The Precession of Mercury‘s Perihelion in View of the (Lorentz) Covariant Gravitational Field Theory: an update

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Abstract

Based on the (Lorentz) Covariant Gravitational Field Theory (Kling 2017a) Mercury's orbit and in particular the precession of its perihelion is investigated. It is demonstrated that Mercury is not only exposed to a centrally directed force toward the Sun but also to a torque which causes a small but continuous exchange of angular momentum with the gravitational field. Mercury's intrinsic perihelion shift, excluding the influence of the other planets in our solar system, is determined to amount thirteen twelfths of the General Relativity prediction. The difference of 3.58 arc sec per century is caused by the property of the gravitational field energy to be equivalent to a mass and to act as a source of gravitation which is a key assumption of the Covariant Gravitational Field Theory. If this property was ignored, the predictions of the Covariant Gravitational Field Theory and of General Relativity were the same within the framework of the applied approximations. Consequently, the precise observation of the precession of Mercury's perihelion provides a means in order to find out if the gravitational field energy is equivalent to a mass or not. The current observations and the available calculations of the Newtonian influence of the other planets onto Mercury's perihelion support the prediction of the Covariant Gravitational Field Theory. We suggest to verify this result by means of a detailed analysis of the available observational data, in particular radiometric range measurements to spacecrafts within the framework of the Covariant Gravitational Field Theory.

Key words: Gravitation – Gravity – Gravitational field – Gravitomagnetism – Covariant Gravitational Field Theory – Mercury – Perihelion – Periastron – Precession.

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1. Introduction

A study presented by Le Verrier in 1859 marks the start of the awareness of the so-called anomalous precession of Mercury's perihelion (Le Verrier 1859a,b). Le Verrier calculated the Newtonian influence of the other planets in our solar system onto Mercury's orbit around the Sun which results in a precession of its perihelion. When comparing this influence with the observed precession he found a deviation of 38 arc sec per century which became famous as anomalous perihelion shift. In order to explain this anomaly, which later on was adjusted to about 43 arc sec per century (Newcomb 1882), scientists came up with a series of proposals such as the request for another hypothetical planet near the Sun (Le Verrier 1859b), the postulate of a magnetic gravitational component (Holzmüller 1870, Tisserand 1872, 1890) according to Weber's electrodynamic law (Weber 1846), or the assignment of the anomaly to a slight oblateness of the Sun being the origin of a solar gravitational quadrupole moment (Newcomb 1895 - 1898). The existence of an additional planet could not be confirmed, Weber's law was given up in favour of Maxwell's theory, the solar quadrupole moment turned out to be far too small in order to account for the discrepancy. The quest for an explanation came to an end when Einstein managed to explain Mercury's perihelion shift by General Relativity (Einstein 1915). He calculated the excess precession to be 43"/cy (arc sec per century) which was a remarkable agreement with the difference of the observations and the calculations of the Newtonian influence of the other planets at that time (Newcomb 1882)[2]. The importance of this agreement has been reflected by the fact that the explanation of the precession of Mercury's perihelion is the first of three – later on called 'classical' - tests of General Relativity proposed by Einstein (1916, 1919). According to Weinberg (1972) 'This is by far the most import experimental verification of general relativity …'. More detailed information on the history can be found e.g. in (Roseveare 1982) or (Treschman 2014).

Motivation for the present work is to check the impact of a recently presented Lorentz Covariant Gravitational Field Theory (Kling 2017a) on the precession of Mercury's perihelion. This gravitational field theory is based on a set of axiomatic principles and derived from Heaviside's field equations (Heaviside 1893) which can be characterised as a gravitational counterpart to the Maxwell equations. A key feature of this theory is the assumption that the equivalence of energy and mass is unconditionally valid without any exception which implies that also gravitational field energy is equivalent to a mass and consequently acts as a source of gravitation.

[2] Interestingly, the expression deduced by Einstein by means of General Relativity had been presented by Gerber (1898, 1902) already 17 years earlier. Gerber derived this formula by other means which have been exposed to criticism (e.g. von Laue 1917) and which shall not be discussed here.

As this particular feature marks a significant difference from General Relativity (Kling 2017a), it is of dedicated interest to check if the property of the gravitational field to act as a field generator has a measurable impact to the precession of Mercury's perihelion. In case this turns out to be true, the observation of Mercury's perihelion might provide a means of clarifying the underlying question if gravitation really radically differs from all other fundamental forces. Consequently we analyse the intrinsic precession of Mercury's perihelion in the gravitational field of the Sun in the framework of the Covariant Gravitational Field Theory. The calculation of the influence of the other planets onto Mercury's orbit is not subject of the present work.

In section 2 expressions for Mercury's and the Sun's gravitational fields are derived. Starting from these fields the force onto Mercury by the Sun is calculated in section 3. In section 4 the relativistic motion of Mercury is investigated which enables us to conclude on the precession of Mercury's perihelion in section 5. In section 6 our result is compared with the available observational data. A summary in section 7 completes this work.

Some conventions:

Due to the formal analogy of the gravitational field theory used in the present work with Maxwell's theory of electrodynamics, we will – as in (Kling 2017a) - use the same terms and characters which are commonly used in electrodynamics. As the present study focuses on gravitation, we will for simplicity even refrain from adding a specific index in order to indicate that we are considering a gravitational quantity. Consequently, E describes the gravitational field strength, B the gravitomagnetic induction and so forth. In full analogy to the electrical case the terms 'gravitoelectric' and 'gravitomagnetic' will be used.

2. Mercury's and the Sun's Gravitational Fields

In this section we investigate the gravitational fields of Mercury and the Sun within the framework of the Lorentz Covariant Gravitational Field Theory presented by Kling (2017a). As in the next section the motion of Mercury will be analysed from the perspective of an inertial frame S in which the Sun's centre of mass is at rest, expressions for the gravitational field components in this frame are needed. The mass density distributions of both the Sun and Mercury are assumed to be spherically symmetric in their individual rest frame.

2.1 Mercury's Gravitational Field

Mercury's gravitational field is characterized by the gravitoelectric field strength $\boldsymbol{E}_M$ and the gravitomagnetic induction $\boldsymbol{B}_M$ which result from the gravitational field equations (Kling 2017a)

$$\nabla \cdot \boldsymbol{E}_M = -4\pi G \rho_M - 4\pi G \rho_{Mf} \quad , \quad (1)$$

$$\nabla \times \boldsymbol{E}_M = -\frac{\partial \boldsymbol{B}_M}{\partial t} \quad , \quad (2)$$

$$\nabla \cdot \boldsymbol{B}_M = 0 \quad , \quad (3)$$

$$c^2 \nabla \times \boldsymbol{B}_M = -4\pi G\, \boldsymbol{j}_M - 4\pi G\, \boldsymbol{j}_{Mf} + \frac{\partial \boldsymbol{E}_M}{\partial t} \quad , \quad (4)$$

where ρ_M describes Mercury's mass density distribution, $\boldsymbol{j}_M$ its mass current density which is defined as the mass flow per unit time across a unit area perpendicular to the flow, ρ_{Mf} and $\boldsymbol{j}_{Mf}$ the mass equivalent of the energy density and of the related energy flux density of the gravitational field bound to Mercury, ∇ the nabla operator, $\partial/\partial t$ the derivative with respect to time, G the gravitational constant and c the vacuum speed of the gravitational field which is assumed to coincide with the vacuum speed of light. Bold characters indicate vectors. If $\boldsymbol{v}$ describes Mercury's velocity in our frame $\mathcal{S}$, the associated mass current density is

$$\boldsymbol{j}_M = \rho_M\, \boldsymbol{v} \quad . \quad (5)$$

The mass density ρ_M and the mass current density $\boldsymbol{j}_M$ cover Mercury's mass excluding the mass equivalent of the gravitational field energy which is described by ρ_{Mf} and $\boldsymbol{j}_{Mf}$ and included via the respective source terms in the equations (1) and (4). A convenient way to solve this set of differential equations is to make use of its covariance under Lorentz transformations which has been proven in (Kling 2017a) and to start from Mercury's rest frame $\mathcal{S}$'. The origin of the coordinate system in $\mathcal{S}$' shall be located at Mercury's centre of mass, its orientation shall be the same as in $\mathcal{S}$. All quantities related to $\mathcal{S}$' will be marked by a prime. In $\mathcal{S}$' Mercury's velocity is zero, and there is no mass current and also no gravitomagnetic field, and the mass equivalent of the energy density of Mercury's gravitational field is (Kling 2013)

$$\rho'_{Mf} = -\frac{1}{8\pi G c^2}\, \boldsymbol{E'_M}^{\mathbf{2}} \quad . \quad (6)$$

Consequently our set of field equations is simplified to

$$\nabla' \cdot \boldsymbol{E}'_M = -4\pi G \rho'_M + \frac{1}{2c^2}\, \boldsymbol{E'_M}^{\mathbf{2}} \quad , \quad (7)$$

$$\nabla' \times \boldsymbol{E}'_M = 0 \quad . \quad (8)$$

Spherical symmetry implies that the mass density distribution is solely a function of the distance r' from Mercury's centre, i.e. $\rho'_M = \rho'_M(r')$. Furthermore, Mercury's mass shall be concentrated inside a sphere of a radius a_M, Mercury's radius, which implies that $\rho'_M(r') = 0$ for $r' > a_M$. For this type of arrangement the solution

$$\boldsymbol{E}'_M = E'_M(r')\,\boldsymbol{e}_{r'} = -\kappa'_M\, G\,\frac{m}{r'^2}\;\boldsymbol{e}_{r'} \tag{9a}$$

$(r' \geq a_M)$, with

$$\kappa'_M = \frac{1}{1+\frac{r_{gM}}{a_M}-\frac{r_{gM}}{r'}} \tag{9b}$$

has already been given in (Kling 2013), where the parameter $\boldsymbol{e}_{r'}$ denotes the radial unit vector,

$$m = 4\pi\int_0^{a_M} \rho(r')\; r'^2\, dr' \tag{10}$$

describes Mercury's rest mass and $r_{gM} = Gm/2c^2$ Mercury's characteristic radius (Kling 2013). The correction factor κ'_M is a consequence of the consideration of the mass equivalent of the gravitational field energy as a field generator. In (Kling 2017b) the gravitational field components from the perspective of the frame $\mathcal{S}$, which is moving at the velocity $-\boldsymbol{v}$ along the z'-axis with respect to $\mathcal{S}$', have been derived by means of the Lorentz transformations where the contribution of the gravitational field itself has been neglected, i.e. where the correction factor has been assumed to be approximately $\kappa'_M \approx 1$. If we proceed in exactly the same way while respecting the mass equivalent of the gravitational field energy as a field generator, we achieve for the gravitational field components at the position $\boldsymbol{x} = (x,y,z)$ $(|\boldsymbol{x}-\boldsymbol{v}t| > a_M)$ at the time t in our inertial frame $\mathcal{S}$ (see also Kling 2017a, 2020 update)

$$\boldsymbol{E}_M(\boldsymbol{x},t) = -G\,m\,\gamma\,\kappa_M\,\frac{\boldsymbol{x}-\boldsymbol{v}t}{[x^2+y^2+(z-vt)^2\gamma^2]^{3/2}} \qquad , \tag{11}$$

$$\boldsymbol{B}_M(\boldsymbol{x},t) = \frac{\boldsymbol{v}}{c^2}\times\boldsymbol{E}_M(\boldsymbol{x},t) \qquad , \tag{12}$$

where

$$\kappa_M = \frac{1}{1+\frac{r_{gM}}{a_M}-\frac{r_{gM}}{[x^2+y^2+(z-vt)^2\gamma^2]^{1/2}}} \qquad , \tag{13}$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \qquad , \tag{14}$$

and $v = |\boldsymbol{v}|$. The time axis has been defined in such a way that Mercury's centre of mass is positioned at the origin of our coordinate system at the time $t = 0$; the z-axis is assumed to point into the same direction as the z'-axis. As in our case Mercury will not travel strictly along the z-axis, we have to generalise this result. If $\boldsymbol{x}_M(t)$ describes the position vector of the origin of the coordinate system in $\mathcal{S}$', which is nothing else than the position vector of Mercury's centre of mass, we achieve for the gravitational field components at the position $\boldsymbol{x}$ $(|\boldsymbol{x}-\boldsymbol{x}_M(t)|>a_M)$

$$\boldsymbol{E}_M(\boldsymbol{x},t)=-G\,m\,\gamma\,\kappa_M\frac{\boldsymbol{x}-\boldsymbol{x}_M(t)}{[|(\boldsymbol{x}-\boldsymbol{x}_M(t))_\perp|^2+|(\boldsymbol{x}-\boldsymbol{x}_M(t))_\parallel|^2\gamma^2]^{3/2}}\qquad,\qquad(15a)$$

$$=-G\,m\,\gamma\,\kappa_M\,\frac{\boldsymbol{x}-\boldsymbol{x}_M(t)}{|\boldsymbol{x}-\boldsymbol{x}_M(t)|^3}\left[1+\left|\frac{\boldsymbol{x}-\boldsymbol{x}_M(t)}{|\boldsymbol{x}-\boldsymbol{x}_M(t)|}\cdot\frac{\boldsymbol{v}}{c}\right|^2\gamma^2\right]^{-3/2}\qquad,\qquad(15b)$$

where

$$\kappa_M=\frac{1}{1+\frac{r_{gM}}{a_M}}\;\frac{1}{1-\left(\frac{r_{gM}}{1+r_{gM}/a_M}\right)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}(t)|}\left[1+\left|\frac{\boldsymbol{x}-\boldsymbol{x}_M(t)}{|\boldsymbol{x}-\boldsymbol{x}_M(t)|}\cdot\frac{\boldsymbol{v}}{c}\right|^2\gamma^2\right]^{-1/2}}\quad.\;(16)$$

In equation (15a) the parameters $(\boldsymbol{x}-\boldsymbol{x}_M(t))_\parallel$ and $(\boldsymbol{x}-\boldsymbol{x}_M(t))_\perp$ describe the components of $\boldsymbol{x}-\boldsymbol{x}_M(t)$ along and orthogonal to the direction of Mercury's motion. Let us now have a look at the ratio of Mercury's characteristic and Mercury's physical radius. Making use of the mass $m=3{,}3011\cdot10^{23}$ kg and the radius $a_\mathrm{M}=2439{,}7$ km of Mercury (Williams 2016a) we achieve $r_{g\mathrm{M}}/a_\mathrm{M}\approx\;5\cdot10^{-11}\ll1$. As the correction induced by the factor $\left(1+r_{gM}/a_M\right)^{-1}$ is far below the precision at which Mercury's mass is known, we will not make a relevant fault when assuming

$$\frac{m}{1+\frac{r_{gM}}{a_M}}\;\approx\;m\qquad.\qquad(17)$$

and consequently

$$\frac{r_{gM}}{1+\frac{r_{gM}}{a_M}}\;\approx\;r_{gM}\qquad.\qquad(18)$$

As we are only interested in the gravitational field at positions $\boldsymbol{x}$ with $|\boldsymbol{x}-\boldsymbol{x}_M(t)|>a_\mathrm{M}$, also the condition $r_{gM}/|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}(t)|\ll1$ holds. According to Williams (2016a) Mercury's mean orbital velocity is $47{,}36$ km/s. Consequently we have $v^2/c^2\approx2.5\cdot10^{-8}\ll1$. As both v^2/c^2 and $r_{gM}/|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}(t)|$ are much smaller than 1, we expand the right side of equation (15b) with respect to powers of v^2/c^2 and $r_{gM}/|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}(t)|$ and achieve

$$\boldsymbol{E}_\mathrm{M}(\boldsymbol{x},t)=-Gm\gamma\frac{\boldsymbol{x}-\boldsymbol{x}_M(t)}{|\boldsymbol{x}-\boldsymbol{x}_M(t)|^3}\left[1+\frac{r_{gM}}{|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}(t)|}-\frac{3}{2}\left|\frac{\boldsymbol{x}-\boldsymbol{x}_M(t)}{|\boldsymbol{x}-\boldsymbol{x}_M(t)|}\cdot\frac{\boldsymbol{v}}{c}\right|^2\right]$$

$$+\,O\left(\left(\frac{v^2}{c^2}\right)^2,\left(\frac{v^2}{c^2}\right)\frac{r_{gM}}{|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}|},\left(\frac{r_{gM}}{|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}|}\right)^2\right)\qquad.\qquad(19)$$

In our investigation to follow we will always restrict ourselves to contributions up to the first order in v^2/c^2 and $r_{gM}/|\boldsymbol{x}-\boldsymbol{x_M}(t)|$, i.e. all terms of the order $(v^2/c^2)^i\left(r_{gM}/|\boldsymbol{x}-\boldsymbol{x_M}|\right)^j$ with $\mathrm{i}+\mathrm{j}\ \geq 2$ will be neglected.

The factor γ – in full analogy to the respective electrical case (see e.g. Feynman, Leighton & Sands 1966) – naturally came in via the Lorentz transformation. Different from other statements (see e.g. Oas 2005a/b) undoubtedly Mercury's velocity-dependent relativistic gravitational mass is to be used and not its rest mass.

2.2 The Sun's Gravitational Field

Our inertial frame $\mathcal{S}$ has been defined as the Sun's rest system. The origin of the associated coordinate system is located at the Sun's centre of mass. The Sun's mass density distribution is assumed to be spherically symmetric, i.e. its slight oblateness is neglected as well as its rotation. Consequently, the Sun's gravitational field components can be directly derived from the equations (11-14) which describe Mercury's gravitational field. By setting $\boldsymbol{v}$=0 and replacing all quantities related to Mercury by their solar counterpart we obtain for the Sun's gravitational field strength $\boldsymbol{E}_\odot(\boldsymbol{x})$ at the position $\boldsymbol{x}$ ($|\boldsymbol{x}|>a_\odot$)

$$\boldsymbol{E}_\odot(\boldsymbol{x}) = -G\, M_\odot\, \kappa_\odot\, \frac{\boldsymbol{x}}{r^3} \qquad , \tag{20}$$

with

$$\kappa_\odot = \frac{1}{1+\frac{r_{g\odot}}{a_\odot}-\frac{r_{g\odot}}{r}} \qquad , \tag{21}$$

where $r=|\boldsymbol{x}|$ describes the distance from the origin of our coordinate system, $a_\odot$ the Sun's radius, $M_\odot$ its mass and $r_{g\odot}=GM_\odot/2c^2$ its characteristic radius. As in the frame $\mathcal{S}$ the Sun is at rest, there is no gravitomagnetic field component. The Sun's gravitational field represents an energy distribution which is equivalent to the mass density distribution (Kling 2013)

$$\rho_{\odot,field}(\boldsymbol{x}) = -\frac{\boldsymbol{E}_\odot(\boldsymbol{x})^2}{8\pi G c^2} \qquad \text{for } |\boldsymbol{x}|>a_\odot \qquad , \tag{22}$$

which is spherically symmetric and consequently just a function of the radial distance r. By means of the equations (20 – 21) we get

$$\rho_{\odot,field}(r) = -\frac{GM_\odot^2}{8\pi c^2}\,\frac{1}{\left(1+\frac{r_{g\odot}}{a_\odot}-\frac{r_{g\odot}}{r}\right)^2}\,\frac{1}{r^4} \qquad \text{for r} > a_\odot \qquad . \tag{23}$$

In the next section the total mass $\widehat{M}_\odot(r)$ which is enclosed by a concentric sphere of a radius $r > a_\odot$ will be needed. It is

$$\widehat{M}_\odot(r) = \int_{0 \leq |\boldsymbol{x}| \leq r} \rho_{\odot,tot}(\boldsymbol{x}')\ d^3x' \qquad , \tag{24}$$

with

$$\rho_{\odot,tot}(\boldsymbol{x}) = \rho_\odot(\boldsymbol{x}) + \rho_{\odot,field}(\boldsymbol{x}) \qquad , \tag{25}$$

where $\rho_\odot(\boldsymbol{x})$ describes the solar mass density distribution excluding the contribution of the gravitational field. Equation (24) leads us to

$$\widehat{M}_\odot(r) = M_\odot + 4\pi \int_{a_\odot}^{r} \rho_{\odot,field}(r')\ r'^2 dr' \tag{26a}$$

$$= \frac{M_\odot}{1 + \frac{r_{g\odot}}{a_\odot}} \frac{1}{1 - \frac{r_{g\odot}}{r} / \left(1 + \frac{r_{g\odot}}{a_\odot}\right)} \tag{26b}$$

$$= \frac{M_\odot}{1 + \frac{r_{g\odot}}{a_\odot}} \left[1 + \frac{r_{g\odot}}{r} / \left(1 + \frac{r_{g\odot}}{a_\odot}\right) + \cdots\right] \qquad . \tag{26c}$$

for $r \geq a_\odot$. In (Kling 2013) the ratio $r_{g\odot}/a_\odot$ has been identified to be ~10^{-6}. If we now, analogue to Mercury's case above, make use of the fact that $r_{g\odot}/a_\odot \ll 1$, approximately assume

$$\frac{M_\odot}{1 + \frac{r_{g\odot}}{a_\odot}} \approx M_\odot \qquad , \tag{27}$$

and consequently

$$r_{g\odot} / \left(1 + \frac{r_{g\odot}}{a_\odot}\right) \approx r_{g\odot} \qquad , \tag{28}$$

and furthermore consider only terms up to the first order in $r_{g\odot}/r$ we end up at

$$\widehat{M}_\odot(r) = M_\odot \left[1 + \frac{r_{g\odot}}{r}\right] \qquad . \tag{29}$$

This expression will be useful in the next section.

3. The Force onto Mercury by the Gravitational Field of the Sun

Let us consider the force which Mercury is exerting on a probe mass M_p which is at rest and located at the position $\boldsymbol{x}_p = (x_p, y_p, z_p)$. As Mercury is accompanied by a gravitoelectric as well as a gravitomagnetic field, Mercury's force $\boldsymbol{F}_p(t)$ onto the probe mass has both a gravitoelectric and a gravitomagnetic component. It is

$$\boldsymbol{F}_p(t) = M_p \left[\boldsymbol{E}_M(\boldsymbol{x}_p, t) + \boldsymbol{v}_p(t) \times \boldsymbol{B}_M(\boldsymbol{x}_p, t)\right] \quad , \tag{30}$$

where $\boldsymbol{v}_p$ denotes the relative velocity of the probe mass referred to the gravitomagnetic field component which has been explained in Kling (2017b) when analysing the deflection of a photon by the Sun's gravitational field[3]. As Mercury's gravitomagnetic field is moving at the velocity $\boldsymbol{v}$, we have

$$\boldsymbol{v}_p = -\boldsymbol{v} \quad , \tag{31}$$

where $\boldsymbol{v}$ again describes Mercury's velocity. Hence we achieve

$$\boldsymbol{F}_p(\boldsymbol{x}_p, t) = M_p \left[\boldsymbol{E}_M(\boldsymbol{x}_p, t) - \boldsymbol{v}(t) \times \boldsymbol{B}_M(\boldsymbol{x}_p, t)\right] \quad . \tag{32}$$

With equation (12) we get

$$\boldsymbol{F}_p(\boldsymbol{x}_p, t) = M_p \left[\left(1 + \frac{v(t)^2}{c^2}\right)\boldsymbol{E}_M(\boldsymbol{x}_p, t) - \left(\boldsymbol{v}(t) \cdot \boldsymbol{E}_M(\boldsymbol{x}_p, t)\right)\frac{\boldsymbol{v}(t)}{c^2}\right] \quad . \tag{33}$$

Conversely, the probe mass acts on Mercury by the force $\boldsymbol{F}_M(t)$, where

$$\boldsymbol{F}_M(t) = -\boldsymbol{F}_p(t) \quad . \tag{34}$$

As Mercury is moving in its orbit, its position, its velocity, its gravitational field components, and also the related forces are time-dependent. For simplicity we will not explicitly indicate this dependency in the analysis to follow. The loop way via the force of Mercury onto a probe mass has been chosen as it offers a convenient possibility to take into account the characteristics of Mercury's gravitational field.

Being interested in the force which the Sun is exerting onto Mercury our task is to solve the field equations (1-4) for two bodies, i.e. the Sun and Mercury. As a consequence of the non-linearities in the equations (1) and (4) the classical superposition principle in general is not

[3] Some annotations: The electromagnetic Lorentz force behaves in the same way. The relevant velocity is the relative velocity between the probe charge and the respective magnetic field. This statement is independent from the magnet's state of motion. If a magnet is moving in a way that the magnetic field does not change, the magnetic field does not move. As an example such a motion happens if a rod magnet rotates along its axis. According to Maxwell's equations a moving magnetic field always implies that the curl of the electrical field does not vanish. Conversely, a curl free electrical field always implies a constant magnetic field.

valid anymore. In (Kling 2016) this problem has been investigated on a very simple scenario of two spherically symmetric mass distributions. It has been shown that the error, which is introduced if the superposition principle is assumed to be approximately valid, is small as long as we restrict our interest to distances $r \gg r_g$ and as long as the condition $r_g/a \ll 1$ holds. (The parameter r_g is again the characteristic radius, a the radius of the mass distribution). As both the Sun and Mercury meet the latter condition and the distances relevant to our investigation are definitely much longer than the Sun's and Mercury's characteristic radius, we adopt this result and make use of the superposition principle as an approximation[4].

If we replace the probe mass in the equations (33, 34) by the Sun's total mass density distribution $\rho_{\odot,tot}(\boldsymbol{x})$ and approximately neglect the Sun's rotation[5], we achieve

$$\boldsymbol{F}_M = -\int \rho_{\odot,tot}(\boldsymbol{x}) \left[\left(1+\frac{v^2}{c^2}\right)\boldsymbol{E}_M(\boldsymbol{x}) - \left(\boldsymbol{v}\cdot\boldsymbol{E}_M(\boldsymbol{x})\right)\frac{\boldsymbol{v}}{c^2}\right] d^3x \qquad (35a)$$

$$= -\left(1+\frac{v^2}{c^2}\right)\int \rho_{\odot,tot}(\boldsymbol{x})\ \boldsymbol{E}_M(\boldsymbol{x})\, d^3x + \frac{\boldsymbol{v}}{c^2}\left(\boldsymbol{v}\cdot\int \rho_{\odot,tot}(\boldsymbol{x})\ \boldsymbol{E}_M(\boldsymbol{x})\, d^3x\right) \qquad (35b)$$

for the Sun's force onto Mercury. Using Mercury's gravitational field strength given in equation (19) we get

$$\int \rho_{\odot,tot}(\boldsymbol{x})\ \boldsymbol{E}_M(\boldsymbol{x})\, d^3x$$

$$= -G\, m\, \gamma \int \rho_{\odot,tot}(\boldsymbol{x})\ \frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|^3}\left[1+\frac{r_{gM}}{|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}|}-\frac{3}{2}\left|\frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|}\cdot\frac{\boldsymbol{v}}{c}\right|^2\right] d^3x \quad , \qquad (36a)$$

where higher order terms, as indicated in section 2.1, have been neglected. In our investigation to follow we will extend this procedure and generally neglect all terms of the order $(v^2/c^2)^{j_1}\left(r_{gM}/|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}|\right)^{j_2}\left(r_{g\odot}/|\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{M}}|\right)^{j_3}$ with $j_1+j_2+j_3 \geq 2$. For a detailed discussion we split the expression on the right hand side as follows

[4] This approximation is consistent with the rule described at the end of this page.

[5] The rotation of the Sun delivers every mass element of the Sun an additional velocity component which will be neglected for the following reasons. (a) For each mass element of the Sun there is another mass element having a velocity component which is equal in magnitude but pointing into the opposite direction due to the Sun's rotation. So if the gravitomagnetic induction of Mercury was constant for the whole solar sphere these velocity components would balance each other out due to equation (32). Just a rest effect remains because of the small gradient of the gravitomagnetic induction over the Sun's sphere. (b) The rotational speed of each mass element is considerably smaller than Mercury's orbital velocity which furthermore scales down the rest effect from (a).

$$\int \rho_\odot(\boldsymbol{x})\ \boldsymbol{E}_M(\boldsymbol{x})\ d^3x$$

$$= -\,Gm\gamma \int \rho_{\odot,tot}(\boldsymbol{x})\ \frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|^3}\ \ d^3x$$

$$-\,Gm\gamma \int\limits_{|\boldsymbol{x}|\,\le\, a_\odot} \rho_\odot(\boldsymbol{x})\ \frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|^3}\ \frac{r_{gM}}{|\boldsymbol{x}-\boldsymbol{x}_M|}\ d^3x$$

$$+\,\frac{3}{2}Gm\gamma \int\limits_{|\boldsymbol{x}|\,\le\, a_\odot} \rho_\odot(\boldsymbol{x})\ \frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|^3}\ \left|\frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|}\cdot\frac{\boldsymbol{v}}{c}\right|^2\ d^3x$$

$$-\,Gm\gamma \int\limits_{|\boldsymbol{x}|\,>\, a_\odot} \rho_{\odot,field}(\boldsymbol{x})\ \frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|^3}\left[\frac{r_{gM}}{|\boldsymbol{x}-\boldsymbol{x}_M|}-\frac{3}{2}\left|\frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|}\cdot\frac{\boldsymbol{v}}{c}\right|^2\right] d^3x \qquad . \qquad (36b)$$

When evaluating the first of these four integrals we make use of the spherical symmetry of the Sun's mass distribution $\rho_{\odot,tot}(r)$ which implies that (e.g. Binney & Tremaine 1994)

$$\int \rho_{\odot,tot}(\boldsymbol{x})\ \frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|^3}\ \ d^3x = -\ \frac{\boldsymbol{x}_M}{r_M^3}\ \int\limits_{|\boldsymbol{x}|\,\le\, r_M} \rho_{\odot,tot}(\boldsymbol{x})\ d^3x$$

$$= -\ \frac{\boldsymbol{x}_M}{r_M^3}\ \widehat{M}_\odot(r_M) \qquad , \qquad (37)$$

where $r_M = |\boldsymbol{x}_M|$. The mass $\widehat{M}_\odot(r_M)$ which is enclosed by a sphere of the radius r_M has already been determined in section 2.2 and can be taken from equation (29). The integration range of the second and the third integral is restricted to $|\boldsymbol{x}| \le a_\odot$ where the condition $|\boldsymbol{x}| \ll |\boldsymbol{x}_M|$ holds at good accuracy. So we have

$$\frac{r_{gM}}{|\boldsymbol{x}-\boldsymbol{x}_M|} \approx \frac{r_{gM}}{r_M} \qquad , \qquad (38)$$

and

$$\left|\frac{\boldsymbol{x}-\boldsymbol{x}_M}{|\boldsymbol{x}-\boldsymbol{x}_M|}\cdot\frac{\boldsymbol{v}}{c}\right|^2 \approx \left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2 \qquad . \qquad (39)$$

These approximations are all the more tolerable since $r_{gM}/r_M \ll r_{gM}/a_M \approx\ 5\cdot 10^{-11} \ll 1$ (relevant to the second integral) and $v^2/c^2 \approx 2.5\cdot 10^{-8} \ll 1$ (relevant to the third integral) as discussed in section 2.1. The fourth integral provides only terms which are to be neglected according to the rule described above. In this way we get

$$\int \rho_{\odot,tot}(\boldsymbol{x})\ \boldsymbol{E}_M(\boldsymbol{x})\ d^3x$$

$$= G\ \widehat{M}_\odot(r_M)\ m\,\gamma\ \frac{\boldsymbol{x}_M}{r_M^3} + G\ M_\odot\ m\,\gamma\ \frac{r_{gM}}{r_M}\frac{\boldsymbol{x}_M}{r_M^3} - \frac{3}{2}\ G\ M_\odot\ m\,\gamma\ \frac{\boldsymbol{x}_M}{r_M^3}\left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2 \qquad (40a)$$

$$= \frac{G\ M_\odot m\,\gamma}{r_M^2}\left[1+\frac{r_{g\odot}+r_{g\mathrm{M}}}{r_M} - \frac{3}{2}\left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2\right]\frac{\boldsymbol{x}_M}{r_M} \qquad . \qquad (40b)$$

Apparently, $\int \rho_{\odot,tot}(\boldsymbol{x})\ \boldsymbol{E}_M(\boldsymbol{x},t)\ d^3x$ points in the direction of $\boldsymbol{x}_M$. If we insert this expression into equation (35b), we get the total force $\boldsymbol{F}_M$ onto Mercury as

$$\boldsymbol{F}_M = -\frac{G\ M_\odot m\,\gamma}{r_M^2}\left[1+\frac{r_{g\odot}+r_{g\mathrm{M}}}{r_M} - \frac{3}{2}\left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2\right]\left[\left(1+\frac{v^2}{c^2}\right)\frac{\boldsymbol{x}_M}{r_M} - \frac{\boldsymbol{v}}{c^2}\left(\boldsymbol{v}\cdot\frac{\boldsymbol{x}_M}{r_M}\right)\right] \qquad . \qquad (41)$$

On top of the classical Newtonian force there are contributions caused by the gravitation of the Sun’s and Mercury’s gravitational field and several relativistic terms which can be attributed to Mercury’s relativistic mass, its relativistic gravitational field and the gravitomagnetic contribution to the Lorentz force onto Mercury. Being interested in bound orbits let us exclude the specific case that Mercury is centrally falling onto the Sun and assume $\boldsymbol{x}_M \times \boldsymbol{v} \neq \boldsymbol{0}$. As all components of $\boldsymbol{F}_M$ are either centrally directed or pointing into the direction of the velocity vector $\boldsymbol{v}$, Mercury is moving in a dedicated orbital plane which is defined by $\boldsymbol{x}_M$ and $\boldsymbol{v}$. Remarkably there is on top of the centrally directed radial component also another – angular – component which is caused by the gravitomagnetic part of the Lorentz force.

In the next section we will analyse the relativistic equation of motion of Mercury both in the direction along the radial unit vector $\boldsymbol{x}_M/r_M$ and the direction perpendicular to it. As a preparation of this analysis let us now split $\boldsymbol{F}_M(t)$ into the respective components. We get

$$\frac{\boldsymbol{x}_M}{r_M}\cdot\boldsymbol{F}_M(t) = -\frac{G\ M_\odot m\,\gamma}{r_M^2}\left[1+\frac{r_{g\odot}+r_{g\mathrm{M}}}{r_M} - \frac{3}{2}\left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2\right]\left[1+\frac{v^2}{c^2} - \left(\frac{\boldsymbol{v}}{c}\cdot\frac{\boldsymbol{x}_M}{r_M}\right)^2\right]. \qquad (42)$$

and

$$\frac{\boldsymbol{x}_M}{r_M}\times\boldsymbol{F}_M(t) = \frac{G\ M_\odot m\,\gamma}{r_M^2}\left[1+\frac{r_{g\odot}+r_{g\mathrm{M}}}{r_M} - \frac{3}{2}\left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2\right]\frac{1}{c^2}\left(\boldsymbol{v}\cdot\frac{\boldsymbol{x}_M}{r_M}\right)\left(\frac{\boldsymbol{x}_M}{r_M}\times\boldsymbol{v}\right) . \qquad (43)$$

Introducing Mercury’s relativistic angular momentum

$$\boldsymbol{L}_M = m\,\gamma\,(\boldsymbol{x}_M \times \boldsymbol{v}) \qquad (44)$$

these equations can be re-written as

$$\frac{\boldsymbol{x}_M}{r_M}\cdot\boldsymbol{F}_M(t) = -\frac{G\ M_\odot m\,\gamma}{r_M^2}\left(1+\frac{r_{g\odot}+r_{g\mathrm{M}}}{r_M} - \frac{3}{2}\left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2\right)\left(1+\frac{{L_M}^2}{m^2\gamma^2c^2r_M^2}\right) \qquad (45)$$

and

$$\frac{\boldsymbol{x}_M}{r_M} \times \boldsymbol{F}_M(t) = \frac{G\,M_\odot}{r_M^3}\left(1 + \frac{r_{g\odot} + r_{g\mathrm{M}}}{r_M} - \frac{3}{2}\left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2\right)\left(\boldsymbol{v}\cdot\frac{\boldsymbol{x}_M}{r_M}\right)\frac{\boldsymbol{L}_M}{c^2} \qquad , \tag{46}$$

where $L_M = |\boldsymbol{L}_M|$. If we again consider only contributions up to the first order of v^2/c^2, $r_{g\odot}/r_M$ and $r_{g\mathrm{M}}/r_M$, we end up at

$$\frac{\boldsymbol{x}_M}{r_M}\cdot \boldsymbol{F}_M(t) = -\frac{G\,M_\odot m\,\gamma}{r_M^2}\left(1 + \frac{r_{g\odot} + r_{g\mathrm{M}}}{r_M} - \frac{3}{2}\left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2 + \frac{{L_M}^2}{m^2c^2r_M^2}\right) \tag{47}$$

and

$$\frac{\boldsymbol{x}_M}{r_M} \times \boldsymbol{F}_M(t) = \frac{G\,M_\odot}{r_M^3}\left(\boldsymbol{v}\cdot\frac{\boldsymbol{x}_M}{r_M}\right)\frac{\boldsymbol{L}_M}{c^2} \qquad . \tag{48}$$

In the next section these expressions will be used in order to analyse Mercury's equation of motion.

4. Relativistic Equation of Motion

We investigate Mercury's motion within the Sun's rest frame. The origin of our coordinate system is located at the centre of the Sun. The z-axis is defined to be orthogonal to Mercury's orbital plane. We will make use of the cylinder coordinates R, φ and z in order to describe Mercury's position.

Mercury's relativistic equation of motion is given as

$$\frac{d\boldsymbol{p}_M}{dt} = \boldsymbol{F}_M \qquad , \tag{49}$$

where

$$\boldsymbol{p}_M = m\,\gamma\,\boldsymbol{v} \tag{50}$$

denotes Mercury's relativistic momentum. As already mentioned we analyse this equation of motion separately in the radial and the angular direction.

4.1 Angular Motion

Let us start with the direction perpendicular to $\boldsymbol{x}_M$**.** Here we have

$$\boldsymbol{x}_M \times \frac{d\boldsymbol{p}_M}{dt} = \boldsymbol{x}_M \times \frac{d\boldsymbol{p}_M}{dt} + \frac{d\boldsymbol{x}_M}{dt} \times \boldsymbol{p}_M = \frac{d}{dt}(\boldsymbol{x}_M \times \boldsymbol{p}_M) = \frac{d}{dt}\boldsymbol{L}_M \qquad , \tag{51}$$

where $\boldsymbol{L}_M$ describes Mercury's angular momentum defined in equation (44). Equation (49) leads us to

$$\frac{d}{dt}\boldsymbol{L}_M = \boldsymbol{x}_M \times \boldsymbol{F}_M \qquad . \qquad (52)$$

By means of equation (48) we have

$$\frac{d}{dt}\boldsymbol{L}_M = \frac{G\, M_\odot}{r_M^2}\left(\boldsymbol{v}\cdot\frac{\boldsymbol{x}_M}{r_M}\right)\frac{\boldsymbol{L}_M}{c^2} \qquad . \qquad (53)$$

In cylinder coordinates Mercury's position vector is

$$\boldsymbol{x}_M = R\ \boldsymbol{e}_R \qquad , \qquad (54)$$

$(|\boldsymbol{x}_M| = r_M = R)$, its velocity

$$\boldsymbol{v} = \dot{R}\ \boldsymbol{e}_R + R\dot{\varphi}\ \boldsymbol{e}_\varphi \qquad , \qquad (55)$$

and its angular momentum

$$\boldsymbol{L}_M = m\,\gamma\, R^2\dot{\varphi}\ \boldsymbol{e}_z \qquad , \qquad (56)$$

respectively

$$L_M = |\boldsymbol{L}_M| = m\,\gamma\, R^2\dot{\varphi} \qquad , \qquad (57)$$

where $\boldsymbol{e}_R$ denotes the radial, $\boldsymbol{e}_{\boldsymbol{\varphi}}$ the azimuthal unit vector and $\boldsymbol{e}_z$ the unit vector in the z direction. An over-dot above a letter indicates the derivative with respect to time. As we are solely interested in the shape of Mercury's orbit, we use equation (57) in order to replace the time t by the angle φ as the independent variable. Equation (57) implies

$$\frac{d}{dt} = \frac{L_M}{m\,\gamma\, R^2}\frac{d}{d\varphi} \qquad . \qquad (58)$$

Consequently, we have

$$\frac{d}{dt}\boldsymbol{L}_M = \frac{L_M}{m\,\gamma\, R^2}\frac{d}{d\varphi}\ \boldsymbol{L}_M \qquad , \qquad (59)$$

and

$$\dot{R} = \frac{d}{dt}R = \frac{L_M}{m\,\gamma\, R^2}\frac{d}{d\varphi}\ R = -\ \frac{L_M}{m\,\gamma}\ \frac{d}{d\varphi}\left(\frac{1}{R}\right) \qquad . \qquad (60)$$

Making use of these expressions in equation (53) leads us to the differential equation

$$\frac{1}{L_M}\frac{d}{d\varphi}\ L_M = -\frac{G\, M_\odot}{c^2}\ \frac{d}{d\varphi}\left(\frac{1}{R}\right) \qquad , \qquad (61)$$

the solution of which is

$$L_M = L_{M0}\, e^{-\frac{G\,M_{\odot}}{R\,c^2}} \qquad , \tag{62}$$

where L_{M0} is an integration constant. For $c \to \infty$ we achieve $L_M = L_{M0}$. Apparently, our integration constant is nothing else than Mercury's angular momentum in the Newtonian limit in which L_M is constant. The gravitomagnetic forces, however, cause a torque onto Mercury which leads to a continuous exchange of angular momentum between Mercury and the gravitational field. This behaviour is a novel phenomenon which is a consequence of the gravitomagnetic contribution to the Lorentz force. Re-writing equation (62) by means of the Sun's characteristic radius $r_{g\odot}$ as

$$L_M = L_{M0}\, e^{-2\frac{r_{g\odot}}{R}} \tag{63}$$

gives an impression of the magnitude of this effect. Estimating R by Mercury's semi-major axis - adopted from (Williams 2016a) - and taking the Sun's mass from (Williams 2016b) we achieve $r_{g\odot}/R \approx 1.3 \cdot 10^{-8} \ll 1$. Furthermore, as a consequence of the small eccentricity of Mercury's orbit, which is ~ 0.2 (Williams 2016a), the variation of R in Mercury's orbit is limited. Consequently, the variation of Mercury's angular momentum is rather small. However, later on we will see that despite of its smallness this radial variation will give a relevant contribution to the precession of Mercury's perihelion.

4.2 Radial Motion

In the radial domain equation (49) provides

$$\frac{\boldsymbol{x}_M}{r_M} \cdot \frac{d\boldsymbol{p}_M}{dt} = \frac{\boldsymbol{x}_M}{r_M} \cdot \boldsymbol{F}_M \qquad . \tag{64}$$

Let us now have a closer look at the left hand side of this equation. Making use of cylindrical coordinates yields

$$\frac{\boldsymbol{x}_M}{r_M} \cdot \frac{d\boldsymbol{p}_M}{dt}$$

$$= \frac{\boldsymbol{x}_M}{r_M} \cdot \frac{d}{dt}(m\,\gamma\,\boldsymbol{v}) \tag{65a}$$

$$= m\,\boldsymbol{e}_R \cdot \left(\dot{\gamma}\left[\dot{R}\,\boldsymbol{e}_R + R\dot{\varphi}\,\boldsymbol{e}_{\boldsymbol{\varphi}}\right] + \gamma\left[\ddot{R}\,\boldsymbol{e}_R + \dot{R}\,\dot{\boldsymbol{e}}_R + \dot{R}\dot{\varphi}\,\boldsymbol{e}_{\boldsymbol{\varphi}} + R\ddot{\varphi}\boldsymbol{e}_{\boldsymbol{\varphi}} + R\dot{\varphi}\dot{\boldsymbol{e}}_{\boldsymbol{\varphi}}\right]\right) \tag{65b}$$

$$= m\left(\dot{\gamma}\,\dot{R} + \gamma\left[\ddot{R} - R\dot{\varphi}^2\right]\right) \tag{65c}$$

$$= m\frac{d}{dt}\left(\gamma\dot{R}\right) - \frac{{L_M}^2}{m\,\gamma\,R^3} \qquad , \tag{65d}$$

where the relations $\dot{\boldsymbol{e}}_R = \dot{\varphi}\,\boldsymbol{e}_{\boldsymbol{\varphi}}$ and $\dot{\boldsymbol{e}}_{\boldsymbol{\varphi}} = -\dot{\varphi}\,\boldsymbol{e}_R$ and furthermore equation (57) have been utilized. Changing the independent variable from the time t to the angle φ provides by means of the equations (58) and (60)

$$\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{d\boldsymbol{p}_M}{dt} = m\,\frac{L_M}{m\,\gamma\,R^2}\frac{d}{d\varphi}\left[-\gamma\frac{L_M}{m\,\gamma}\,\frac{d}{d\varphi}\left(\frac{1}{R}\right)\right] - \frac{{L_M}^2}{m\,\gamma\,R^3} \qquad . \tag{66}$$

If we furthermore make use of the equations (61, 63), we get

$$\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{d\boldsymbol{p}_M}{dt} = -\;\frac{{L_{M0}}^2 e^{-4\frac{r_{g\odot}}{R}}}{m\,\gamma\,R^2}\left[\frac{d^2}{d\varphi^2}\left(\frac{1}{R}\right) + \frac{1}{R} - 2\,r_{g\odot}\left(\frac{d}{d\varphi}\left(\frac{1}{R}\right)\right)^2\right] \qquad . \tag{67}$$

Introducing as an abbreviation the radius

$$R_c = \frac{{L_{M0}}^2}{GM_\odot m^2} \qquad , \tag{68}$$

whose physical meaning will become clear later on, leads us via equation (64) to

$$\frac{d^2}{d\varphi^2}\left(\frac{R_c}{R}\right) + \frac{R_c}{R} - 2\frac{r_{g\odot}}{R_c}\left(\frac{d}{d\varphi}\left(\frac{R_c}{R}\right)\right)^2 = -\frac{\gamma\,R^2}{GM_\odot m}e^{4\frac{r_{g\odot}}{R}}\left(\frac{\boldsymbol{x}_M}{r_M}\cdot\boldsymbol{F}_M\right) \qquad . \tag{69}$$

Inserting the expression for the radial component of the force onto Mercury given in equation (47) yields

$$\frac{d^2}{d\varphi^2}\left(\frac{R_c}{R}\right) + \frac{R_c}{R} - 2\frac{r_{g\odot}}{R_c}\left(\frac{d}{d\varphi}\left(\frac{R_c}{R}\right)\right)^2$$

$$= \gamma^2 e^{4\frac{r_{g\odot}}{R}}\left(1 + \frac{r_{g\odot} + r_{g\mathrm{M}}}{r_M} - \frac{3}{2}\left|\frac{\boldsymbol{x}_M}{r_M}\cdot\frac{\boldsymbol{v}}{c}\right|^2 + \frac{{L_M}^2}{m^2c^2r_M^2}\right) \tag{70a}$$

$$= \gamma^2\left[e^{4\frac{r_{g\odot}}{R}}\left(1 + \frac{r_{g\odot} + r_{g\mathrm{M}}}{r_M} - \frac{3}{2}\left(\frac{\dot{R}}{c}\right)^2\right) + \frac{{L_{M0}}^2}{m^2c^2R^2}\right] \tag{70b}$$

As γ is velocity dependent, the factor γ^2 requires further attention. In terms of cylindrical co-ordinates we have

$$\gamma^2 = (1 - v^2/c^2)^{-1} = 1 + v^2/c^2 \qquad + O((v^2/c^2)^2) \tag{71a}$$

$$= 1 + \left(\dot{R}^2 + R^2\dot{\varphi}^2\right)/\,c^2 \qquad + O((v^2/c^2)^2) \tag{71b}$$

$$= 1 + \left(\dot{R}/c\right)^2 + \frac{{L_M}^2}{m^2c^2R^2} \qquad + O((v^2/c^2)^2) \qquad . \tag{71c}$$

Neglecting the higher order terms gives

$$\frac{d^2}{d\varphi^2}\left(\frac{R_c}{R}\right)+\frac{R_c}{R}-2\frac{r_{g\odot}}{R_c}\left(\frac{d}{d\varphi}\left(\frac{R_c}{R}\right)\right)^2$$

$$=\left[1+\left(\frac{\dot{R}}{c}\right)^2+\frac{{L_{M0}}^2}{m^2c^2R^2}e^{-4\frac{r_{g\odot}}{R}}\right]\left[e^{4\frac{r_{g\odot}}{R}}\left(1+\frac{r_{g\odot}+r_{g\mathrm{M}}}{R}-\frac{3}{2}\left(\frac{\dot{R}}{c}\right)^2\right)+\frac{{L_{M0}}^2}{m^2c^2R^2}\right] \quad (72a)$$

$$=\left[1+\left(\frac{\dot{R}}{c}\right)^2\right]e^{4\frac{r_{g\odot}}{R}}\left(1+\frac{r_{g\odot}+r_{g\mathrm{M}}}{R}-\frac{3}{2}\left(\frac{\dot{R}}{c}\right)^2\right)+\frac{2\,{L_{M0}}^2}{m^2c^2R^2}+\cdots \quad (72b)$$

$$=\left[1+\left(\frac{\dot{R}}{c}\right)^2\right]\left(1+\;4\,\frac{r_{g\odot}}{R}\right)\left(1+\frac{r_{g\odot}+r_{g\mathrm{M}}}{R}-\frac{3}{2}\left(\frac{\dot{R}}{c}\right)^2\right)+\frac{2\,{L_{M0}}^2}{m^2c^2R^2}+\cdots \quad (72c)$$

$$=1+5\,\frac{r_{g\odot}}{R}+\frac{r_{g\mathrm{M}}}{R}-\frac{1}{2}\left(\frac{\dot{R}}{c}\right)^2+\frac{2\,{L_{M0}}^2}{m^2c^2R^2}+\cdots \quad , \quad (72d)$$

where the exponential factor in equation (72b) has been expanded into a Taylor series. If we utilize equation (60) for $\dot{R}$ and express L_{M0} by $r_{g\odot}$ and R_c, we obtain

$$\frac{d^2}{d\varphi^2}\left(\frac{R_c}{R}\right)+\frac{R_c}{R}=1+\frac{5r_{g\odot}+r_{g\mathrm{M}}}{R}+4\frac{r_{g\odot}\,R_c}{R^2}+\frac{r_{g\odot}}{R_c}\left(\frac{d}{d\varphi}\left(\frac{R_c}{R}\right)\right)^2+\cdots \quad . \quad (73)$$

The dots indicate higher order terms which according to our rule defined in section 3 will be neglected. Before we solve this differential equation, let us have a brief look at the Newtonian limit. For $c\to\infty$ we have $r_{g\odot}\to 0$ and $r_{g\mathrm{M}}\to 0$ arriving at the classical differential equation associated with the Kepler problem

$$\frac{d^2}{d\varphi^2}\left(\frac{R_c}{R}\right)+\frac{R_c}{R}=1 \quad , \quad (74)$$

the general solution of which is

$$\frac{R_c}{R(\varphi)}=1+e\cos(\varphi-\varphi_0) \quad , \quad (75)$$

respectively,

$$R(\varphi)=\frac{R_c}{1+e\cos(\varphi-\varphi_0)} \quad , \quad (76)$$

where e and φ_0 are integration constants. Without loss of generality we define our coordinate system in such a way that $\varphi_0=0$. This relation is the equation of a conic section with one focus at the origin and the eccentricity e. We will not discuss details of the solutions of the

Kepler problem here, but refer to (Binney & Tremaine 1994) or to text books on classical mechanics such as (Goldstein, Poole & Safko 2002). In the present work our interest is restricted to bound orbits where the eccentricity e meets the condition $0 \leq e < 1$. The orbit's semi-major axis is defined as

$$a = \frac{R_c}{1 - e^2} \qquad . \tag{77}$$

Apparently, our radius R_c, defined in equation (68), is the radius of a circular Keplerian orbit in the special case $e = 0$.

Let us now get back to equation (73). As $r_{g\odot}/R_c \ll 1$ and $r_{g\mathrm{M}}/R_c \ll 1$, this differential equation describes the Kepler problem which is only slightly perturbed by the terms associated with $r_{g\odot}/R_c$ and $r_{g\mathrm{M}}/R_c$ on the right hand side. We solve this equation by mainly applying the technique described by Lemmon & Mondragon (2016). Substituting for

$$\frac{1}{\sigma} = \frac{R_c}{R} - 1 \tag{78}$$

equation (73) transforms into

$$\frac{d^2}{d\varphi^2}\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma}\left(1 - 13\frac{r_{g\odot}}{R_c} - \frac{r_{g\mathrm{M}}}{R_c}\right) = 9\frac{r_{g\odot}}{R_c} + \frac{r_{g\mathrm{M}}}{R_c} + 4\frac{r_{g\odot}}{R_c}\frac{1}{\sigma^2} + \frac{r_{g\odot}}{R_c}\left(\frac{d}{d\varphi}\left(\frac{1}{\sigma}\right)\right)^2 \qquad . \tag{79}$$

Actually, the variable $1/\sigma$ measures the deviation of R_c/R from a circular Keplerian orbit. By introducing the parameter

$$\sigma_c' = \left(1 - 13\frac{r_{g\odot}}{R_c} - \frac{r_{g\mathrm{M}}}{R_c}\right) / \left(9\frac{r_{g\odot}}{R_c} + \frac{r_{g\mathrm{M}}}{R_c}\right) \tag{80}$$

equation (79) can be re-written as

$$\left(1 - 13\frac{r_{g\odot}}{R_c} - \frac{r_{g\mathrm{M}}}{R_c}\right)^{-1} \frac{d^2}{d\varphi^2}\left(\frac{\sigma_c'}{\sigma}\right) + \frac{\sigma_c'}{\sigma} = 1 + \frac{1}{9 + r_{g\mathrm{M}}/r_{g\odot}}\left(\frac{4}{\sigma^2} + \left[\frac{d}{d\varphi}\left(\frac{1}{\sigma}\right)\right]^2\right) \qquad , \tag{81a}$$

$$= \left(1 + \frac{5e^2/2}{9 + r_{g\mathrm{M}}/r_{g\odot}}\right)(1 + P) \qquad , \tag{81b}$$

with the perturbation

$$P = \frac{1}{9 + r_{g\mathrm{M}}/r_{g\odot} + 5e^2/2}\left(\frac{4}{\sigma^2} + \left[\frac{d}{d\varphi}\left(\frac{1}{\sigma}\right)\right]^2 - \frac{5}{2}e^2\right) \qquad . \tag{82}$$

The rationale of this particular definition of P will become transparent later on.

Defining

$$\sigma_c = \frac{\sigma_c'}{1+\frac{5e^2/2}{9+r_{g\mathrm{M}}/r_{g\odot}}} \tag{83}$$

we get

$$\left(1-13\frac{r_{g\odot}}{R_c}-\frac{r_{g\mathrm{M}}}{R_c}\right)^{-1}\frac{d^2}{d\varphi^2}\left(\frac{\sigma_c}{\sigma}\right)+\frac{\sigma_c}{\sigma} = 1+P \qquad . \tag{84}$$

If we approximately neglect the perturbation P – the related implications will be checked later on - and furthermore change the independent variable φ by introducing the new variable

$$\psi = \varphi\sqrt{1-\left(13r_{g\odot}+r_{g\mathrm{M}}\right)/R_c} \qquad , \tag{85}$$

we arrive at

$$\frac{d^2}{d\psi^2}\left(\frac{\sigma_c}{\sigma}\right)+\frac{\sigma_c}{\sigma} = 1 \qquad , \tag{86}$$

with $\sigma = \sigma(\psi)$. Equation (86) is formally identical to equation (74). Analogue to the Keplerian solution described above the general solution of this differential equation is

$$\frac{\sigma_c}{\sigma(\psi)} = 1+e_\sigma\cos(\psi-\psi_0) \qquad , \tag{87}$$

where e_σ and ψ_0 are integration constants. Again without loss of generality we define our coordinate system in such a way that $\psi_0 = 0$. Transforming back to $R = R(\varphi)$ provides the solution we are looking for

$$R(\varphi) = \frac{\hat{R}_c}{1+\hat{e}\cos(\hat{\alpha}\varphi)} \qquad , \tag{88}$$

with

$$\hat{\alpha} = \sqrt{1-\left(13r_{g\odot}+r_{g\mathrm{M}}\right)/R_c} \approx 1-\left(13r_{g\odot}+r_{g\mathrm{M}}\right)/2R_c \qquad , \tag{89}$$

$$\hat{R}_c = \frac{1-\left(13r_{g\odot}+r_{g\mathrm{M}}\right)/R_c}{1-(4-5e^2/2)r_{g\odot}/R_c}\; R_c \approx R_c-(9+5e^2/2)r_{g\odot}-r_{g\mathrm{M}} \qquad , \tag{90}$$

and

$$\hat{e} = \frac{\left[(9+5e^2/2)r_{g\odot}+r_{g\mathrm{M}}\right]/R_c}{1-(4-5e^2/2)r_{g\odot}/R_c}\; e_\sigma \qquad . \tag{91}$$

The eccentricity e, which is an integration constant, is restored in the Newtonian limit ($c \to \infty$) if we replace the integration constant e_σ by $e/\left[(9+5e^2/2)r_{g\odot}/R_c + r_{g\mathrm{M}}/R_c\right]$. Thus we achieve

$$\hat{e} = \frac{e}{1-(4-5e^2/2)r_{g\odot}/R_c} \qquad . \tag{92}$$

Let us now check the impact of the neglect of the perturbation P. If we insert our solution via equation (78) into equation (82), we get

$$P = \frac{e^2}{6}\cos(2\hat{\alpha}\varphi) + O\left(\frac{r_{g\odot}}{R_c},\frac{r_{g\mathrm{M}}}{R_c}\right) \qquad . \tag{93}$$

Now the rationale of the definition of the perturbation P becomes transparent. Actually in the equations (81b, 82) P has been defined in such a way that all zero order terms in $r_{g\odot}/R_c$ and r_{gM}/R_c which are independent from φ have been excluded from P. Estimating R_c by Mercury's semi-major axis and taking Mercury's and the Sun's mass from (Williams 2016a,b) we obtain $r_{g\odot}/R_c \approx 1.3\cdot 10^{-8} \ll 1$ and $r_{gM}/R_c \approx 2.1\cdot 10^{-15} \ll 1$. Recalling that equation (84) represents a differential equation for the variable $1/\sigma$, which measures the deviation of R_c/R from a circular Keplerian orbit, it is appropriate to neglect non-zeroth order terms in $r_{g\odot}/R_c$ and r_{gM}/R_c. In order to get an understanding of the relevance of the residual perturbation, we insert our solution according to equation (88) on the right hand side of equation (84), i.e. we use the expression in equation (93) for our perturbation. By means of the substitution according to equation (85) we get the differential equation

$$\frac{d^2}{d\psi^2}\left(\frac{\sigma_c}{\sigma}\right) + \frac{\sigma_c}{\sigma} = 1 + \frac{e^2}{6}\cos(2\psi) \qquad , \tag{94}$$

the solution of which is

$$\frac{\sigma_c}{\sigma(\psi)} = 1 + e_\sigma\cos(\psi) - \frac{e^2}{18}\cos(2\psi) \qquad . \tag{95}$$

Transforming back to $R = R(\varphi)$ yields

$$\frac{\hat{R}_c}{R(\varphi)} = 1 + \hat{e}\cos(\hat{\alpha}\varphi) + \hat{f}\cos(2\hat{\alpha}\varphi) \qquad , \tag{96}$$

where

$$\hat{f} = -\frac{e^2}{18\,(1+\sigma_c)} \approx -\frac{e^2}{2}\left[\left(1+\frac{5}{18}e^2\right)\frac{r_{g\odot}}{R_c} + \frac{r_{g\mathrm{M}}}{9\,R_c}\right] \qquad . \tag{97}$$

So our perturbation introduces a small, periodic 'wiggle' (Rindler 2006) to Mercury's orbit. Importantly, this wiggle has twice the frequency of our solution according to equation (88) and does not alter the precession of Mercury's perihelion. This behaviour is not a specific characteristic of our approach. It also appears when analysing Mercury's orbit by means of General Relativity (e.g. Rindler 2006).

5. Perihelion Precession of Mercury

Mercury's orbit according to equation (88) differs from equation (76) by a slight change of the eccentricity and the semi-major axis, and above all by a change in the angular scaling caused by the parameter $\hat{\alpha}$ in the argument of the cosine. This change makes the elliptical orbit continuously rotate whenever the factor $\hat{\alpha}$ is different from 1. In such a case we have a precession of the orbit's periapsis. Actually, Mercury is always in a perihelion position if $\hat{\alpha}\varphi$ equals a multiple of 2π. The angular shift $\Delta\varphi$ between two subsequent perihelia compared to the Newtonian ellipse is

$$\Delta\varphi = 2\pi\left(\frac{1}{\hat{\alpha}} - 1\right) \quad . \tag{98}$$

By means of equation (89) we get

$$\Delta\varphi = 2\pi\left(\left[1 - \frac{13r_{g\odot} + r_{g\mathrm{M}}}{R_c}\right]^{-1/2} - 1\right) \tag{99}$$

$$= 13\pi\frac{r_{g\odot}}{R_c}\left(1 + \frac{1}{13}\frac{m}{M_\odot}\right) + \cdots \quad . \tag{100}$$

If we neglect higher order contributions in $r_{g\odot}/R_c$ and $r_{g\mathrm{M}}/R_c$ and recall equation (77), we arrive at

$$\Delta\varphi = \frac{13}{2}\pi\frac{GM_\odot}{ac^2(1-e^2)}\left(1 + \frac{1}{13}\frac{m}{M_\odot}\right) \quad . \tag{101}$$

Apparently, the angular shift $\Delta\varphi$ is greater than zero which means that there is a continuous advance in Mercury's perihelion. According to Williams (2016a,b) Mercury's and the Sun's masses are $m = 3{,}3011\cdot 10^{23}$ kg and $M_\odot = 1.988500\cdot 10^{30}$ kg, and we have $m/M_\odot \approx 1.7\cdot 10^{-7}$. Thus the second summand inside the brackets does not give a relevant contribution and can be approximately neglected. Finally, we achieve a shift of

$$\Delta\varphi = \frac{13}{2}\pi\frac{GM_\odot}{ac^2(1-e^2)} \tag{102}$$

per revolution. This result differs from the prediction of General Relativity[6] (e.g. Weinberg 1972, Rindler 2006)

$$\Delta\varphi_{GR} = 6\pi \frac{GM_\odot}{ac^2(1-e^2)} \tag{103}$$

by an excess perihelion advance of

$$\Delta\varphi - \Delta\varphi_{GR} = \frac{1}{2}\pi \frac{GM_\odot}{ac^2(1-e^2)} = \Delta\varphi_{GR}/12 \quad . \tag{104}$$

Usually the perihelion shift is given in arc seconds measured over a period of a century. Taking $e = 0.20563069$, $a = 0.38709893\,AU$ and a tropical orbit period of 87,968 days from the NASA Mercury Fact Sheet (Williams 2016a), $GM_\odot = 1.32712440018(8) \cdot 10^{20}\,m^3\,s^{-2}$ from (Park 2017), and considering a Julian century to be 36525 days we achieve $\Delta\varphi = 46.56''/cy$, $\Delta\varphi_{GR} = 42.98''/cy$ and $\Delta\varphi - \Delta\varphi_{GR} = 3.58''/cy$.

Which mechanisms are causing this perihelion shift? From our analysis of Mercury's orbit in the previous section the contributors to the precession can be identified. They are listed in table 1 and are as follows:

- One sixth of the General Relativity shift $\Delta\varphi_{GR}$ can be assigned to Mercury's relativistic inertial mass.
- Another one sixth of $\Delta\varphi_{GR}$ is caused by Mercury's relativistic gravitational mass. As already pointed out in section 2.1 the gravitational field of a moving mass is stimulated by its relativistic and not by its rest mass.
- Two third of $\Delta\varphi_{GR}$ can be attributed to the gravitomagnetic component of the Lorentz force.
- One twelfth of the General Relativity shift is a consequence of the phenomenon that the energy density of the gravitational field is equivalent to a mass density distribution and consequently acts as a source of gravitation by itself.

Origin	Contribution/$\Delta\varphi_{GR}$
relativistic inertial mass	1/6
relativistic gravitational mass	1/6
gravitomagnetic component of the Lorentz force	2/3
gravitation of the mass equivalent of the gravitational field energy	1/12
total	13/12

Table 1: Overview of the contributors to Mercury's perihelion precession.

[6] Also in the analysis by means of General Relativity equation (103) is an approximate solution.

Now the root cause of the often reported incorrect statement that a vector theory – our Covariant Gravitational Field Theory is a vector theory – would necessarily produce a perihelion shift which equals one sixth of the General Relativity result becomes visible. This conclusion usually is achieved (e.g. Giulini 2008) by disregarding the relativistic gravitational mass, all gravitomagnetic effects as well as the mass equivalence of the gravitational field energy.

We would end up exactly at the General Relativity result if we ignored the equivalence of the energy density of the Sun's gravitational field to a mass density distribution which acts as a source of gravitation by itself. As General Relativity does not take this feature of the gravitational field into account – actually in General Relativity the gravitational field energy cannot be localised by principle (Kling 2017a) – we should not be surprised to come across a difference. Importantly, the difference in the predictions of General Relativity and the Covariant Gravitational Field Theory should be big enough to be verified or excluded by observation. In this way an observational clarification of the question should be possible if the gravitational field energy acts as a source of gravitation.

6. Observations

The purpose of this section is to check how the prediction of our Covariant Gravitational Field Theory on Mercury's perihelion precession complies with the available observations. Traditionally the motion of celestial bodies has been traced by Earth-bound observatories. This implies that the observed precession of Mercury's perihelion is superimposed by the motion, in particular the precession, of the Earth. Consequently, the precession of Mercury's perihelion with respect to an absolute coordinate system is not a direct observable but has to be reconstructed from the observed precession and the precession of the Earth. The latter one can be separated into the so-called 'precession of the equator' and the 'precession of the ecliptic' the combination of which is called 'general precession'[7]. Both components are not constant but subject to a continuous change over time. So when comparing these data a normalization with respect to a certain epoch is needed. Once the absolute precession of Mercury's perihelion has been derived from Earth-bound measurements and measurements of the Earth's general precession another problem pops up. As Mercury is not the only object orbiting around the Sun, the other planets – by principle all other objects - are affecting its orbit and stimulating Mercury's perihelion to precede. So in order to conclude on the intrinsic precession derived in the previous section the influence of the other planets needs to be determined and subtracted from its total (absolute) precession. Nowadays these Earth-bound observations are comple-

[7] Historically the Earth's precession was called the 'precession of the equinoxes'.

mented by radiometric range measurements gained by means of spacecrafts such as the MESSENGER (MErcury Surface, Space ENvironment, GEochemistry, and Ranging) spacecraft.

Almost since the detection of the discrepancy between the observed precession of Mercury's perihelion and the prediction of the Newtonian gravitational theory the oblateness of the Sun has been identified as a possible root of this anomaly (Newcomb 1895 - 1898). However, Clemence (1949) estimated the contribution of the solar gravitational quadrupole moment to Mercury's precession at 0.01(2)"/cy which is too small in order to be relevant to our investigation. As this conclusion has been confirmed by later studies (see e.g. Pireaux, Rozelot & Godier 2003), we will neglect this contribution to Mercury's perihelion precession in our analysis.

Reference	Epoch	Observed (Earth-bound) Precession of Mercury's Perihelion arc sec/cy	General Precession arc sec/cy	Mercury's Total Precession (Observed - General Precession) arc sec/cy	Newtonian Component induced by other Planets arc sec/cy	Delta (Mercury's Total Precession - Newtonian Component induced by other Planets) arc sec/cy
Le Verrier (1859)	J1850	5591,38	5025,48	565,9	526,7	39,2
Newcomb (1898)	J1900	5599,76	5024,93	574,83	531,46	43,37
Doolittle (1912)	J1850			575,1(19)	529,7	45,4
Clemence (1949)	J1850	5599,74(41)	5025,65(30)	574,10(51)	531,52(28)	42,56(50) a)
Bretagnon (1982)	J2000	5603,33 b)	5029,10 c)	574,23 d)		
Narlikar & Rana (1985)		5603,33(211) b)	5029,10(222) c)	574,24	528,95(5)	45,29 e)
Rana (1987)		5603,30(2)		571,91(2) f)	528,93(2)	42,9807(6) g)
Simon et al (1994)		5603,04	5028,82 h)	574,22 d)		
Park (2017)			5028,83(4)			
Roy (2014)/Steward (2005)				575(3)	528,9 i)	46,1(30)
Park et al. (2017)				575,3100(15) j)		

a) Clemence (1949) considered a contribution of 0,01(2)"/cy caused by the solar gravitational quadrupole moment, b) calculated for J2000 from Bretagnon's (1982) formula, c) Bretagnon (1982), adopted from Lieske et al. (1977), d) calculated from the observed precession of Mercury and the general precession, e) when assuming the prediction of General Relativity (42,98"/cy) Narlikar & Rana concluded on a deviation of 2,31"/cy, f) calculated by Rana (1987) from the Newtonian and the General Relativity contribution, g) General Relativity precession derived from the calculation by Lestrade & Bretagnon (1982), Bretagnon (1982), h) taken from Williams et al. (1991), i) Steward's (2005) value modified by Roy (2014), j) extracted from the post-fit numerically integrated ephemeris (PPN formalism based model) derived from an analysis of radiometric range measurements to the MESSENGER spacecraft.

Table 2: Observational data and calculations of the Newtonian component.

Table 2 shows a list of Earth-bound observational data of Mercury's precession, general precession data, the resulting total precession, calculations of the influence of the other planets as well as the difference of both. It shows the development of the relevant parameters since the detection of the anomaly of Mercury's precession by Le Verrier. In our discussion we will, however, concentrate on recent data assigned to the epoch J2000 in table 2.

The effect of the other planets onto Mercury's precession, commonly called Newtonian component, is calculated by means of Newton's law of gravitation. Modern studies usually take the laws of General Relativity for granted and estimate this influence by means of the parameterized Post-Newtonian (PPN) formalism (Will & Nordtvedt 1972). As our approach is based on the Covariant Gravitational Field Theory and not on General Relativity, and it is essential not to mix different gravitational theories, these results cannot directly be used for our needs and consequently are not included in table 2. Strictly taken we would have to solve the field equations (1-4) for our solar system as a whole in order to identify the influence of the other planets. As an approximation the tugs of the other planets at Mercury could be estimated by means of the planetary forces analogue to equation (30). As both options are beyond the scope of the present work, we will restrict ourselves to the consideration of the available calculations of the Newtonian component listed in table 2. This way of proceeding is justified as the velocities of the other planets relative to Mercury are much smaller than the vacuum speed of light, and consequently the gravitomagnetic contribution to the respective gravitational Lorentz force will be small compared to the gravitoelectric component. The results of the latest calculations of the Newtonian component by Narlikar & Rana (1985), Rana (1987) and Roy (2014) /Steward (2005) are quite close to one another and differ only in the second digit after the comma. So we take the average of these results (528.93"/cy) as a base for our check.

With the exception of the early value by Le Verrier all values for Mercury's total precession listed in table 2 are within the window between 574.1"/cy and 575.31"/cy. Assessing these data we refer to a statement by Rana (1987). Rana points out that 'the geometry of the orbits of Mercury and Earth suggests that the low precision of the geocentric angular data having an error of 1" are incapable of giving the rate of motion of the perihelion of Mercury to better than $3'' cy^{-1}$.' Furthermore he argues 'that accurate ranging (rather than angular positions) of Mercury to within 1km or so, over a carefully selected portion of the orbit and accumulated over a period of at least one decade, will enable us to determine the apse rate to an accuracy of about $0''.1 cy^{-1}$.' Table 2 does not only show data which have been gained from Earth-bound observations but also a highly precise value reported by Park et al. (2017) who use an analysis of radiometric range measurements to the MESSENGER spacecraft. Park et al. made use of the PPN formalism, fitted a PPN parameter as well as the solar quadrupole moment to the MESSENGER ranging data and extracted Mercury's precession rate from the post-fit numerically integrated ephemeris. As this fit is based on the PPN formalism, the fitted ephemeris by principle is not only a response to the ranging data but also to the constraints of the PPN formalism. However, Park et al. point out that this method 'is simply a process to reduce the numerical error of the extracted precession rate'. Furthermore they state that 'The combination of parameters, regardless of constraints used, must give the same precession rate in order to fit the MESSENGER range data.' So we conclude that the precession rate thus extracted (575.31"/cy) gives the most reliable value also for our needs.

Taking all of these considerations into account we end up at an estimation of the observed anomalous precession of $46.38"/cy$ which is fairly close to the prediction of the Covariant Gravitational Field Theory ($46.56"/cy$). How accurate is this value? Considering the uncertainty of the general precession as well as the overall scatter of all of the calculations of the Newtonian component listed in table 2 we estimate the uncertainty of the difference of both with all due caution to be at least <$\pm\ 3"/cy$. Although this uncertainty has been estimated with extreme care, we have to recognize that the value predicted by General Relativity is not included in the respective window.

7. Summary

Based on the Lorentz Covariant Gravitational Field Theory (Kling 2017a) Mercury's orbit has been analysed, and the associated intrinsic precession of Mercury's perihelion has been derived. The analysis has been restricted to the consideration of contributions up to the first order of the Sun's and Mercury's characteristic radius and up to the first order of v^2/c^2. Starting from the related field equations the Sun's and Mercury's gravitational field components as well as the Sun's force onto Mercury have been determined. It has been demonstrated that Mercury is not only exposed to a centrally directed force toward the Sun but also to a torque which causes a small but continuous exchange of angular momentum with the gravitational field. Mercury's radial and angular equations of motion have been established and solved. Mercury's intrinsic perihelion shift, excluding the influence of the other planets, has been determined to equal thirteen twelfths of the prediction of General Relativity. The sources of this perihelion shift have been identified. It has been shown that the Lorentz Covariant Gravitational Field Theory would predict the same perihelion shift as General Relativity if the equivalence of the gravitational field energy to a mass and its property to act as a source of gravitation was ignored. The consistent application of the equivalence of energy and mass, however, makes a difference of one twelfth of the General Relativity value ($3.58"cy^{-1}$). We conclude that a precise observation of Mercury's perihelion shift has the potential to clarify the fundamental question if in terms of the equivalence of energy and mass the gravitational interaction behaves similar to all other fundamental forces or if its nature is really radically different. The current observational data as well as the available calculations of the Newtonian influence of the other planets in our solar system onto Mercury's perihelion precession indicate a perihelion shift which is very close to the prediction of the Covariant Gravitational Field Theory and somewhat away from the value of General Relativity. We suggest to check this result by a detailed analysis of the available observational data, in particular radiometric range measurements to spacecrafts within the frame work of the Covariant Gravitational Field Theory (analogue to the study by Park et al. (2017) which has been done on the base of the PPN formalism).

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