 = (0, 0, 1)$, $p_2 = (1, a, b)$, $p_4 = (1, 1, 1)$ and with $p_5 = (0, 1, 0)$.

The involution $t \mapsto -t$ of $L_2(5)$ is the elation of $PG(2, q)$ defined by the matrix

$$
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

and maps $p_2$ onto $p_3$, so $p_3 = (1 + b, a + b, b)$.

The element $t \mapsto t + 1$ of $L_2(5)$ is the projective transformation with matrix

$$
\begin{pmatrix}
0 & 0 & 1 \\
0 & a & 1 \\
b & 0 & b
\end{pmatrix}
$$

and must map $p_2$ onto $p_3$. This only happens if $b^2 + b + 1 = 0 = a^2 + a + 1$.

Hence, $a$ and $b$ are the distinct solutions of $X^2 + X + 1 = 0$ and $a, b \in GF(4)$. Therefore, the arc $K$ is a 6-arc in a subplane $PG(2, 4)$ and such an arc is fixed by $A_6$ (Proposition 7). The proof of Proposition 13 can again be used to show that there is no second orbit of size 6.

(b) Suppose now that there is a 10-arc $K$ in $PG(2, 2^{2h})$, $h \geq 1$, fixed by a group $A_5$ acting primitively on the 10 points. The action of $A_5$ on the 10 points of $K$ can be represented by the natural action of $A_5$ on the unordered pairs of $\{1, \ldots, 5\}$. The elements $\alpha = (1 \ 2)(3 \ 4)(5)$ and $\beta = (1 \ 3)(2 \ 4)(5)$ generate the Klein 4-group as a subgroup of $A_5$. Both $\alpha$ and $\beta$ define elations in $PG(2, 2^{2h})$, $h \geq 1$, and since they commute, they must have a common axis (see proof of Proposition 1). Since $\alpha$ fixes the unordered pairs $\{1, 2\}, \{3, 4\}$, the axis of $\alpha$, in $PG(2, 2^h)$, is the line $L_1$ passing through the 2 points of $K$ represented by $\{1, 2\}$ and $\{3, 4\}$. Analogously, $\beta$ fixes the pairs $\{1, 3\}, \{2, 4\}$, so the axis $L_2$ of the elation $\beta$, acting on $PG(2, 2^h)$, is the line passing through the points of $K$ represented by $\{1, 3\}, \{2, 4\}$. But since $L_1$ and $L_2$ coincide, this would imply that 4 points of the 10-arc $K$ are collinear. This gives a contradiction.

Remark 2. Hexagons $\mathcal{H}$ fixed by $A_5$ were studied in detail by Dye [5]. These hexagons occur when $q \equiv \pm 1 \pmod{10}$, $q = 5^h$ or $q = 2^{2h}$, $h \geq 1$.

When $q = 2^{2h}$, $h \geq 1$, then $\mathcal{H}$ is a hyperoval in a subplane $PG(2, 4)$ and $\mathcal{H}$ is fixed by $A_6$ (Proposition 14).

From now on, assume $q$ odd. If $q = 5^h$, then $\mathcal{H}$ is a conic in a subplane $PG(2, 5)$ of $PG(2, q)$, $A_5 \equiv L_2(5)$, and $\mathcal{H}$ is fixed by $S_5 \equiv PGL_2(5)$, but $\mathcal{H}$ is not contained in a conic when $q \equiv \pm 1 \pmod{10}$. In both cases, this
hexagon is called the Clebsch hexagon [5]. One of its particular properties is that it has exactly 10 Brianchon-points, i.e., 10 points on exactly 3 biseccants to $\mathcal{H}$. If $q = 5^h$, these Brianchon-points are the internal points of the conic $\mathcal{H}$ in the subplane $\PG(2,5)$. The 10 Brianchon-points constitute a 10-arc if $q \equiv \pm 1 \pmod{10}$ (Proposition 12).

With this hexagon correspond 5 triangles whose edges partition $\mathcal{H}$ and on which $A_5$ acts in a natural way. These 5 triangles are self-polar w.r.t. a unique conic $C$. When $q = 5^h$, $\mathcal{H} = C$. The 10 Brianchon-points belong to $C$ if and only if $q = 3^{2h}$, $h \geq 1$, and in this case, $C$ is a conic in a subplane $\PG(2,9)$. Equivalently, when $q = 3^{2h}$, $h \geq 1$, the 10-arc $K_1$ (Proposition 12) is a conic in a subplane $\PG(2,9)$.

**Proposition 15.** Let $K$ be a $k$-arc of $\PG(2,q)$, fixed by an almost simple group $G_K$ acting primitively on $K$, then $K$ is one of the following arcs:

1. $K$ is a conic in a subplane $\PG(2,q')$ of $\PG(2,q)$;
2. $K$ is a 5- or 6-arc in a subplane $\PG(2,4)$ of $\PG(2,2^h)$, $h \geq 1$;
3. $K$ is a 6- or 10-arc in $\PG(2,q)$, $q \equiv \pm 1 \pmod{10}$.

**Proof.** This follows from the preceding propositions.

This completes the classification.

### 5. Complete 2-Transitive Arcs

As an immediate consequence of the classification of primitive arcs made in Sections 3 and 4, the following list of complete 2-transitive arcs is obtained.

**Proposition 16.** If $K$ is a complete $k$-arc of $\PG(2,q)$, fixed by a 2-transitive projective group $G_K$, then $K$ is one of the following arcs:

1. $K$ is a 4-arc in $\PG(2,2)$ or $\PG(2,3)$;
2. $K$ is a conic in $\PG(2,q)$, $q$ odd, $q > 3$;
3. $K$ is the unique 6-arc in $\PG(2,4)$;
4. $K$ is the unique 6-arc in $\PG(2,9)$ fixed by $A_5$.

**Proof.** This follows from the preceding classification.

The completeness of the 6-arc, fixed by $A_5$, in $\PG(2,9)$, is proved in [7, p. 414] while a 6-arc in $\PG(2,q)$, $q \geq 11$, is incomplete [7, p. 205].
REFERENCES