FUZZY ARITHMETIC WITHOUT USING THE METHOD OF α - CUTS

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Abstract: In this article, an alternative method to evaluate the arithmetic operations on fuzzy number has been developed, on the assumption that the Dubois-Prade left and right reference functions of a fuzzy number are *distribution function* and *complementary distribution function* respectively. Using the method, the arithmetic operations of fuzzy numbers can be done in a very simple way. This alternative method has been demonstrated with the help of numerical examples.

Key words and phrases: Fuzzy membership function, Dubois-Prade reference function, distribution function, Set superimposition, Glivenko-Cantelli theorem.

1. INTRODUCTION

The standard method of α-cuts to the membership of fuzzy number does not always yield results. For example, the method of α -cuts fails to find the fuzzy membership function (fmf) of even the simple function $\sqrt{X}$ when X is fuzzy. Indeed, for $\sqrt{X}$ in particular, Chou (2009) has forwarded a method of finding the fmf for a triangular fuzzy number X. We shall in this article, put forward an alternative method for dealing with the arithmetic of fuzzy numbers which are not necessarily triangular.

Dubois and Prade (see e.g. Kaufmann and Gupta (1984)) have defined a fuzzy number $X=[a,b,c]$ with membership function

$$\mu_X(x)=\begin{cases} L(x), & a\le x\le b \\ R(x), & b\le x\le c \\ 0, & otherwise \end{cases} \qquad (1.1)$$

$L(x)$ being continuous non-decreasing function in the interval [a, b], and $R(x)$ being a continuous non-increasing function in the interval [b, c], with $L(a)=R(c)=0$ and $L(b)=R(b)=1$. Dubois and Prade named $L(x)$ as left reference function and $R(x)$ as right reference function of the concerned fuzzy number. A continuous non-decreasing function of this type is also called a distribution function with reference to a Lebesgue-Stieltjes measure (de Barra (1987). pp-156).

In this article, we are going to demonstrate the easiness of applying our method in evaluating the arithmetic of fuzzy numbers if start from the simple assumption that the Dubois-Prade left reference function is a *distribution function*, and similarly the Dubois-Prade right reference function is a *complementary distribution function*. Accordingly, the functions $L(x)$ and $(1-R(x))$ would have to be associated with densities $\frac{d}{dx}(L(x))$ and $\frac{d}{dx}(1-R(x))$ in [a,b] and [b,c] respectively (Baruah (2010 a, b)).

2. SUPERIMPOSITION OF SETS

The superimposition of sets defined by Baruah (1999), and later used successfully in recognizing periodic patterns (Mahanta et al. (2008)), the operation of set superimposition is defined as follows: if the set A is superimposed over the set B, we get

$$A(S)\,B = (A-B) \cup (A \cap B)^{(2)} \cup (B-A) \qquad (2.1)$$

where S represents the operation of superimposition, and $(A \cap B)^{(2)}$ represents the elements of $(A \cap B)$ occurring twice. It can be seen that for two intervals $[a_1, b_1]$ and $[a_2, b_2]$ superimposed gives

$$[a_1, b_1]\,(S)\,[a_2, b_2]$$

$$= [a_{(1)}, a_{(2)}] \cup [a_{(2)}, b_{(1)}]^{(2)} \cup [b_{(1)}, b_{(2)}]$$

where $a_{(1)} = \min(a_1, a_2)$, $a_{(2)} = \max(a_1, a_2)$, $b_{(1)} = \min(b_1, b_2)$, and $b_{(2)} = \max(b_1, b_2)$.

Indeed, in the same way if $[a_1, b_1]^{(1/2)}$ and $[a_2, b_2]^{(1/2)}$ represent two uniformly fuzzy intervals both with membership value equal to half everywhere, superimposition of $[a_1, b_1]^{(1/2)}$ and $[a_2, b_2]^{(1/2)}$ would give rise to

$$[a_1, b_1]^{(1/2)}\,(S)\,[a_2, b_2]^{(1/2)}$$

$$= [a_{(1)}, a_{(2)}]^{(1/2)} \cup [a_{(2)}, b_{(1)}]^{(1)} \cup [b_{(1)}, b_{(2)}]^{(1/2)}. \qquad (2.2)$$

So for n fuzzy intervals $[a_1, b_1]^{(1/n)}$, $[a_2, b_2]^{(1/n)} \ldots [a_n, b_n]^{(1/n)}$ all with membership value equal to 1/n everywhere,

$$[a_1, b_1]^{(1/n)}(S)\,[a_2, b_2]^{(1/n)}(S) \ldots\ldots\ldots (S)\,[a_n, b_n]^{(1/n)}$$

$$= [a_{(1)}, a_{(2)}]^{(1/n)} \cup [a_{(2)}, a_{(3)}]^{(2/n)} \cup \ldots\ldots\ldots \cup [a_{(n-1)}, a_{(n)}]^{((n-1)/n)} \cup [a_{(n)}, b_{(1)}]^{(1)} \cup [b_{(1)}, b_{(2)}]^{((n-1)/n)} \cup \ldots\ldots\ldots \cup [b_{(n-2)}, b_{(n-1)}]^{(2/n)} \cup [b_{(n-1)}, b_{(n)}]^{(1/n)}, \qquad (2.3)$$

where, for example, $[b_{(1)}, b_{(2)}]^{((n-1)/n)}$ represents the uniformly fuzzy interval $[b_{(1)}, b_{(2)}]$ with membership $((n-1)/n)$ in the entire interval, $a_{(1)}, a_{(2)}, \ldots\ldots\ldots, a_{(n)}$ being values of $a_1, a_2, \ldots\ldots\ldots, a_n$ arranged in increasing order of magnitude, and $b_{(1)}, b_{(2)}, \ldots\ldots\ldots, b_{(n)}$ being values of $b_1, b_2, \ldots\ldots\ldots, b_n$ arranged in increasing order of magnitude.

We now define a *random* vector $X = (X_1, X_2, \ldots\ldots, X_n)$ as a family of X_k, $k = 1, 2, \ldots\ldots, n$, with every X_k inducing a sub-σ field so that X is measurable. Let $(x_1, x_2 \ldots x_n)$ be a particular realization of X, and let $X_{(k)}$ realize the value $x_{(k)}$ where $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ are *ordered* values of $x_1, x_2, \ldots\ldots, x_n$ in increasing order of magnitude. Further let the sub-σ fields induced by X_k be independent and identical. Define now

$$\Phi_n(x) = 0, \text{ if } x < x_{(1)},$$

$$= (r-1)/n, \text{ if } x_{(r-1)} \leq x \leq x_{(r)},\ r = 2, 3, \ldots, n,$$

$$= 1, \text{ if } x \geq x_{(n)}; \qquad (2.4)$$

$\Phi_n(x)$ here is an empirical distribution function of a theoretical distribution function $\Phi(x)$.

As there is a one to one correspondence between a Lebesgue-Stieltjes measure and the distribution function, we would have

$$\Pi(a, b) = \Phi(b) - \Phi(a) \qquad (2.5)$$

where Π is a measure in (Ω, A, Π), *A* being the σ- field common to every x_k.

Now the Glivenko-Cantelli theorem (see e.g. Loeve (1977), pp-20) states that

$\Phi_n(x)$ converges to $\Phi(x)$ uniformly in x. This means,

$$\sup \left| \Phi_n(x) - \Phi(x) \right| \to 0 \qquad (2.6)$$

Observe that (r-1)/n in (2.4), for $x_{(r-1)} \leq x \leq x_{(r)}$, are membership values of $[a_{(r-1)}, a_{(r)}]^{((r-1)/n)}$ and $[b_{(n-r+1)}, b_{(n-r)}]^{((r-1)/n)}$ in (2.3), for r = 2, 3, …, n. Indeed this fact found from superimposition of uniformly fuzzy sets has led us to look into the possibility that there could possibly be a link between distribution functions and fuzzy membership.

In the sections 3, 4, 5 and 6 we are going to discuss the arithmetic of fuzzy numbers.

3. ADDITION OF FUZZY NUMBERS

Consider $X = [a,b,c]$ and $Y = [p,q,r]$ be two triangular fuzzy numbers. Suppose $Z = X + Y = [a+p, b+q, c+r]$ be the fuzzy number of $X+Y$. Let the fmf of X and Y be $\mu_X(x)$ and $\mu_Y(y)$ as mentioned below

$$\mu_X(x) = \begin{cases} L(x), & a \leq x \leq b \\ R(x), & b \leq x \leq c \\ 0, & otherwise \end{cases} \qquad (3.1)$$

and

$$\mu_Y(y) = \begin{cases} L(y), & a \leq y \leq b \\ R(y), & b \leq y \leq c \\ 0, & otherwise \end{cases} \qquad (3.2)$$

where $L(x)$ and $L(y)$ are the left reference functions and $R(x)$ and $R(y)$ are the right reference functions respectively. We assume that $L(x)$ and $L(y)$ are *distribution function* and $R(x)$ and $R(y)$ are *complementary distribution function*. Accordingly, there would exist some density functions for the distribution functions $L(x)$ and $(1-R(x))$. Say,

$$f(x) = \frac{d}{dx}(L(x)), a \leq x \leq b$$

and

$$g(x) = \frac{d}{dx}(1-R(x)), b \leq x \leq c$$

We start with equating $L(x)$ with $L(y)$, and $R(x)$ with $R(y)$. And so, we obtain $y = \phi_1(x)$ and $y = \phi_2(x)$ respectively. Let $z = x + y$, so we have $z = x + \phi_1(x)$ and $z = x + \phi_2(x)$, so that $x = \psi_1(z)$ and $x = \psi_2(z)$, say. Replacing x by $\psi_1(z)$ in $f(x)$, and by $\psi_2(z)$ in $g(x)$, we obtain $f(x) = \eta_1(z)$ and $g(x) = \eta_2(z)$ say.

Now let, $$\frac{dx}{dz} = \frac{d}{dz}(\psi_1(z)) = m_1(z)$$

and $$\frac{dx}{dz} = \frac{d}{dz}(\psi_2(z)) = m_2(z)$$

The distribution function for $X+Y$, would now be given by

$$\int_{a+p}^{x} \eta_1(z) m_1(z) dz, \; a+p \leq x \leq b+q$$

and the complementary distribution function would be given by

$$1 - \int_{b+q}^{x} \eta_2(z) m_2(z) dz, \; b+q \leq x \leq c+r$$

We claim that this distribution function and the complementary distribution function constitute the fuzzy membership function of $X+Y$ as,

$$\mu_{X+Y}(x) = \begin{cases} \int_{a+p}^{x} \eta_1(z) m_1(z) dz, & a+p \leq x \leq b+q \\ 1 - \int_{b+q}^{x} \eta_2(z) m_2(z) dz, & b+q \leq x \leq c+r \\ 0 & , otherwise \end{cases}$$

4. SUBTRACTION OF FUZZY NUMBERS

Let $X=[a,b,c]$ and $Y=[p,q,r]$ be two fuzzy numbers with fuzzy membership function as in (3.1) and (3.2). Suppose $Z=X-Y$. Then the fuzzy membership function of $Z=X-Y$ would be given by $Z=X+(-Y)$.

Suppose $(-Y)=[-r,-q,-p]$ be the fuzzy number of $(-Y)$. We assume that the Dubois-Prade reference functions $L(y)$ and $R(y)$ as distribution and complementary distribution function respectively. Accordingly, there would exist some density functions for the distribution functions $L(y)$ and $(1-R(y))$. Say,

$$f(y)=\frac{d}{dy}(L(y)),\ p\le y\le q$$

and $$g(y)=\frac{d}{dy}(1-R(y)),\ q\le y\le r$$

Let $t=-y$ so that $\frac{dy}{dt}=-1=m(t)$, say. Replacing $y=-t$ in $f(y)$ and $g(y)$, we obtain $f(y)=\eta_1(t)$ and $g(y)=\eta_2(t)$, say. Then the fmf of $(-Y)$ would be given by

$$\mu_{-Y}(y)=\begin{cases}\int_{-r}^{y}\eta_2(t)m(t)dt, & -r\le y\le -q\\ 1-\int_{-q}^{y}\eta_1(t)m(t)dt, & -q\le y\le -p\\ 0 & ,otherwise\end{cases}$$

Then we can easily find the fmf of $X-Y$ by addition of fuzzy numbers X and $(-Y)$ as described in the earlier section.

5. MULTIPLICATION OF FUZZY NUMBERS

Let $X=[a,b,c]$, $(a,b,c>0)$ and $Y=[p,q,r]$, $(p,q,r>0)$ be two triangular fuzzy numbers with fuzzy membership function as in (3.1) and (3.2). Suppose $Z=X.Y=[a.p,b.q,c.r]$ be the fuzzy number of $X.Y$. $L(x)$ and $L(y)$ are the left reference functions and $R(x)$ and $R(y)$ are the right reference functions respectively. We assume that $L(x)$ and $L(y)$ are distribution function and $R(x)$ and $R(y)$ are complementary distribution function. Accordingly, there would exist some density functions for the distribution functions $L(x)$ and $(1-R(x))$. Say,

$$f(x)=\frac{d}{dx}(L(x)),\ a\le x\le b$$

and $$g(x)=\frac{d}{dx}(1-R(x)),\ b\le x\le c$$

We again start with equating $L(x)$ with $L(y)$, and $R(x)$ with $R(y)$. And so, we obtain $y=\phi_1(x)$ and $y=\phi_2(x)$ respectively. Let $z=x.y$, so we have $z=x.\phi_1(x)$ and $z=x.\phi_2(x)$, so that $x=\psi_1(z)$ and $x=\psi_2(z)$, say. Replacing x by $\psi_1(z)$ in $f(x)$, and by $\psi_2(z)$ in $g(x)$, we obtain $f(x)=\eta_1(z)$ and $g(x)=\eta_2(z)$ say.

Now let, $$\frac{dx}{dz}=\frac{d}{dz}(\psi_1(z))=m_1(z)$$

and $$\frac{dx}{dz}=\frac{d}{dz}(\psi_2(z))=m_2(z)$$

The distribution function for $X.Y$, would now be given by

$$\int_{ap}^{x} \eta_1(z) m_1(z) dz, \ ap \le x \le bq$$

and the complementary distribution function would be given by

$$1 - \int_{bq}^{x} \eta_2(z) m_2(z) dz, \ bq \le x \le cr$$

We are claiming that this distribution function and the complementary distribution function constitute the fuzzy membership function of $X.Y$ as,

$$\mu_{XY}(x) = \begin{cases} \int_{ap}^{x} \eta_1(z) m_1(z) dz, & ap \le x \le bq \\ 1 - \int_{bq}^{x} \eta_2(z) m_2(z) dz, & bq \le x \le cr \\ 0 & , otherwise \end{cases}$$

6. DIVISION OF FUZZY NUMBERS

Let $X = [a,b,c]$, $(a,b,c > 0)$ and $Y = [p,q,r]$, $(p,q,r > 0)$ be two triangular fuzzy numbers with fuzzy membership function as in (3.1) and (3.2). Suppose $Z = \frac{X}{Y}$. Then the fuzzy membership function of $Z = \frac{X}{Y}$ would be given by $Z = X.Y^{-1}$.

At first, we have to find the fmf of Y^{-1}. Suppose $Y^{-1} = [r^{-1}, q^{-1}, p^{-1}]$ be the fuzzy number of Y^{-1}. We assume that the Dubois-Prade reference functions $L(y)$ and $R(y)$ as distribution and complementary distribution function respectively. Accordingly, there would exist some density functions for the distribution functions $L(y)$ and $(1 - R(y))$. Say,

$$f(y) = \frac{d}{dy}(L(y)), \ p \le y \le q$$

and $$g(y) = \frac{d}{dy}(1 - R(y)), \ q \le y \le r$$

Let $t = y^{-1}$ so that $\frac{dy}{dt} = -\frac{1}{t^2} = m(t)$, say.

Replacing $y = t^{-1}$ in $f(y)$ and $g(y)$, we obtain $f(y) = \eta_1(t)$ and $g(y) = \eta_2(t)$, say. Then the fmf of (Y^{-1}) would be given by

$$\mu_{Y^{-1}}(y) = \begin{cases} \int_{r^{-1}}^{y} \eta_2(t) m(t) dt, & r^{-1} \le y \le q^{-1} \\ 1 - \int_{q^{-1}}^{y} \eta_1(t) m(t) dt, & q^{-1} \le y \le p^{-1} \\ 0 & , otherwise \end{cases}$$

Next, we can easily find the fmf of $\frac{X}{Y}$ by multiplication of fuzzy numbers X and Y^{-1} as described in the earlier section.

In the next section we are going to cite some numerical examples for the above discussed methods.

7. NUMERICAL EXAMPLES

Example 1:

Let $X = [1,2,4]$ and $Y = [3,5,6]$ be two triangular fuzzy numbers with fmf

$$\mu_X(x) = \begin{cases} x - 1, & 1 \le x \le 2 \\ \frac{4 - x}{2}, & 2 \le x \le 4 \\ 0 & , otherwise \end{cases} \qquad (7.1)$$

And $$\mu_Y(y)=\begin{cases}\frac{y-3}{2}, & 3\le y\le 5\\ 6-y, & 5\le y\le 6\\ 0 & ,otherwise\end{cases} \quad (7.2)$$

Here $X+Y=[4,7,10]$. Equating the distribution function and complementary distribution function, we obtain $y=\phi_1(x)=2x+1$ and $y=\phi_2(x)=\frac{x+8}{2}$. Let $z=x+y$, so we shall have

$z=x+\phi_1(x)=3x+1$ and $z=x+\phi_2(x)=\frac{3x+8}{2}$,

so that $x=\psi_1(z)=\frac{z-1}{3}$ and $x=\psi_2(z)=\frac{2z-8}{3}$, respectively. Replacing x by $\psi_1(z)$ and $\psi_2(z)$ in the density functions $f(x)$ and $g(x)$ respectively, we have $f(x)=1=\eta_1(z)$ and $g(x)=\frac{1}{2}=\eta_2(z)$.

Now $m_1(z)=\frac{d}{dz}(\psi_1(z))=\frac{1}{3}$

and $m_2(z)=\frac{d}{dz}(\psi_2(z))=\frac{2}{3}$.

Then the fmf of $X+Y$ would be given by,

$$\mu_{X+Y}(x)=\begin{cases}\frac{x-4}{3}, & 4\le x\le 7\\ \frac{10-x}{3}, & 7\le x\le 10\\ 0 & ,otherwise\end{cases}$$

Example 2:

Let $X=[1,2,4]$ and $Y=[3,5,6]$ be two triangular fuzzy numbers with membership functions as in (7.1) and (7.2). Suppose, $Z=X-Y$ or $Z=X+(-Y)$.

Now, $-Y=[-6,-5,-3]$ be the fuzzy number of $(-Y)$. Let $t=-y$ so that $y=-t$, which implies $m(t)=-1$. Then the density function $f(y)$ and $g(y)$ would be, say,

$f(y)=\frac{d}{dy}\left(\frac{y-3}{2}\right)=\frac{1}{2}=\eta_1(t),\ 3\le y\le 5$ and

$g(y)=\frac{d}{dy}(1-(6-y))=1=\eta_2(t),\ 5\le y\le 6$.

Then the fmf of $(-Y)$ would be given by

$$\mu_{-Y}(y)=\begin{cases}\frac{6+y}{6-5}, & -6\le y\le -5\\ \frac{3+y}{3-5}, & -5\le y\le -3\\ 0 & ,otherwise\end{cases}$$

Then by addition of fuzzy numbers $X=[1,2,4]$ and $(-Y)=[-6,-5,-3]$ the fmf of $X-Y$ is given by,

$$\mu_{X+(-Y)}(x)=\begin{cases}\frac{x+5}{2}, & -5\le x\le -3\\ \frac{1-x}{4}, & -3\le x\le 1\\ 0 & ,otherwise\end{cases}$$

Example 3:

Let $X=[1,2,4]$ and $Y=[3,5,6]$ be two triangular fuzzy numbers with membership functions as in (7.1) and (7.2). Suppose, $X.Y=[3,10,24]$ be the fuzzy number of $X.Y$. Equating the distribution function and complementary distribution function, we obtain $y=\phi_1(x)=2x+1$ and $y=\phi_2(x)=\frac{x+8}{2}$.

Let $z = x.y$, so we shall have $z = x.\phi_1(x) = 2x^2 + x$ and

$$z = x.\phi_2(x) = \frac{8x + x^2}{2}$$, so that

$$x = \psi_1(z) = \frac{-1 \pm \sqrt{1+8z}}{4}$$

and $x = \psi_2(z) = -4 \pm \sqrt{16+2z}$. Replacing x by $\psi_1(z)$ and $\psi_2(z)$ in the density functions $f(x)$ and $g(x)$ respectively, we have $f(x) = 1 = \eta_1(z)$ and $g(x) = \frac{1}{2} = \eta_2(z)$.

Now $m_1(z) = \frac{d}{dz}(\psi_1(z))$ and $m_2(z) = \frac{d}{dz}(\psi_2(z))$.

Then the fmf of $X.Y$ would be given by

$$\mu_{X.Y}(x) = \begin{cases} \frac{\sqrt{1+8x}-5}{4}, & 3 \le x \le 10 \\ \frac{8-\sqrt{16+2x}}{2}, & 10 \le x \le 24 \\ 0 & ,otherwise \end{cases}$$

Example 4:

Let $X = [1,2,4]$ and $Y = [3,5,6]$ be two triangular fuzzy numbers with membership functions as in (7.1) and (7.2). Suppose, $Z = \frac{X}{Y}$ or $Z = X.Y^{-1}$.

Then the fmf of $Y^{-1} = [6^{-1}, 5^{-1}, 3^{-1}]$ is given as

$$\mu_{\frac{1}{Y}}(y) = \begin{cases} \frac{6-\frac{1}{y}}{6-5}, & \frac{1}{6} \le y \le \frac{1}{5} \\ \frac{\frac{1}{y}-3}{5-3}, & \frac{1}{5} \le y \le \frac{1}{3} \\ 0 & ,otherwise \end{cases}$$

Then by multiplication of fuzzy numbers $X = [1,2,4]$ and $Y^{-1} = [6^{-1}, 5^{-1}, 3^{-1}]$ the fuzzy membership function of $\frac{X}{Y}$ would be given by,

$$\mu_{\frac{X}{Y}}(x) = \begin{cases} \frac{6x-1}{x+1}, & \frac{1}{6} \le x \le \frac{2}{5} \\ \frac{4-3x}{2(x+1)}, & \frac{2}{5} \le x \le \frac{4}{3} \\ 0 & ,otherwise \end{cases}$$

Example 5:

Let $X = [2,4,5]$ be a triangular fuzzy number and $Y = [4,16,25]$ which is a non-triangular fuzzy number with membership functions respectively as,

$$\mu_X(x) = \begin{cases} \frac{x-2}{2}, & 2 \le x \le 4 \\ 5-x, & 4 \le x \le 5 \\ 0 & ,otherwise \end{cases}$$

and $$\mu_Y(y) = \begin{cases} \frac{\sqrt{y}-2}{2}, & 4 \le y \le 16 \\ 5-\sqrt{y}, & 16 \le y \le 25 \\ 0 & ,otherwise \end{cases}$$

We can find the fmf of $X+Y$ which is given by,

$$\mu_{X+Y}(x) = \begin{cases} \dfrac{\sqrt{1+4x}-2}{4}, & 6 \le x \le 20 \\ \dfrac{11-\sqrt{1+4x}}{2}, & 20 \le x \le 30 \\ 0 & , otherwise \end{cases}$$

All four demonstrations above can be verified to be true, using the method of α -cuts.

9. CONCLUSION

The standard method of α -cuts to the membership of a fuzzy number does not always yield results. We have demonstrated that an assumption that the Dubois-Prade left reference function is a distribution function and that the right reference function is a complementary distribution function leads to a very simple way of dealing with fuzzy arithmetic. Further, this alternative method can be utilized in the cases where the method of α -cuts fails, e.g. in finding the fmf of $\sqrt{X}$.

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