Coupled coincidence point results on generalized distance in ordered cone metric spaces

Hassen Aydi · Erdal Karapınar · Zead Mustafa

Received: 26 March 2012 / Accepted: 8 November 2012
© Springer Basel 2012

Abstract In this paper, we establish some coupled coincidence point results in the setting of partially ordered cone metric spaces via \( c \)-distance. We also proved some related coupled common fixed point theorems for such mappings. Our results generalize, extend, and unify several well-known comparable results in the literature regarding this topic. We present some examples to illustrate the concepts and results.

Keywords \( c \)-distance · Cone metric space · Ordered set · Coupled coincidence point · Common coupled fixed point · Mixed \( g \)-monotone property

Mathematics Subject Classification (2000) 54H25 · 47H10 · 54E50

1 Introduction and preliminaries

In 1996, Kada et al. [17] introduced the notion of \( w \)-distance on a metric space and proved a non-convex minimization theorem which generalizes Caristi’s fixed point theorem and the variational principle. On the other hand, Huang and Zhang [15]...
reintroduced the notion of cone metric spaces by replacing the set of real numbers with an ordered Banach space. The authors also proved some fixed point theorems of contractive mappings on complete cone metric spaces with the assumption of normality of a cone. Following this celebrated paper, a number of fixed point theorems have been reported in normal or non-normal cone metric spaces (see e.g. [1–21] and [23–36] and references therein).

For the sake of completeness, we recall some basic definitions, notations and necessary results from the literature. Throughout the manuscript, we denote by \( \mathbb{R}_+ \) and \( \mathbb{N}^* \) the set of non-negative real numbers and the set of positive integers, respectively. Let \( E \) be a real Banach space and \( 0_E \) is the zero vector of \( E \).

**Definition 1.1** A non-empty subset \( P \) of \( E \) is called a cone if the following conditions hold:

(i) \( P \) is closed and \( P \neq \{0_E\} \),

(ii) \( a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \implies ax + by \in P \),

(iii) \( x \in P, \; -x \in P \implies x = 0_E \).

For a given cone \( P \subset E \), a partial ordering \( \leq_E \) with respect to \( P \) is defined by \( x \leq_E y \) if and only if \( y - x \in P \), for \( x, y \in E \). We shall write \( x <_E y \) to show that \( x \leq_E y \) but \( x \neq y \). Moreover, \( x \ll_E y \) will stand for \( y - x \in \text{int}P \), where \( \text{int}P \) denotes the interior of \( P \). A cone \( P \) is called solid if \( \text{int}P \) is non-empty.

**Definition 1.2** [15]. Let \( X \) be a non-empty set and \( d: X \times X \rightarrow P \) satisfies

(i) \( d(x, y) = 0_E \) if and only if \( x = y \),

(ii) \( d(x, y) = d(y, x) \) for all \( x, y \in X \),

(iii) \( d(x, y) \leq_E d(x, z) + d(z, y) \) for all \( x, y, z \in E \).

Then \( d \) is called a cone metric on \( X \) and \((X, d)\) is called a cone metric space.

**Definition 1.3** [15]. Let \((X, d)\) be a cone metric space, \( \{x_n\} \) is a sequence in \( X \) and \( x \in X \).

(i) If for every \( c \in E \) with \( 0_E \ll_E c \), there is \( N \in \mathbb{N} \) such that \( d(x_n, x) \ll_E c \) for all \( n \geq N \), then \( \{x_n\} \) is said to be convergent to \( x \). This limit is denoted by \( \lim_{n \rightarrow +\infty} x_n = x \) or \( x_n \rightarrow x \) as \( n \rightarrow +\infty \).

(ii) If for every \( c \in E \) with \( 0_E \ll_E c \), there is \( N \in \mathbb{N} \) such that \( d(x_n, x_m) \ll_E c \) for all \( n, m > N \), then \( \{x_n\} \) is called a Cauchy sequence in \( X \).

(iii) If every Cauchy sequence in \( X \) is convergent in \( X \), then \((X, d)\) is called a complete cone metric space.

**Definition 1.4** A cone \( P \subset E \) is **normal** if there is a number \( k \geq 1 \) such that for all \( x, y \in P : 0 \leq x \leq y \) implies \( \|x\| \leq k \|y\| \). The least positive number satisfying the previous condition is called the **normal constant** of \( P \).

Very recently, Cho et al. [11] and Wang et al. [33] introduced the notion of \( c \)-distance on cone metric space \((X, d)\) that is an extention of \( w \)-distance of Kada et al. [17], as follows.
**Definition 1.5** [11,33]. Let \((X, d)\) be a cone metric space. Then a function \(q : X \times X \to E\) is called a \(c\)-distance on \(X\) if the following are satisfied:

1. \(q(x, y) \geq_E 0_E\) for all \(x, y \in X\),
2. \(q(x, z) \leq_E q(x, y) + q(y, z)\) for all \(x, y, z \in X\),
3. for each \(x \in X\) and \(n \geq 1\), if \(q(x, y_n) \leq_E u\) for some \(u = u_x \in P\), then \(q(x, y) \leq_E u\) whenever \(\{y_n\}\) is a sequence in \(X\) converging to a point \(y \in X\),
4. for all \(c \in E\) with \(0_E \ll_E c\), there exists \(e \in E\) with \(0_E \ll_E e\) such that \(q(z, x) \ll_E e\) and \(q(z, y) \ll_E e\) imply \(d(x, y) \ll_E c\).

**Remark 1** [11,33]. The \(c\)-distance \(q\) is a \(w\)-distance on \(X\) if we take \((X, d)\) is a metric space, \(E = \mathbb{R}_+\), \(P = [0, \infty)\) and \((q3)\) is replaced by the following condition: For any \(x \in X\), \(q(x, \cdot) : X \to \mathbb{R}_+\) is lower semi-continuous. Moreover, \((q3)\) holds whenever \(q(x, \cdot)\) is lower semi-continuous. Thus, if \((X, d)\) is a metric space, \(E = \mathbb{R}_+\) and \(P = [0, \infty)\), then every \(w\)-distance is a \(c\)-distance. But the converse is not true in general case. Therefore, the \(c\)-distance is a generalization of the \(w\)-distance.

**Example 1.1** [11,33]. Let \((X, d)\) be a cone metric space and \(P\) be a normal cone. Define a mapping \(q : X \times X \to E\) by \(q(x, y) = d(x, y)\) for all \(x, y \in X\). Then \(q\) is a \(c\)-distance.

**Example 1.2** [11,33]. Let \((X, d)\) be a cone metric space and \(P\) be a normal cone. Define a mapping \(q : X \times X \to E\) by \(q(x, y) = d(u, y)\) for all \(x, y \in X\), where \(u\) is a fixed point in \(X\). Then \(q\) is a \(c\)-distance.

**Remark 2** [11,33]. Let \((X, d)\) be a cone metric space and a function \(q : X \times X \to E\) is a \(c\)-distance on \(X\). Then,

1. \(q(x, y) = q(y, x)\) does not necessarily hold for all \(x, y \in X\),
2. \(q(x, y) = 0_E\) is not necessarily equivalent to \(x = y\) for all \(x, y \in X\).

**Lemma 1.1** [11,33]. Let \((X, d)\) be a cone metric space and \(q\) be a \(c\)-distance on \(X\). Let \(\{x_n\}\) be a sequence in \(X\). Suppose that \(\{u_n\}\) and \(\{v_n\}\) are sequences in \(P\) converging to \(0_E\). Then, we have:

1. If \(q(x_n, y) \leq_E u_n\) and \(q(x_n, z) \leq_E u_n\), then \(y = z\).
2. If \(q(x_n, y_n) \leq_E u_n\) and \(q(x_n, z) \leq_E u_n\), then \(\{y_n\}\) converges to \(z\).
3. If \(q(x_n, x_m) \leq_E u_n\) for all \(m > n\), then \(\{x_n\}\) is a Cauchy sequence in \(X\).
4. If \(q(y, x_n) \leq_E u_n\), then \(\{x_n\}\) is a Cauchy sequence in \(X\).

Let \((X, \leq)\) be a partially ordered set. In 2006, Bhaskar and Lakshmikantham [8] introduced the concept of a mixed monotone property of the mapping \(F : X \times X \to X\).

**Definition 1.6** [8]. Let \((X, \leq)\) be a partially ordered set and \(F : X \times X \to X\). Then the map \(F\) is said to have mixed monotone property if \(F(x, y)\) is monotone non-decreasing in \(x\) and is monotone non-increasing in \(y\); that is, for any \(x, y \in X\),

\[ x_1 \leq x_2 \implies F(x_1, y) \leq F(x_2, y) \text{ for all } y \in X \]
and
\[ y_1 \preceq y_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \quad \text{for all } x \in X. \]

Lakshmikantham and Ćirić in [22] introduced the concept of a \( g \)-mixed monotone mapping which is a natural extension of Definition 1.6.

**Definition 1.7** [22]. Let \((X, \preceq)\) be a partially ordered set. Given \(F : X \times X \to X\) and \(g : X \to X\). Then the map \(F\) is said to have mixed \(g\)-monotone property if \(F(x, y)\) is monotone \(g\)-non-decreasing in \(x\) and is monotone \(g\)-non-increasing in \(y\); that is, for any \(x, y \in X\),
\[ gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \quad \text{for all } y \in X \]
and
\[ gy_1 \preceq gy_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \quad \text{for all } x \in X. \]

**Definition 1.8** [8]. An element \((x, y) \in X \times X\) is called a coupled fixed point of a mapping \(F : X \times X \to X\) if
\[ F(x, y) = x \text{ and } F(y, x) = y. \]

**Definition 1.9** [22]. An element \((x, y) \in X \times X\) is called a coupled coincidence point of the mappings \(F : X \times X \to X\) and \(g : X \to X\) if
\[ F(x, y) = gx \text{ and } F(y, x) = gy. \]

**Definition 1.10** [22]. Let \(X\) be a non-empty set. Then we say that the mappings \(F : X \times X \to X\) and \(g : X \to X\) are commutative if
\[ gF(x, y) = F(gx, gy) \quad \text{for all } x, y \in X. \]

The purpose of this paper is to give some coupled coincidence point theorems in the context of ordered cone metric spaces by the help of the concept of a \(c\)-distance. Also, an illustrative example is presented.

**2 Main results**

In this section, we prove coincidence point results in the framework of partially ordered cone metric spaces in terms of a \(c\)-distance.

**Theorem 2.1** Let \((X, \preceq)\) be a partially ordered set and \((X, d)\) be a complete cone metric space. Let \(q\) be a \(c\)-distance on \(X\). Let \(F : X \times X \to X\) and \(g : X \to X\) be two functions such that there are two nonnegative real numbers \(a\) and \(b\) with \(a + b < 1\) such that
Let $\{x, y, u, v \in X \text{ with } (gx \leq gu, gv \leq gy) \text{ or } (gu \leq gx, gy \leq gv)\). Assume that $F$ and $g$ satisfy the following conditions:

(C1) $F(X \times X) \subseteq g(X)$,  
(C2) $F$ and $g$ commute,  
(C3) $F$ and $g$ are continuous, and  
(C4) $F$ has the mixed $g$-monotone property.

If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$, then $F$ and $g$ have a coupled coincidence point, say $(u, v)$. Moreover, $q(gu, gu) = q(gv, gv) = 0$.

**Proof** Let $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Since $F$ has the mixed $g$-monotone property, we have $gx_0 \leq gx_1 \leq gx_2$ and $gy_2 \leq gy_1 \leq gy_0$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $gx_n = F(x_{n-1}, y_{n-1}) \leq gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_{n-1}, x_{n-1}) \leq gy_n = F(y_n, x_n)$.

Since $gx_{n-1} \leq gx_n$ and $gy_n \leq gy_{n-1}$ hold for all $n \in \mathbb{N}^*$, by inequality (1), we have

$$
q(gx_n, gx_{n+1}) = q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \leq_E a q(gx_{n-1}, gx_n) + bq(gy_{n-1}, gy_n)
$$

and

$$
q(gy_n, gy_{n+1}) = q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \leq_E a q(gy_{n-1}, gy_n) + bq(gx_{n-1}, gx_n).
$$

From (2) and (3), we have

$$
q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \leq_E (a + b)[q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)].
$$

Therefore, continuing this process we will have

$$
q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \leq_E (a + b)^n[q(gx_0, gx_1) + q(gy_0, gy_1)] \quad \text{for all } n \geq 0.
$$

This implies that

$$
q(gx_n, gx_{n+1}) \leq_E (a + b)^n[q(gx_0, gx_1) + q(gy_0, gy_1)] \quad \text{for all } n \geq 0,
$$

and

$$
q(gy_n, gy_{n+1}) \leq_E (a + b)^n[q(gx_0, gx_1) + q(gy_0, gy_1)] \quad \text{for all } n \geq 0.
$$
Let $m > n$. By (q2), we have
\[
q(gx_n, gx_m) \leq E \sum_{i=n}^{m-1} q(gx_i, gx_{i+1}) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(gx_0, gx_1) + q(gy_0, gy_1)],
\]
and
\[
q(gy_n, gy_m) \leq E \sum_{i=n}^{m-1} q(gy_i, gy_{i+1}) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(gx_0, gx_1) + q(gy_0, gy_1)].
\]

Since $a + b < 1$, so by Lemma 1.1.(3), the sequences \{gx_n\} and \{gy_n\} are Cauchy in \((X, d)\). Since \(X\) is complete, there are \(u, v \in X\) such that \(gx_n \to u\) and \(gy_n \to v\), that is,
\[
\lim_{n \to +\infty} d(gx_n, u) = \lim_{n \to +\infty} d(gy_n, v) = 0.
\]
Using the continuity of \(g\), we get \(g(gx_n) \to gu\) and \(g(gy_n) \to gv\). Also, by continuity of \(F\) and commutativity of \(F\) and \(g\), we have
\[
gu = \lim_{n \to \infty} g(gx_{n+1}) = \lim_{n \to \infty} g(F(x_n, y_n)) = \lim_{n \to \infty} F(gx_n, gy_n) = F(u, v),
\]
and
\[
gv = \lim_{n \to \infty} g(gy_{n+1}) = \lim_{n \to \infty} g(F(y_n, x_n)) = \lim_{n \to \infty} F(gy_n, gx_n) = F(v, u).
\]
Hence, \((u, v)\) is a coupled coincidence point of \(F\) and \(g\). Moreover, by (1),
\[
q(gu, gu) = q(F(u, v), F(u, v)) \leq E aq(gu, gu) + bq(gv, gv),
\]
and
\[
q(gv, gv) = q(F(v, u), F(v, u)) \leq E aq(gv, gv) + bq(gu, gu).
\]
By summation, we get that
\[
q(gu, gu) + q(gv, gv) \leq E (a + b)[q(gu, gu) + q(gv, gv)],
\]
which holds unless \(q(gu, gu) = g(gv, gv) = 0\). Since \(a + b < 1\). This completes the proof. \(\Box\)
Corollary 2.1 Let \((X, \preceq)\) be a partially ordered set and \((X, d)\) be a complete cone metric space. Let \(q\) be a \(c\)-distance on \(X\). Let \(F : X \times X \to X\) be continuous and having the mixed monotone property. Assume there exists \(k \in [0, 1)\) such that

\[
q(F(x, y), F(u, v)) \leq_E \frac{k}{2}[q(x, u) + q(y, v)]
\]

for all \(x, y, u, v \in X\) with \((x \preceq u, y \preceq v)\) or \((u \preceq x, y \preceq v)\). If there exist \(x_0, y_0 \in X\) such that \(x_0 \preceq F(x_0, y_0)\) and \(F(y_0, x_0) \preceq y_0\), then \(F\) has a coupled fixed point, say \((u, v)\). Moreover, \(q(u, u) = q(v, v) = 0\).

Proof It follows by taking \(a = b = \frac{k}{2}\) and \(g = I_X\), the identity on \(X\), in Theorem 2.1. \(\square\)

Now, we present the following useful lemma.

Lemma 2.1 Let \((X, d)\) be a cone metric space and \(q\) be a \(c\)-distance on \(X\). Let \(\{x_n\}\) be a sequence in \(X\). Suppose that \(\{\alpha_n\}\) and \(\{\beta_n\}\) are sequences in \(P\) converging to \(0_E\). If \(q(x_n, y) \leq_E \alpha_n\) and \(q(x_n, z) \leq_E \beta_n\), then \(y = z\).

Proof Let \(c \gg 0_E\) be arbitrary. Since \(\alpha_n \to 0_E\), so there exists \(N_1 \in \mathbb{N}\) such that \(\alpha_n \ll_E \frac{c}{2}\) for all \(n \geq N_1\). Similarly, there exists \(N_2 \in \mathbb{N}\) such that \(\beta_n \ll_E \frac{c}{2}\) for all \(n \geq N_2\). Thus, for all \(N \geq \max\{N_1, N_2\}\), we have

\[
q(x_n, y) \ll_E \frac{c}{2} \quad \text{and} \quad q(x_n, z) \ll_E \frac{c}{2}.
\]

Take \(e = \frac{c}{2}\), so by (q4), we get that \(d(y, z) \ll_E c\) for each \(c \gg 0_E\), hence \(y = z\). \(\square\)

Let \((X, \preceq)\) be a partially ordered set. We endow the product set \(X \times X\) with the partial order given by

for all \((x, y), (u, v) \in X \times X\), \((x, y) \triangleright (u, v) \iff x \preceq u, v \preceq y\).

We say that \((x, y)\) and \((u, v)\) are comparable if \((x, y) \triangleright (u, v)\) or \((u, v) \triangleright (x, y)\).

Theorem 2.2 In addition to the hypotheses of Theorem 2.1, suppose that for every \((x, y), (x^*, y^*) \in X \times X\), there exists \((a, b) \in X \times X\) such that \((F(a, b), F(b, a))\) is comparable with \((F(x, y), F(y, x))\) and \((F(x^*, y^*), F(y^*, x^*))\). Then \(F\) and \(g\) have a unique coupled common fixed point, that is, there exists a unique \((x, y) \in X \times X\) such that

\[
u = gu = F(u, v) \quad \text{and} \quad v = gv = F(v, u).
\]

Proof From Theorem 2.1, the set of coupled coincidences is non-empty. We shall show that if \((x, y)\) and \((x^*, y^*)\) are coupled coincidence points, that is, if \(g(x) = F(x, y)\), \(g(y) = F(y, x), gx^* = F(x^*, y^*)\) and \(gy^* = F(y^*, x^*)\), then

\[
gx = gx^* \quad \text{and} \quad gy = gy^*, \tag{7}
\]
By assumption, there exists \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable with \((F(x, y), F(y, x))\) and \((F(x^*, y^*), F(y^*, x^*))\). Without restriction to the generality, we can assume that \((F(x, y), F(y, x)) \succ (F(u, v), F(v, u))\) and \((F(x^*, y^*), F(y^*, x^*)) \succ (F(u, v), F(v, u))\). Put \(u_0 = u, v_0 = v\) and choose \(u_1, v_1 \in X\) such that \(gu_1 = F(u, v_0)\) and \(gv_1 = F(v_0, u_0)\). Then, similarly as in the proof of Theorem 2.1, we can inductively define sequences \(\{gu_n\}\) and \(\{gv_n\}\) in \(X\) by
\[
\begin{align*}
gu_{n+1} &= F(u_n, v_n) \quad \text{and} \quad gv_{n+1} = F(v_n, u_n).
\end{align*}
\]
Further, set \(x_0 = x, y_0 = y, x_0^* = x^*, y_0^* = y^*\) and, on the same way, define the sequences \(\{gx_n\}, \{gy_n\}, \{gx_n^*\}\) and \(\{gy_n^*\}\). Since \((F(x, y), F(y, x)) = (gx_1, gy_1) = (gx, gy) \succ (F(u, v), F(v, u)) = (gu_1, gv_1)\), then \(gx \preceq gu_1\) and \(gv_1 \preceq gy\). Using that \(F\) is a mixed \(g\)-monotone mapping, one can show easily that \(gx \preceq gu_n\) and \(gv_n \preceq gy\) for all \(n \geq 1\). Thus, from (1), we get
\[
\begin{align*}
q(gu_{n+1}, gx) &= q(F(u_n, v_n), F(x, y)) \\
&\leq_E aq(gu_n, gx) + bq(gv_n, gy, gy)
\end{align*}
\]
and
\[
\begin{align*}
q(gv_{n+1}, gy) &= q(F(v_n, u_n), F(y, x)) \\
&\leq_E aq(gv_n, gy) + bq(gu_n, gx).
\end{align*}
\]
Adding, we get
\[
q(gu_{n+1}, gx) + q(gv_{n+1}, gy) \leq_E (a + b)[q(gu_n, gx) + q(gv_n, gy, gy)].
\]
Continuing this process, it follows that
\[
q(gu_n, gx) + q(gv_n, gy) \leq_E (a + b)^n[q(gu_0, gx) + q(gv_0, gy, gy)],
\]
for each \(n \geq 0\). This implies that
\[
q(gu_n, gx) \leq E \alpha_n := (a + b)^n[q(gu_0, gx) + q(gv_0, gy)], \tag{8}
\]
and
\[
q(gv_n, gy) \leq E \alpha_n := (a + b)^n[q(gu_0, gx) + q(gv_0, gy)]. \tag{9}
\]
Similarly, one can show that
\[
q(gu_n, gx^*) \leq E \beta_n := (a + b)^n[q(gu_0, gx^*) + q(gv_0, gy^*)], \tag{10}
\]
Coupled coincidence point results on generalized distance

\[ q(gv_n, gy^*) \leq E \beta_n := (a + b)^n [q(gu_0, gx^*) + q(gv_0, gy^*)]. \] \hspace{5cm} (11)

Since \( \alpha_n \to 0 \) and \( \beta_n \to 0 \), so by (8), (10) and Lemma 2.1, we get that \( gx = gx^* \).

Similarly, by (9), (11) and Lemma 2.1, we get that \( gy = gy^* \). Thus, (7) holds.

Since \( gx = F(x, y) \) and \( gy = F(y, x) \), by commutativity of \( F \) and \( g \), we have

\[ g(gx) = g(F(x, y)) = F(gx, gy) \quad \text{and} \quad g(gy) = g(F(y, x)) = F(gy, gx). \] \hspace{5cm} (12)

Denote \( gx = z \) and \( gy = w \), then by (12), we get

\[ gz = F(z, w) \quad \text{and} \quad gw = F(w, z). \] \hspace{5cm} (13)

Thus, \((z, w)\) is a coincidence point of \( F \) and \( g \). Then, from (7) with \( x^* = z \) and \( y^* = w \), we have \( gx = gz \) and \( gy = gw \), that is,

\[ gz = z \quad \text{and} \quad gw = w. \] \hspace{5cm} (14)

From (13) and (14), we get

\[ z = gz = F(z, w) \quad \text{and} \quad w = gw = F(w, z). \]

Then, \((z, w)\) is a coupled common fixed point of \( F \) and \( g \). To prove the uniqueness, assume that \((p, q)\) is another coupled common fixed point. Then by (7), we have

\[ p = gp = gz = z \quad \text{and} \quad q = gq = gw = w. \]

The proof is completed. \( \square \)

In next results (Theorem 2.3 and Theorem 2.4), we omit the hypotheses of continuity of \( F \) and \( g \), and the commutativity of \( F \) and \( g \).

**Theorem 2.3** Let \((X, \preceq)\) be a partially ordered set. Let \((X, d)\) be a cone metric space and \( P \) be a normal cone with normal constant \( K \). Let \( q \) be a \( c \)-distance on \( X \). Let \( F : X \times X \to X \) and \( g : X \to X \) be two functions such that there are two nonnegative real numbers \( a \) and \( b \) with \( a + b < 1 \) such that

\[ q(F(x, y), F(u, v)) \leq aq(gx, gu) + bq(gy, gv) \] \hspace{5cm} (15)

for all \( x, y, u, v \in X \) with \((gx \preceq gu, gv \preceq gy)\) or \((gu \preceq gx, gy \preceq gv)\). Assume that \( F \) and \( g \) satisfy the following conditions:

1. \( F(X \times X) \subseteq g(X) \),
2. \( (g(X), d) \) is complete,
3. \( F \) has the mixed \( g \)-monotone property,
4. \( \inf\{\|q(F(x, y), gu)\| + \|q(F(y, x), gv)\| + \|q(gx, gu)\| + \|q(gy, gv)\| + \|q(gx, F(x, y))\| + \|q(gy, F(y, x))\|, \ x, y \in X\} > 0 \) for all \( gu \neq F(u, v) \) or \( gv \neq F(v, u) \).
If there exist \(x_0, y_0 \in X\) such that \(g(x_0) \preceq F(x_0, y_0)\) and \(F(y_0, x_0) \preceq g(y_0)\), then \(F\) and \(g\) have a coupled coincidence point, say \((u, v)\). Moreover, \(q(gu, gu) = q(gv, gv) = 0\).

**Proof** Proceeding exactly as in Theorem 2.1, we have that \(\{g x_n\}\) and \(\{g y_n\}\) are Cauchy sequences in the complete cone metric space \((g(X), d))\). Then, there exist \(u, v \in X\) such that \(g x_n \to gu\) and \(g y_n \to gv\). As (4) and (5), we have for all \(m > n \geq 1\)

\[
q(g x_n, g x_m) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(g x_0, g x_1) + q(g y_0, g y_1)],
\]

and

\[
q(g y_n, g y_m) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(g x_0, g x_1) + q(g y_0, g y_1)].
\]

By (q3) and the fact that \(g x_m \to gu\) and \(g y_m \to gv\) as \(m \to \infty\), we get that

\[
q(g x_n, gu) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(g x_0, g x_1) + q(g y_0, g y_1)],
\]

and

\[
q(g y_n, gv) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(g x_0, g x_1) + q(g y_0, g y_1)].
\]

Since \(P\) is a normal cone with normal constant \(K\), from (16) and (17), it follows that

\[
\|q(g x_n, gu)\| \leq K \frac{(a + b)^n}{1 - (a + b)} \|q(g x_0, g x_1) + q(g y_0, g y_1)\|,
\]

and

\[
\|q(g y_n, gv)\| \leq K \frac{(a + b)^n}{1 - (a + b)} \|q(g x_0, g x_1) + q(g y_0, g y_1)\|.
\]

Again, for \(m = n + 1\), we have

\[
\|q(g x_n, g x_{n+1})\| \leq K \frac{(a + b)^n}{1 - (a + b)} \|q(g x_0, g x_1) + q(g y_0, g y_1)\|,
\]

and

\[
\|q(g y_n, g y_{n+1})\| \leq K \frac{(a + b)^n}{1 - (a + b)} \|q(g x_0, g x_1) + q(g y_0, g y_1)\|.
\]
Suppose $gu \neq F(u, v)$ or $gv \neq F(v, u)$. Then, by hypothesis, (18)–(21), we have

$$0 < \inf \{\|q(F(x, y), gu)\| + \|q(F(y, x), gv)\| + \|q(gx, gu)\| + \|q(gy, gv)\| + \|q(gx, F(x, y))\| + \|q(gy, F(y, x))\|, \ x, y \in X\}$$

$$\leq \inf \{\|q(F(x_n, y_n), gu)\| + \|q(F(y_n, x_n), gv)\| + \|q(gx_n, gu)\| + \|q(gy_n, gv)\| + \|q(gx_n, F(x_n, y_n))\| + \|q(gy_n, F(y_n, x_n))\|, \ n \geq 1\}$$

$$= \inf \{\|q(gx_{n+1}, gu)\| + \|q(gy_{n+1}, gv)\| + \|q(gx_n, gu)\| + \|q(gy_n, gv)\| + \|q(gx_n, gxn+1)\| + \|q(gy_n, gyn+1)\|, \ n \geq 1\}$$

$$\leq \inf \left\{6K \frac{(a + b)^n}{1 - (a + b)} \|q(gx_0, gx_1) + q(gy_0, gy_1)\|, \ n \geq 1\right\} = 0,$$

which is a contradiction. We deduce that $gu = F(u, v)$ and $gv = F(v, u)$, that is, $(u, v)$ is a coupled coincidence point of $F$ and $g$. Also, adjusting as the proof of Theorem 2.1, we get that

$$q(gu, gu) = q(gv, gv) = 0.$$

$\square$

**Theorem 2.4** Let $(X, \leq)$ be a partially ordered set and $(X, d)$ be a cone metric space. Let $q$ be a $c$-distance on $X$. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that there are two nonnegative real numbers $a$ and $b$ with $a + b < 1$ such that

$$q(F(x, y), F(u, v)) \leq aq(gx, gu) + bq(gy, gv)$$

(22)

for all $x, y, u, v \in X$ with $(gx \leq gu, gv \leq gy)$ or $(gu \leq gx, gy \leq gv)$. Assume that $F$ and $g$ satisfy the following conditions:

(E1) $F(X \times X) \subseteq g(X)$,

(E2) $(g(X), d)$ is complete,

(E3) $F$ has the mixed $g$-monotone property.

Also, suppose that $X$ satisfies the following property

(E4) if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n$,

(E5) if a non-increasing sequence $x_n \to x$, then $x \leq x_n$ for all $n$.

If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$, then $F$ and $g$ have a coupled coincidence point, say $(u, v)$. Moreover, $q(gu, gu) = q(gv, gv) = 0$.

**Proof** As in the proof of Theorem 2.1, we have that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the complete cone metric space $(g(X), d)$. Then, there exist $u, v \in X$ such that $gx_n \to gu$ and $gy_n \to gv$. Similarly, for all $m > n \geq 1$, we have

$$q(gx_n, gx_m) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(gx_0, gx_1) + q(gy_0, gy_1)],$$
and
\[ q(g_{yn}, g_{ym}) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(g_{x0}, g_{x1}) + q(g_{y0}, g_{y1})]. \]

By (q3), we get that
\[ q(g_{xn}, g_{u}) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(g_{x0}, g_{x1}) + q(g_{y0}, g_{y1})], \tag{23} \]
and
\[ q(g_{yn}, g_{v}) \leq E \frac{(a + b)^n}{1 - (a + b)} [q(g_{x0}, g_{x1}) + q(g_{y0}, g_{y1})]. \tag{24} \]

By summation, we get that
\[ q(g_{xn}, g_{u}) + q(g_{yn}, g_{v}) \leq E 2 \frac{(a + b)^n}{1 - (a + b)} [q(g_{x0}, g_{x1}) + q(g_{y0}, g_{y1})]. \tag{25} \]

Since \{g_{xn}\} is non-decreasing and \{g_{yn}\} is non-increasing, using the properties (4), (5) of \(X\), we have
\[ g_{xn} \preceq g_{u} \quad \text{and} \quad g_{v} \preceq g_{yn} \quad \text{for all} \quad n \geq 0. \]

From this and (22), we have
\[ q(g_{xn}, F(u, v)) = q(F(x_{n-1}, y_{n-1}), F(u, v)) \leq E a q(g_{xn-1}, g_{u}) + b q(g_{yn-1}, g_{v}), \tag{26} \]
and
\[ q(g_{yn}, F(v, u)) = q(F(y_{n-1}, x_{n-1}), F(v, u)) \leq E a q(g_{yn-1}, g_{v}) + b q(g_{xn-1}, g_{u}). \tag{27} \]

Therefore
\[ q(g_{xn}, F(u, v)) + q(g_{yn}, F(v, u)) \leq E (a + b)[q(g_{xn-1}, g_{u}) + q(g_{yn-1}, g_{v})]. \tag{28} \]

By (25), we have
\[
q(g_{xn}, F(u, v)) + q(g_{yn}, F(v, u)) \leq E (a + b)[q(g_{xn-1}, g_{u}) + q(g_{yn-1}, g_{v})] \\
\leq E (a + b) \frac{2(a + b)^{n-1}}{1 - (a + b)} [q(g_{x0}, g_{x1}) + q(g_{y0}, g_{y1})] \\
= \frac{2(a + b)^n}{1 - (a + b)} [q(g_{x0}, g_{x1}) + q(g_{y0}, g_{y1})].
\]
This implies that
\[ q(gx_n, F(u, v)) \leq E \frac{2(a + b)^n}{1 - (a + b)} [q(gx_0, gx_1) + q(gy_0, gy_1)], \] (29)
and
\[ q(gy_n, F(v, u)) \leq E \frac{2(a + b)^n}{1 - (a + b)} [q(gx_0, gx_1) + q(gy_0, gy_1)]. \] (30)

By (23), (29) and Lemma 2.1, we obtain \( gu = F(u, v) \). Similarly, by (24), (30) and Lemma 2.1, we obtain \( gv = F(v, u) \). Also, adjusting as the proof of Theorem 2.1, we get that
\[ q(gu, gu) = q(gv, gv) = 0. \]

\[ \square \]

**Corollary 2.2** Let \((X, \leq)\) be a partially ordered set and \((X, d)\) be a complete cone metric space. Let \( q \) be a c-distance on \( X \). Let \( F : X \times X \to X \) be having the mixed monotone property. Assume there exists \( k \in [0, 1) \) such that
\[ q(F(x, y), F(u, v)) \leq E \frac{k}{2} [q(x, u) + q(y, v)] \] (31)
for all \( x, y, u, v \in X \) with \((x \leq u, v \leq y)\) or \((u \leq x, y \leq gv)\). Also, suppose that \( X \) satisfies the following property
(4) if a non-decreasing sequence \( x_n \to x \), then \( x_n \leq x \) for all \( n \),
(5) if a non-increasing sequence \( x_n \to x \), then \( x \leq x_n \) for all \( n \).

If there exist \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( F(y_0, x_0) \leq y_0 \), then \( F \) has a coupled fixed point, say \((u, v)\). Moreover, \( q(u, u) = q(v, v) = 0 \).

**Proof** It follows by taking \( a = b = \frac{k}{2} \) and \( g = I_X \) in Theorem 2.4. \( \square \)

**Remark 3** Corollary 2.1 corresponds to Theorem 3.1 of a very recent paper of Cho et al. [12]. Corollary 2.2 improves Theorem 3.3 of Cho et al. [12] since there the contractive contractive was
\[ q(F(x, y), F(u, v)) \leq E \frac{k}{4} [q(x, u) + q(y, v)] \]
while here it is the inequality (22). This is due to Lemma 2.1.

Finally, we present the following example.

**Example 2.1** Let \( E = C^1_{\mathbb{R}}[0, 1] \) with \( \|x\|_E = \|x\|_\infty + \|x\|_\infty \) and \( P = \{x \in E, x(t) \geq 0, t \in [0, 1]\} \). Let \( X = [0, \infty) \) (with usual order) and let \( d : X \times X \to E \) be defined by \( d(x, y)(t) = |x - y|2^t \). Then \((X, d)\) is an ordered cone metric space. Let,
further, \( q : X \times X \to E \) be defined by \( q(x, y)(t) = 2^t y \). It is easy to check that \( q \) is a \( c \)-distance. Consider the mappings \( F : X \times X \to X \) and \( g : X \to X \) by
\[
F(x, y) = \begin{cases} 
\frac{x^2 - y^2}{3} & \text{if } x \geq y \\
0 & \text{if } x < y
\end{cases}
and \ g(x) = x^2.
\]

Take \( a = b = \frac{1}{3} \). The mapping \( F \) has the mixed \( g \)-monotone property and \( (g(X), d) \) is complete. For all \( x, y, u, v \in X \) with \( (gx \preceq gu, gy \preceq gv) \) or \( (gu \preceq gx, gy \preceq gv) \), we have
\[
q(F(x, y), F(u, v)) = F(u, v)2^t \leq \frac{|u^2 - v^2|}{6} 2^t \leq \frac{u^2}{3} 2^t + \frac{v^2}{3} 2^t = aq(gx, gu) + bq(gy, gv)
\]

We have \( g(0) \preceq F(0, 1) \) and \( g(1) \geq F(1, 0) \). All hypotheses of Theorem 2.4 are satisfied, and \( (u, v) = (0, 0) \) is a coincidence point (coupled common fixed point) of \( F \) and \( g \).

References