Interaction of superpositions of coherent states of light with two-level atoms

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(Received 9 January 1992 and accepted 5 February 1992)

Abstract. We investigate some of the basic features of the interaction of superpositions of coherent states of light with two-level atoms in the framework of the Jaynes-Cummings model. We compare the behaviour of the system in the case of having a coherent superposition state and a statistical mixture of coherent states as an initial field. We investigate the collapses and revivals of the atomic inversion by studying the evolution of the $Q$ function of the cavity field. We also establish the connection between the purity of the field and the collapses and revivals of the atomic inversion.

1. Introduction

There has been considerable interest in the properties of the so-called superposition states of light (SS) [1–6]. One particularly interesting case is the superposition of two (or more) coherent states [2–6]. Due to the quantum interference, the properties of such a superposition are very different from the properties of the constituent states (coherent states), as well as from the incoherent superposition or statistical mixture (SM) of coherent states. For instance, the superposition exhibits squeezing [5], higher-order squeezing [7], sub-Poissonian photon statistics [2] and oscillations in the photon-number distribution [5, 6], and these properties clearly differentiate the superposition state and the statistical mixture of two coherent states. Because by using coherent states one could produce superpositions of macroscopically distinguishable states (or Schrödinger cat-like states) [8], the problem is of importance for the quantum theory of measurement. There have already been proposed several schemes in order to produce superpositions of coherent states; the nonlinear interaction of the field in a coherent state with a Kerr-like medium can produce an SS [9]. Another possible way would be through the non-unitary evolution of the field either in the interaction of a coherent state with two-level atoms [10, 11] or in quantum non-demolition measurements [12]. The last method would allow us to produce the states known as the even coherent state:

$$|\alpha\rangle_e = A_{e}^{1/2}(|\alpha\rangle + |\bar{\alpha}\rangle), \quad A_{e}^{-1} = 2[1 + \exp(-2|\alpha|^2)],$$

or as the odd coherent state:

$$|\alpha\rangle_o = A_{o}^{1/2}(|\alpha\rangle - |\bar{\alpha}\rangle), \quad A_{o}^{-1} = 2[1 - \exp(-2|\alpha|^2)].$$

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The non-classical properties of such superpositions as well as their sensitivity to
dissipation have already been discussed in a previous work [5], where we concluded
that squeezing has a much slower decay than the oscillations in the photon number
distribution, for instance. In this paper we are going to be concerned with the
interaction of superpositions of coherent states with matter. This is, in our opinion,
another step forward to a better understanding of the consequences of the
superposition principle at a 'macroscopic' level, because it would allow the
utilization of atoms for the field monitoring [13]. We are going to employ one of the
simplest possible models, the so called Jaynes–Cummings model (JCM), which
consists of a single mode of the quantized electromagnetic field interacting with a
two-level atom in a lossless cavity [14]. We are aware of the importance of dissipation
in this problem, but we prefer to analyse the ideal case first in order to identify the
relevant properties. In spite of its simplicity, the JCM exhibits interesting features,
namely, the atomic inversion (the probability of being in the excited state minus the
probability of being in the ground state) is very sensitive to the statistics of the initial
field. If the field is initially prepared in a number state, the inversion will show the
usual Rabi oscillations (also shown when the input field is not quantized). However,
if the input field is prepared in a coherent state, the oscillations are modulated in such
a way they exhibit collapses and revivals [15], which have been already experiment-
ally observed [16]. The sensitivity of the model to the photon statistics will be our
starting point, because the photon distribution for the SS (which shows strong
oscillations) in (1) and (2) is very different from the Poissonian photon distribution
for the corresponding SM of two coherent states [5].

We are going to assume that the initial single mode electromagnetic field inside
the cavity is in a superposition state of the kind:

$$\rho = A[|\alpha\rangle\langle \alpha| + r^2 - \alpha\langle \alpha| - |\alpha\rangle\langle -\alpha| + 2r \exp(-2\alpha^2)]^{-1},$$

where $A = [(1 + r^2 + 2r \exp(-2\alpha^2)]^{-1}$ with $\alpha$ real. The parameter $r$ can assume the
values $-1, 0, 1$, which corresponds to an odd coherent state (2), a coherent state and
an even coherent state (1) respectively. A convenient experimental scheme would be
the one proposed in [12], where atoms very detuned from the cavity resonance
frequency would produce the SS, followed by immediate injection of atoms in
resonance with the cavity field (the case we are going to treat here). As we know,
because the interference terms in (3) have a rapid decay to a SM when we include
dissipation, so we want to see how different would be the behaviour of the system if
the input state is a statistical mixture of the states $|\alpha\rangle$ and $|-\alpha\rangle$, i.e.,

$$\rho = \frac{1}{2}|\alpha\rangle\langle \alpha| + \frac{1}{2}|\alpha\rangle\langle -\alpha|,$$

The JCM can be solved exactly (non-perturbatively) in the rotating wave
approximation, and this allows us to obtain exact expressions for the expectation
values of physical quantities. The first step is to calculate the time-dependent density
operator for the system, $\rho(t)$, and use it to evaluate the atomic inversion in both cases.
We can then use the phase-space approach presented by Eiselt and Risken [17], in
order to explain the pattern of collapses and revivals of the Rabi oscillations. It is
possible to obtain valuable information directly from the time-dependent quasi-
probability distributions [18]. Our analysis will be based on the evolution of the $Q$
function for the cavity field. The $Q$ function is a c-number representation of the
density operator associated with anti-normally ordered operators, and can be written in a convenient form as:

\[ Q(t) = \frac{1}{\pi} \langle \beta | \rho(t) | \beta \rangle. \]  

(5)

The evolution of the \( Q \) function will be clearly connected to the evolution of the collapses and revivals [17], but one has to be careful in associating the revivals with the collisions of component parts of the distributions. This is because the evolution of the \( Q \) function if we start with a SS is very similar to the evolution of the \( Q \) function for the SM as an initial state. However, the additional interference structure present in the case of SS will be responsible for the distinction.

In order to complement our investigations, we are also going to discuss further aspects of the field evolution. We know that even if the field is initially prepared in a pure state, its subsequent evolution will be very complicated, and during most of the time the field will be in a statistical mixture [19]. The most appropriate indicator of the degree of purity of the field is given by the quantum mechanical entropy [19], defined as [20]:

\[ S = -\text{Tr} [\rho(t) \ln \rho(t)]. \]  

(6)

It is easy to see from the following property of the density operator describing a pure state, \( \rho^n = \rho \) (\( n \) integer), which leads to \( \text{Tr} \rho^2 = 1 \) that the quantum mechanical entropy should be zero for a pure state. Otherwise, if the state is a SM, \( \text{Tr} \rho^2 < 1 \), and the entropy will be no longer zero. In our problem, however, it will be enough to compute the quantity \( \zeta = 1 - \text{Tr}_r [\rho^2_r] \), which can serve as a measure of the purity of the field [10] and has a behaviour similar to the entropy. We want to stress, however, that the entropy gives us the most general measure of the purity of a quantum state simply because it contains all the powers of \( \rho \).

Another aspect we would like to discuss briefly is the photon number distribution of the field, defined as:

\[ P_n(t) = \langle n | \rho(t) | n \rangle. \]  

(7)

The complex field evolution is, of course, reflected in the photon distribution, and only at particular times will the photon number distribution resemble that of a pure state. We found that the distribution exhibits a pattern of oscillations (at certain times) very similar to those present when the input field is a coherent state [11].

The paper is organized as follows: in Section 2 we briefly discuss the model and the atomic inversion. In Section 3 we discuss some aspects of the field dynamics, where we show the evolution of the \( Q \) function of the field and provide an explanation for the pattern of collapses and revivals found in Section 2. We conclude the paper by discussing the purity of the state of the field as well as its photon number distribution.

2. Atomic dynamics

2.1. The Jaynes–Cummings model

The Jaynes–Cummings [14] model is one of the few models that can be exactly solved in physics. It is also one of the most elementary ways of studying the interaction of electromagnetic radiation and matter. It consists of a two-level atom with a ground state \( |g\rangle \) and an excited state \( |e\rangle \) placed inside a lossless cavity and interacting with a quantized single mode of the electromagnetic field. For the sake of simplicity we are going to consider the cavity field with a frequency \( \omega \) resonant
with the atomic transition frequency \[ [(E_e - E_g)/\hbar = \omega_0] \], i.e., \( \omega = \omega_0 \). The total Hamiltonian in the rotating wave approximation may be written as:

\[
H = \frac{1}{2} \hbar \omega_0 \sigma_3 + \hbar \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}) + \hbar \lambda (\hat{a} \sigma_+ + \sigma_- \hat{a}^\dagger).
\]  

(8)

The atomic operators are simply \( \sigma_3 = |e\rangle \langle e| - |g\rangle \langle g| \); \( \sigma_+ = |e\rangle \langle g| \); \( \sigma_- = |g\rangle \langle e| \), and the field operators obey the usual commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\). The coupling constant \( \lambda \) depends on the cavity volume, dipole strength and mode frequency.

The model can be solved in several different ways. The most convenient method for our purposes is the density operator method. It is a quite general approach in the sense that we can include statistical mixtures in our analysis. The density operator for the atom–field system will follow a unitary evolution and may be written as:

\[
\rho(t) = U(t) \rho(0) U^\dagger(t),
\]

(9)

where \( U(t) \) is the evolution operator, which, in the interaction picture is simply given by \([19, 21]\):

\[
U^\dagger(t) = \exp \left[ i \lambda (\hat{a}^\dagger \sigma_- + \sigma_+ \hat{a}) t \right].
\]

(10)

It is now convenient to write the evolution operator in the atomic basis. After expanding it in a Taylor series, we obtain:

\[
U^\dagger(t) = \left( \begin{array}{c} C_{n+1} \cr i S_{n+1} \hat{a} \end{array} \right),
\]

(11)

where

\[
C_{n+1} = \cos \left[ \lambda t (\hat{a} \hat{a}^\dagger)^{1/2} \right], \quad C_n = \cos \left[ \lambda t (\hat{a}^\dagger \hat{a})^{1/2} \right],
\]

\[
S_{n+1} = \sin \left[ \lambda t (\hat{a} \hat{a}^\dagger)^{1/2} \right], \quad S_n = \sin \left[ \lambda t (\hat{a}^\dagger \hat{a})^{1/2} \right].
\]

(12)

Now we have to specify the initial state of the system. Let us assume the atom can be prepared in a superposition state in such a way that the initial (system) density operator can be written as:

\[
\rho(0) = \frac{1}{(1 + s^2)^{1/2}} \left( \begin{array}{cc} \rho_0(0) & s \rho_0(0) \\ s^* \rho_0(0) & s^2 \rho_0(0) \end{array} \right),
\]

(13)

which means that the atomic state at \( t = 0 \) will be:

\[
|\psi(0)\rangle_a = \frac{1}{(1 + s^2)^{1/2}} (|e\rangle + s |g\rangle).
\]

(14)

We will consider some particular values for the parameter \( s \), i.e., \( s = \pm 1 \) (atom in a superposition of ground and excited states) and \( s = 0 \) (atom initially in the excited state). In writing (13), we assumed the atom and field are initially uncorrelated.

Then we can proceed to calculate the time dependent density operator for the system in the atomic basis, inserting (11) and (12) into (9). The result is:

\[
\rho(t) = \frac{1}{(1 + s^2)^{1/2}} \left( \begin{array}{cc} \hat{A} \rho_0(0) \hat{A}^\dagger & \hat{A} \rho_0(0) \hat{B}^\dagger \\ \hat{B} \rho_0(0) \hat{A}^\dagger & \hat{B} \rho_0(0) \hat{B}^\dagger \end{array} \right),
\]

(15)

where

\[
\hat{A} = C_{n+1} - i S_{n+1} \hat{a},
\]

\[
\hat{B} = s C_n - i \hat{a}^\dagger S_{n+1}.
\]

(16)

The operators \( \hat{C} \) and \( \hat{S} \) are defined in equations (12).
The density operator in (15) will enable us to calculate all the expectation values and quantities we need in our discussion.

2.2. Evolution of the atomic inversion

Now we are going to investigate a particular aspect of the atomic dynamics, namely, how the field excites and de-excites the atom. A convenient way of doing that is through the atomic inversion, defined as the probability of the atom being in the excited state minus the probability of being in the ground state, i.e.:

$$W(t) = \text{Tr} [\hat{\rho}(t) (|e\rangle\langle e| - |g\rangle\langle g|)].$$

If we insert the density operator (15) in (17) and perform the tracing operation, we obtain:

$$W(t) = \sum_{n=0}^{\infty} \left[ \langle n | \hat{A} \hat{A}^\dagger | n \rangle - \langle n | \hat{B} \hat{B}^\dagger | n \rangle \right],$$

where the operators $\hat{A}$ and $\hat{B}$ are given in equations (16).

If the field is initially prepared in a SM of states $|\alpha\rangle$ and $|\alpha^*\rangle$, with density operator given by equation (4), the atomic inversion will be:

$$W_\alpha(t) = \sum_{n=0}^{\infty} P_n^\alpha \left\{ \cos \left[ 2\lambda t(n+1)^{1/2} \right] - s^2 \cos \left[ 2\lambda t(n)^{1/2} \right] \right\},$$

where $P_n^\alpha = \exp(-\alpha^2) \alpha^{2n} / n!$ is the Poisson distribution, i.e., the photon-number distribution for the initial field (SM).

As we expected, the atomic inversion is the same as if the input field was a coherent state. However, if the field is initially prepared in a superposition of coherent states [equation (3)], the atomic inversion will have the same form as in the former case,

$$W_s(t) = \sum_{n=0}^{\infty} P_n^s \left\{ \cos \left[ 2\lambda t(n+1)^{1/2} \right] - s^2 \cos \left[ 2\lambda t(n)^{1/2} \right] \right\},$$

but with

$$P_n^s = \exp \left( -\alpha^2 \right) \frac{\alpha^{2n} \left[ 1 + r(-1)^n \right]^2}{n! \left[ 1 + r^2 + 2r \exp \left( -2\alpha^2 \right) \right]^{n+1}},$$

which is the photon number distribution for the initial field (SS).

We are going to start our analysis taking $s=0$, i.e., with the atom initially prepared in the upper state. Since the resulting series cannot be analytically performed [14] we will evaluate them numerically.

In figure 1 (a) we plot the atomic inversion in function of $\lambda t$ for the field initially in a SM (equation (19)), and in figure 1 (b) the atomic inversion for the field initially in an SS (even coherent state), given by equation (20) with $r=1$. The most obvious difference, is that for the SS the revival time will be approximately half of the revival time for the SM. This is an effect due to the interference between the two coherent states in the superposition, and can be understood looking at the photon-number distribution of the initial fields. We know that the collapses are caused by the dephasing of the various terms in the sums (19) or (20). Thus we can calculate the time in which the revivals will occur by estimating the time that neighbour terms in the sums will be in phase again (for $n \approx \bar{n}$) [15]:

$$T_{\bar{n}}^s \left[ 2\lambda (\bar{n}+1)^{1/2} - 2\lambda (\bar{n})^{1/2} \right] \approx 2\pi.$$
Figure 1. Atomic inversion for: (a) the field initially prepared in a statistical mixture of coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ and (b) the field initially prepared in a superposition state (even coherent state), and the atom in the excited state in function of $\lambda t$. The initial mean photon number is $\bar{n} = 25$.

This argument is true in the case of SM, but it cannot be applied for the SS, because as we see from equation (21), $P_n^S$ is zero for even (odd) $n$ in case of having an odd (even) coherent state. The difference in phase of two (non-zero) neighbour terms in (20) will be, then:

$$T_R^S(2\lambda(\bar{n} + 2)^{1/2} - 2\lambda(\bar{n})^{1/2}) \approx 2\pi,$$

(23)

$$T_R^S \approx \frac{T_R^M}{2}.$$  

(24)

This is valid for both the even and odd coherent states.

If we take now $s = \pm 1$, i.e., the atom initially prepared in a coherent superposition of the ground $|g\rangle$ and excited $|e\rangle$ states, we can see how the field in a quantum superposition strongly re-phases the atomic inversion in such a way that the revivals of the Rabi oscillations are produced. In the plot in figure 2 (a), we see that if the atom starts in a superposition state and the field in a SM, the interaction has little effect on the atomic inversion [22], and apparently the atom will remain in the superposition state (Zaheer and Zubairy [22] called these states 'trapping states'). However, if the field is also in a SS ($r = \pm 1$), the quantum interference term between the two coherent states will be responsible for the revivals shown in figure 2 (b). Those are exactly the same as in the case when the atom was considered to be prepared in the excited state (with the field in an SS, of course).
As we expected, the interference produces remarkable effects on the atomic dynamics. In all the cases the initial intensity of the field was chosen $\alpha^2 = \bar{n} = 25$. This allows us to have well defined collapses and revivals. In what follows we are going to see how these results can be connected to some aspects of the field dynamics.

3. Field dynamics

3.1. Evolution of the $Q$ function

In the previous section we discussed a particular aspect of the atomic dynamics (collapses and revivals) in the Jaynes–Cummings model with the initial field prepared either in a SM of two coherent states or in a superposition of the same states. Now we are going to try to understand better the behaviour of the system by focusing our attention on the field dynamics. The first step to be taken is the calculation of the reduced density operator of the field $\hat{\rho}_f(t)$. It can be easily obtained from the system (total) density operator [equation (15)] by tracing over the atomic variables, i.e., $\hat{\rho}_f(t) = \text{Tr}_A [\hat{\rho}(t)]$. Performing this operation on (15), we get:

$$\hat{\rho}_f(t) = \hat{A} \hat{\rho}_f(0) \hat{A}^\dagger + \hat{B} \hat{\rho}_f(0) \hat{B}^\dagger,$$

where the operators $\hat{A}$ and $\hat{B}$ are defined in (16). Now we are going to discuss the field behaviour from the point of view of the quasi-probability distributions as presented by Eiselt and Risken [17]. Their discussion is mainly based on the $Q$ function,
defined [18] as the Fourier transform of the 'quantum' characteristic function \(C(\xi, t)\) associated with anti-normally ordered operators:

\[
Q(\beta, t) = \frac{1}{\pi} \int d^2\xi \exp (\beta \xi^* - \xi \beta^*) C(\xi, t),
\]

where the characteristic function is:

\[
C(\xi, t) = \text{Tr} [\hat{\rho}_t(t) \exp (-\xi^* \hat{a}) \exp (\hat{a} \xi^*)].
\]

The \(Q\) function is not only a convenient tool to calculate expectation values of anti-normally ordered products of operators, but also gives us a new insight into the mechanism of interaction in the Jaynes–Cummings model. Once we have the density operator for the field (25) we can calculate its \(Q\) function using expression (5) in a straightforward way. The results are:

\[
Q^M = \frac{1}{2\pi} (|S_1|^2 + |S_2|^2 + |S_3|^2 + |S_4|^2),
\]

for the initial field prepared in a SM, and:

\[
Q^S = \frac{A}{\pi} \left[ |S_1|^2 + |S_2|^2 + r^2(|S_2|^2 + |S_4|^2) + r(S_1 S_1^* + S_2 S_2^* + S_3 S_4^* + S_4 S_3^*) \right]
\]

for the field in an SS. The terms \(S_i\) are given by:

\[
S_1 = \langle \beta | \hat{A} | \alpha \rangle = \sum_{n=0}^{\infty} \frac{\exp \left[ \frac{-1}{2} (|\beta|^2 - \alpha^2) \right]}{n!} (C_{n+1} - i\alpha S_{n+1}),
\]

\[
S_2 = \langle \beta | \hat{\rho} | \alpha \rangle = \sum_{n=0}^{\infty} \frac{\exp \left[ \frac{-1}{2} (|\beta|^2 - \alpha^2) \right]}{n!} (C_{n+1} + i\alpha S_{n+1}),
\]

\[
S_3 = \langle \beta | \hat{A} | \alpha \rangle = \sum_{n=0}^{\infty} \frac{\exp \left[ \frac{-1}{2} (|\beta|^2 - \alpha^2) \right]}{n!} (sC_n - i\beta S_{n+1}),
\]

\[
S_4 = \langle \beta | \hat{\rho} | \alpha \rangle = \sum_{n=0}^{\infty} \frac{\exp \left[ \frac{-1}{2} (|\beta|^2 - \alpha^2) \right]}{n!} (sC_n + i\beta S_{n+1}).
\]

The sums \(S_i\) were evaluated numerically. The result can be appreciated in figure (3). We have taken \(s = 0\), i.e., the atom initially in the excited state. We have also chosen the intensity of the initial field to be \(a_0^2 = 25\), which means that the coherent state components will be distinctly separated. In figure 3 (a) we have plotted the contours in phase space for the \(Q\) function of the field inside the cavity at different times starting with a SS. At \(t = 0\) we clearly see the two peaks (corresponding to each coherent state). After some time \(t_1 = T_R/2\), we see that each peak has split into two, which counter-rotate on the circle \(|\alpha| \approx a_0\). As pointed out by Eiselt and Risken [17], the splitting of the \(Q\) function is connected to the collapse of the Rabi oscillations discussed in the last Section. Now, at \(t_2 = T_R\), when there is a revival [see figure 1 (a)], we also see a recombination of the branches (two by two) at \(\theta = \pi/2\), as we can see in figure 3 (a). Then the \(Q\) function splits again, in such a way that the next recombination occurs at approximately \(t_3 = 2T_R\). This is shown in figure 3 (b), and, again, at this time a second revival is going to occur. Thus, because in the case of the field initially in a SS we have a double peaked \(Q\) function, which will split and collide, then the 'collisions' will be connected to the revivals [17]. However, we have to be careful in our analysis, because if we prepare the field in the SM (equation (4)), its \(Q\)
function will show a behaviour very similar to that for the SS just discussed. In fact, we can take figure 3 as being the same for the SM, since, by using the $Q$ function, the interference structure cannot be seen clearly (for a large number of photons). Thus, in order to explain the patterns of revivals in figure 1, we have to examine in more detail the field dynamics. In figures 4–7 we compare the $Q$ functions when we have an initial field given by a SM and when we have a SS. We see the same overall pattern for both cases, but a more complex structure for the SS, due to the interference terms. So, looking at the $Q$ function for lower field intensities (than for figure 3), we are able to appreciate better these interference patterns. In figure 4 (a) we plot the $Q$ function of the field initially prepared in a SM and in figure 4 (b) for the field in a SS at $t = 0$. In both cases the field intensity is $\alpha_0^2 = 4$. In figure 5 we have the same situation, but at a later time $t \approx 2T_R^S/3$. Because of the relative proximity of the states in phase space, the
splitting of the $Q$ function is not as clear as for higher intensities. However, we see that the interference pattern is always more complex if we start in an SS than if we start in a SM. For instance, at $t=T_R^S$, we see, in figure 6(b), for the SS, pronounced interference not seen in figure 6(a), for the SM. This difference is of course connected to the fact that around this time $T_R^S$ there is a revival in the SS case (see figure 1(b)), while for the SM the atom is still in the collapse region. In order to confirm this connection, we see in figure 7 the $Q$ functions at $t=T_R^M = 2T_R^S$, i.e., when the first revival occurs for the SM and the second for the SS. We see strong interference patterns in both cases.

Figure 4. $Q$ function of the cavity field at $t=0$ for: (a) the field initially prepared in a statistical mixture and (b) the field initially prepared in a superposition state. In both cases the initial intensity (mean photon number) is $\bar{n}=4$. 
Figure 5. $Q$ function of the cavity field at $t = 2T_{\text{p}}/3$ for: (a) the field initially prepared in a statistical mixture and (b) the field initially prepared in a superposition state. In both cases the initial intensity (mean photon number) is $\bar{n} = 4$. 
Figure 6. $Q$ function of the cavity field at $t = T_R^S$ for: (a) the field initially prepared in a statistical mixture and (b) the field initially prepared in a superposition state. In both cases the initial intensity (mean photon number) is $\bar{n} = 4$. 
Figure 7. $Q$ function of the cavity field at $t = 2T_R^S$ for: (a) the field initially prepared in a statistical mixture and (b) the field initially prepared in a superposition state. In both cases the initial intensity (mean photon number) is $\bar{n} = 4$. 

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3.2. Field purity and photon number distribution

In Section 3.1, we showed that the Q functions we obtain for an initial field given by a SM or a SS splits into two components, which at later times recombine, resembling pure states (specifically a superposition of coherent states). In this Section we are going to be concerned with the field purity, so we are going to be interested basically in the entropy and in the photon-number distribution. As we have briefly discussed in the Introduction, the entropy \( S = - \text{Tr} [\rho(t) \ln \rho(t)] \) gives us the most complete information about the field purity [19]. The entropy will be zero if \( \rho(t) \) describes a pure state, and greater than zero for a statistical mixture. In our case, however, it will be enough to calculate the quantity defined as:

\[
\zeta = 1 - \text{Tr} [\rho^2],
\]

which has a behaviour very similar to the entropy. We can obtain \( \text{Tr} [\rho^2(t)] \) directly from equation (25), and we get, for the field initially in a SS:

\[
\zeta_s = 1 - \frac{A^2}{(1 + s^2)^2} (T_1^2 + T_2^2 + 2|T_3|^2),
\]

where

\[
T_1 = \sum_{n=0}^{\infty} P_n^M(C_{n+1}^2 + s^2 \alpha^2 S_{n+1}^2 D^2),
\]

\[
T_2 = \sum_{n=0}^{\infty} P_n^M \left( s^2 C_n^2 E + \frac{n^2}{\alpha^2} S_n^2 D^2 \right),
\]

\[
T_3 = \sum_{n=0}^{\infty} P_n^M(C_{n+1}^2 E - is \alpha S_{n+1} D) \left( sC_n B + \frac{in}{\alpha} S_n D \right),
\]

with \( E = [1 + (-1)^n], D = [1 - (-1)^n] \). The Cs and Ss are defined in equations (12). If the field is initially in a SM, we get:

\[
\zeta_m = 1 - \frac{1}{2(1 + s^2)^2} (T_4^2 + T_5^2 + |T_6|^2 + |T_7|^2 + |T_8|^2 + |T_9|^2),
\]

where

\[
T_4 = \sum_{n=0}^{\infty} P_n^M(C_{n+1}^2 + s^2 \alpha^2 S_{n+1}^2),
\]

\[
T_5 = \sum_{n=0}^{\infty} P_n^M \left( s^2 C_n^2 + \frac{n^2}{\alpha^2} S_n^2 \right),
\]

\[
T_6 = \sum_{n=0}^{\infty} P_n^M(-1)^n(C_{n+1} - is \alpha S_{n+1})^2,
\]

\[
T_7 = \sum_{n=0}^{\infty} P_n^M(-1)^n \left( sC_n - \frac{in}{\alpha} S_n \right)^2,
\]

\[
T_8 = \sum_{n=0}^{\infty} P_n^M(C_{n+1} - is \alpha S_{n+1}) \left( sC_n + \frac{in}{\alpha} S_n \right),
\]

\[
T_9 = \sum_{n=0}^{\infty} P_n^M(-1)^n(C_{n+1} - is \alpha S_{n+1}) \left( sC_n - \frac{in}{\alpha} S_n \right),
\]
Figure 8. Parameter $\zeta = 1 - \text{Tr} \left[ \rho^2(t) \right]$ against $\lambda t$ for (1) the field initially prepared in a SM and (2) for the field initially prepared in a coherent state. The atom is initially prepared in the excited state and $\bar{n} = 25$.

Figure 9. Parameter $\zeta = 1 - \text{Tr} \left[ \rho^2(t) \right]$ against $\lambda t$ for (1) the field initially prepared in a SM and (2) for the field initially prepared in an even coherent state (SS). The atom is initially prepared in the excited state and $\bar{n} = 25$. 
Figure 10. Parameter $\zeta = 1 - \text{Tr}[\hat{\beta}^2(t)]$ against $\lambda t$ for (1) the field initially prepared in a SM and (2) for the field initially prepared in a coherent state. The atom is initially prepared in a superposition state and $\bar{n} = 25$.

Figure 11. Parameter $\zeta = 1 - \text{Tr}[\hat{\beta}^2(t)]$ against $\lambda t$ for (1) the field initially prepared in a SM and (2) for the field initially prepared in an even coherent state. The atom is initially prepared in a superposition state and $\bar{n} = 25$. 
We can now evaluate the expressions (31) and (32) numerically. From the definition of \( \zeta = 1 - \text{Tr}_f [\hat{f}^2] \), we have that \( \zeta = 0 \) for a pure state, and \( \zeta > 0 \) for a statistical mixture. The greater \( \zeta \), the less pure the state will be. In figure 8 we plot \( \zeta_S \) against \( \lambda t \) for \( s=r=0 \), i.e., the atom initially prepared in the excited state and the field prepared in a coherent state, as well as \( \zeta_M \), for \( s = 0 \) and the field in a SM of coherent states. As we expected, if we start in a SM, the field will be basically in a mixed state at all times. If we start with a coherent state, however, the field will be nearly pure in the middle of the collapse region, as was pointed out by Phoenix and Knight [10, 19]. The maxima and minima, however, occur at the same time in both cases. Now we want to see what happens if the field is prepared in a SS. In figure 9 we plot \( \zeta_M \) as well as \( \zeta_S \) for \( r=1 \). In this case \( \zeta_S \) shows a very different behaviour in comparison with \( \zeta \) in the former case. The first difference is that the field is in its 'less mixed state' (after the collapse) at \( t \approx T^S_\lambda \), i.e., at the revival time rather than in the collapse region. Moreover, the state of the field never gets as pure as in the coherent state case, and indeed the state remains in a statistical mixture during most of the time. As we can see in figure 9, there is a very well defined line around \( \zeta_S = 1/2 \), which is the value of \( \zeta \) for the field initially in a statistical mixture and that may be explained because the SS decays to a SM during the interaction, only going nearly to a pure state for some specific times.

If the atom is prepared in a superposition state \( (s=1) \) at \( t=0 \), the variation of \( \zeta \) in time will be very different. In figure 10 we plot it against \( \lambda t \) for the field in a SM as well as in a coherent state \( (r=0) \) at \( t=0 \). We see that for the SM the parameter \( \zeta \) is still smaller (indicating a less mixed state) during the collapse region, although it shows a different pattern. However, if the field is initially in a coherent state, it remains quite pure (but increasingly mixed as time passes). This result is in agreement with the atomic inversion in figure 2 (a), i.e., the atom hardly exhibits Rabi oscillations. So in such a situation the atom and field will evolve independently (at least for high intensity fields) [10], but this is only valid for the particular atomic superposition state considered here (equation (14)). This is not true, however, if the field is prepared in a superposition of coherent states \( (r=1) \). In figure 11 we plot the parameter \( \zeta_S \) in that situation. The oscillations resemble those in figure 8 (coherent state at \( t=0 \)), but with a shift, in such a way that the state is less mixed at the collapse region in this case (see figure 2 (b)).

We have also calculated the atomic entropy, which will be equal to the field entropy [19, 23] if the initial states are pure states. The entropy is plotted in figure 12 for the field initially in a SS \( (r=1) \) and the atom in the upper state \( (s=0) \). We see, as expected, a pattern very similar to that shown in figure 9 for \( \zeta_S \). The well defined line at \( S = \ln 2 \approx 0.7 \) corresponds to the entropy of a statistical mixture of two coherent states. However, it can be seen that the entropy takes values above 0.7, unlike the coherent case, which may be seen as destructive interference due to the nature of the SS.

Another interesting aspect of the field evolution is the photon-number distribution. Due to the complicated evolution of the field density operator (25), the photon-number distribution defined in (7) will also evolve in a complicated way. However, for some particular times, the distribution will assume familiar forms. If the initial field is in a coherent state, it is known [10, 11] that in the middle of the collapse region the field will be approximately in a superposition of two coherent states. In fact, the photon number distribution of the cavity field in such a time resembles the distribution of either an even coherent state (1) or an odd coherent
Figure 12. Atomic entropy (equal to field entropy) against $\lambda t$ for the field initially prepared in an even coherent state (SS) and the atom in the excited state and $\bar{n} = 25$.

Figure 13. Photon-number distribution of the cavity field at $t = T^S_R$ for the field initially prepared in an even coherent state (SS) and the atom in the excited state. The initial intensity of the field is $\bar{n} = 25$.

state (2), depending on the initial mean photon number being odd or even [11]. We can calculate the photon-number distribution in case of having a SS just by inserting equation (25) into (7). In figure 13 we plot the distribution at $t \approx R^S_R \approx T^M_R/2$, that is when the field is 'less mixed', if the initial field is in a SS (even coherent state). Indeed the distribution resembles that of an even coherent state (pure state) irrespectively of the initial state being even or odd coherent states. As in the case of the coherent state, it will depend on the initial mean photon number [11].
Acknowledgments

We would like to thank Professor Peter Knight for helpful and stimulating discussions. This work was supported by CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), the Mexican Consejo Nacional de Ciencia y Tecnologia (CONACyT), the Overseas Research Student Awards Scheme and the United Kingdom Science and Engineering Research Council.

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