

# Direct Boolean Integer and Fractional Order SMC of Switching Systems: Application to a DC-DC Buck Converter <sup>★</sup>

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**Abstract:** In this paper, a strategy based on the Sliding Mode Control(SMC) is used to achieve the Boolean input for DC-DC buck converter. Three different surfaces are proposed in order to design integer order and fractional order controller. Unlike conventional methods, sliding mode controllers designed through this methodology directly produce Boolean control actions, avoiding the usability of pulse-width modulation (PWM) generally used to control power converters. Simulations are carried out using Matlab/SIMULINK and the results show the efficiency of the proposed method to control DC-DC buck converter.

*Keywords:* Fractional order controller; switched system; DC-DC buck converter; Sliding Mode Control; Puls width modulation.

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## 1. INTRODUCTION

Switched systems have gained an increasing place in industrial applications since the middle of the 20th century, especially in the field of power electronics where static converters are used extensively. For that reason, they have been acknowledged as an object of research for several decades. But their study has recently received growing attention from control researchers. Indeed, beyond their industrial interest, such systems are basically characterized by a discontinuous dynamic behavior, which makes them particularly attractive to the emergent hybrid community. From a physical point of view, switched systems are defined as continuous time systems including some components that evolve much faster than the time scale at which their global behavior needs to be analyzed (Richard et al. (2006)).

Dynamic Models of DC-DC converters are in the class of switching systems. Applying digital methods to the control of power converters, in particular board-mounted DC-DC converters, offers a rich set of possibilities from which to create new features, improved performance, and much greater product flexibility, and all at lower cost. DC-DC power converters are employed in a variety of applications, including power supplies for personal computers, office equipment, spacecraft power systems, laptop computers, and telecommunications equipment, as well as DC motor drives. The input to a DC-DC converter is an unregulated DC voltage  $V_g$ . The converter produces a regulated output voltage  $v_c$ , having a magnitude (and possibly polarity) that differs from  $V_g$ .

In recent years, numerous studies and applications of fractional-order systems in many areas of science and engineering have been presented (Podlubny (1999); Hilfer (2001)). Fractional calculus as old as the ordinary differential calculus goes back to times when Leibniz and Newton invented differential calculus. The problem raised by Leibniz in a letter dated September 30, 1695 for a fractional derivative has become an ongoing topic for more than hundreds of years. Emerging of effective methods in differentiation and integration of non-integer order equations makes fractional-order systems more and more attractive for the control systems. Fractional order controllers have been investigated in many papers e.g. an analytical robust stability checking method of fractional-order linear time invariant interval uncertain system (Ahn et al. (2007)), two sets of tuning rules for fractional PIDs (Valério and Costa (2006)), a method for controlling main irrigation canals with variable dynamical parameters based on robust fractional order controllers (Feliu et al. (2007)), and several alternative methods for the control of power electronic buck converters applying fractional order control (FOC) (Calderón et al. (2006)).

Recently, a work has been presented in (Richard et al. (2006)) where the authors proposed a switching control action using the SMC and Boolean law which avoid the usability of pulse-width modulation (PWM). In present paper, we propose a controller based on SMC as,

$$u = \frac{1}{2}(1 - \text{sgn}(S)) \quad (1)$$

where  $S$  is the surface. The stability will be proved using three surfaces, i.e. integer order PD, PID and fractional order PID.

The rest of paper is organized as follows. Basic definitions and preliminaries of fractional order are briefly discussed in section 2. Mathematical model of DC-DC buck converter

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is studied in section 3. In section 4, the switching control laws are designed using the SMC through three different surfaces. Simulation results are discussed in section 5 and finally the paper will be concluded in section 6.

## 2. BASIC DEFINITIONS AND PRELIMINARIES

Fractional calculus is a generalization of the integration and differentiation to the non-integer (fractional) order fundamental operator  ${}_a D_t^q$ , where  $a$  and  $t$  are the limits and  $q$  is the order of the operation. The Grunwald-Letnikov (G-L) fractional differential equation operator of order  $q$  is,

$${}_a D_t^q f(t) = \lim_{h \rightarrow 0} \left( \frac{1}{h} \right)^q \sum_{j=0}^{N-1} (-1)^j \binom{q}{j} f(t-h) \quad (2)$$

where  $h = \frac{t-a}{N}$ . The Riemann-Liouville definition is another common notation of fractional derivative. Accordingly a  $q^{th}$  order fractional derivative of function  $f(t)$  with respect to time  $t$  and the terminal value 0, is given by:

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-q-1} f(\tau) d\tau \quad (3)$$

where  $n$  is the first integer larger than  $q$ , i.e.,  $n-1 \leq q < n$  and is the Gamma function as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (4)$$

The terminal value indicates the lower limit in the integral in (3). It may be a nonzero value in the general definition of the fractional derivative. The Laplace transform of the Riemann-Liouville derivative is given by:

$$\mathcal{L} \left\{ \frac{d^q f(t)}{dt^q} \right\} = s^q \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-k-1} f(0)}{dt^{q-k-1}} \quad (5)$$

$n-1 < q \leq n$

Since the Laplace transformation requires the knowledge of non-integer order derivatives of the function at  $t=0$ , the Riemann-Liouville fractional derivative is unsuitable (Podlubny (1999)) in some cases. The problem will be coped with the Caputo definition, which is sometimes called smooth fractional derivative. This is described by:

$$D_t^q f(t) = \begin{cases} \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q+1-n}} d\tau & n-1 \leq q < n \\ \frac{d^n}{dt^n} f(t) & q = n \end{cases} \quad (6)$$

where  $n$  is the first integer larger than  $q$ . The Laplace transform of the Caputo fractional derivative is given by:

$$\mathcal{L} \left\{ \frac{d^q f(t)}{dt^q} \right\} = s^q \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{q-1-k} f^{(k)}(0) \quad (7)$$

$n-1 < q \leq n \in \mathbb{N}$

Contrary to the Riemann-Liouville, only integer order derivatives of function  $f(\cdot)$  appear in the Laplace transformation. For zero initial conditions, (7) reduces to:

$$\mathcal{L} \left\{ \frac{d^q f(t)}{dt^q} \right\} = s^q \mathcal{L}\{f(t)\} \quad (8)$$

In the rest of the work, notation  $D^q x(t)$  represents the Caputo fractional derivative of order  $q$ .

## 3. MATHEMATICAL MODELS OF DC-DC BUCK CONVERTER

The simplest form of buck converter is schematically represented in Fig. 1. This converter consists of an inductor,  $L$ , a capacitor,  $C$  and a switch, which has two states  $u = 1$  and  $u = 0$ . In general, in order to obtain positive output voltages only two structures are necessary (Fig. 2). The converters connect to a DC power source with a voltage (unregulated),  $V_g$  and provide a regulated voltage,  $v_c$  to the load resistor,  $R$  by controlling the state of the switch. In some situations, the load also could be inductive, for example a DC motor, or approximately, a current load, for example in a cascade configuration. For simplicity, here, only resistive loads is considered.

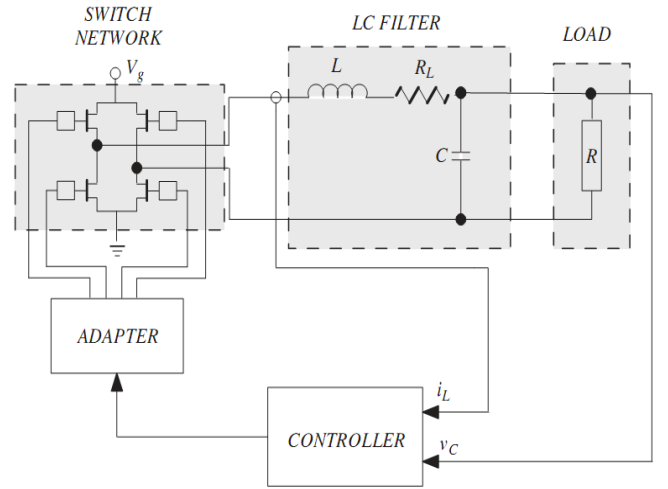


Fig. 1. Buck Converter

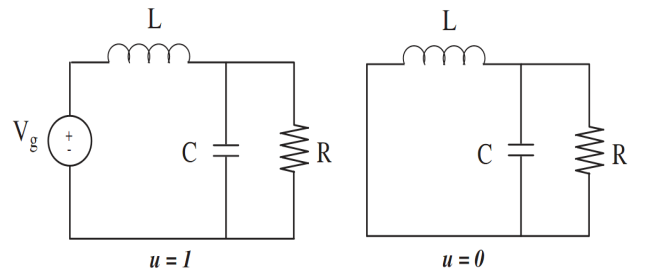


Fig. 2. DC-DC buck converter topologies.

To design the control system of a converter, it is necessary to model the converter dynamic behavior. Unfortunately, modeling of converter dynamic behavior is hampered by the nonlinear time-varying nature of the switching and pulse-width modulation process. The formulation in the form of a bilinear system of a DC-DC buck converter defined on  $\mathbb{R}^n$  is,

$$\begin{bmatrix} \dot{v}_c \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} v_c \\ \dot{v}_c \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ LC \end{bmatrix} u V_g. \quad (9)$$

where  $u \in \{0, 1\}$  and will be designed using sliding mode control.

#### 4. SLIDING MODE CONTROL OF BUCK CONVERTER

*Definition 1.* The fractional integral of the constant function  $f(t) = c$  is given by

$$D^{-v}c = c \frac{t^v}{\Gamma(v+1)}. \quad (10)$$

*Theorem 1.* Consider the dynamics of buck converter (9). The sliding condition is satisfied if the control  $u(t)$  is given by,

$$u(t) = \frac{1}{2} \left[ 1 - \operatorname{sgn} \left( K_d \left[ \frac{1}{RC} (v_c - v_d) + (\dot{v}_c - \dot{v}_d) \right] \right) \right]. \quad (11)$$

**Proof.** Let us consider (12) as a sliding surface: ■

$$S = K_p (v_c - v_d) + K_d (\dot{v}_c - \dot{v}_d), \quad (12)$$

where  $K_p$  and  $K_d$  will be achieved in the following.

Consider the Lyapunov function candidate

$$V = \frac{1}{2} S^2, \quad (13)$$

where  $V$  is a positive semi-definite function. Time derivative of Lyapunov function is,

$$\begin{aligned} \dot{V} &= S \dot{S} \\ &= S [K_p (\dot{v}_c - \dot{v}_d) + K_d (\ddot{v}_c - \ddot{v}_d)]. \end{aligned} \quad (14)$$

Substituting (9) and (11) in (14) yields,

$$\begin{aligned} \dot{V} &= S [K_p (\dot{v}_c - \dot{v}_d) \\ &+ K_d \left( -\frac{1}{LC} v_c - \frac{1}{RC} \dot{v}_c + \frac{V_g}{2LC} [1 - \operatorname{sgn}(S)] - \ddot{v}_d \right)]. \end{aligned} \quad (15)$$

Assuming  $K_p = \frac{K_d}{RC}$  and  $v_d$  as a constant value, (15) can be rewritten as,

$$\dot{V} = S \left[ K_d \left( -\frac{1}{LC} v_c + \frac{V_g}{2LC} [1 - \operatorname{sgn}(S)] \right) \right]. \quad (16)$$

A sufficient condition in sliding mode control is,

$$\dot{V} = S \dot{S} \leq 0 \quad (17)$$

therefore,

$$\begin{aligned} S \dot{S} &= S \left[ K_d \left( -\frac{1}{LC} v_c + \frac{V_g}{2LC} [1 - \operatorname{sgn}(S)] \right) \right] \\ &= \frac{V_g K_d}{2LC} \left( S \left[ 1 - 2 \frac{v_c}{V_g} - \operatorname{sgn}(S) \right] \right). \end{aligned} \quad (18)$$

The sliding condition is satisfied if  $0 < v_c \leq V_g$  and it is obvious from the buck converter that the output voltage is positive and less than input voltage. Regarding to the sliding surface as a PD, there may be some steady state error in the output, this problem can be solve using theorem 2.

*Theorem 2.* Consider the dynamics of buck converter (9). The sliding condition is satisfied if the control  $u(t)$  is given by,

$$u(t) = \frac{1}{2} \left[ 1 - \operatorname{sgn} \left( K_d \frac{1}{RC} (v_c - v_d) + (\dot{v}_c - \dot{v}_d) + \frac{1}{LC} \int (v_c - v_d) \right) \right]. \quad (19)$$

**Proof.** Let us consider (20) as a sliding surface: ■

$$S = K_p (v_c - v_d) + K_d (\dot{v}_c - \dot{v}_d) + K_i \int (v_c - v_d), \quad (20)$$

where  $K_p$ ,  $K_d$  and  $K_i$  will be achieved in the following.

Consider the Lyapunov function candidate

$$V = \frac{1}{2} S^2, \quad (21)$$

where  $V$  is a positive semi-definite function. Time derivative of Lyapunov function is,

$$\begin{aligned} \dot{V} &= S \dot{S} \\ &= S [K_p (\dot{v}_c - \dot{v}_d) + K_d (\ddot{v}_c - \ddot{v}_d) + K_i (v_c - v_d)]. \end{aligned} \quad (22)$$

Substituting (9) and (19) in (22) yields,

$$\begin{aligned} \dot{V} &= S [K_p (\dot{v}_c - \dot{v}_d) + K_i (v_c - v_d) \\ &+ K_d \left( -\frac{1}{LC} v_c - \frac{1}{RC} \dot{v}_c + \frac{V_g}{2LC} [1 - \operatorname{sgn}(S)] - \ddot{v}_d \right)]. \end{aligned} \quad (23)$$

Assuming  $K_p = \frac{K_d}{RC}$ ,  $K_i = \frac{K_d}{LC}$  and  $v_d$  as a constant value, (23) can be rewritten as,

$$\dot{V} = \frac{K_d V_g}{2LC} S \left[ 1 - 2 \frac{v_d}{V_g} - \operatorname{sgn}(S) \right]. \quad (24)$$

The sliding condition is satisfied if  $0 < v_d \leq V_g$ . Now, Theorem 2 will be generalized for fractional order surface in theorem 3.

*Theorem 3.* Consider the dynamics of buck converter (9). The sliding condition is satisfied if the control  $u(t)$  is given by,

$$u(t) = \frac{1}{2} \left[ 1 - \operatorname{sgn} \left( K_d \left[ \frac{1}{RC} D^{\mu-1} (v_c - v_d) + D^{\mu} (v_c - v_d) + \frac{1}{LC} D^{\mu-2} (v_c - v_d) \right] \right) \right]. \quad (25)$$

**Proof.** Let us consider (26) as a sliding surface: ■

$$S = K_p D^{\mu-1} (v_c - v_d) + K_d D^{\mu} (v_c - v_d) + K_i D^{\mu-2} (v_c - v_d), \quad (26)$$

where  $0 < \mu \leq 1$  is the fractional order of differentiator and (26) is similar (20) for  $\mu = 1$ . Therefore, for  $\mu = 1$  fractional order PID surface will be changed to the integer order PID surface.  $K_p$ ,  $K_d$  and  $K_i$  are positive coefficients which will be achieved in the following.

The same previous theorem, consider the Lyapunov function candidate as,

$$V = \frac{1}{2}S^2, \quad (27)$$

where time derivative of Lyapunov function is,

$$\begin{aligned} \dot{V} &= S\dot{S} \\ &= S \left[ K_p D^\mu (v_c - v_d) + K_d D^{\mu-1} (\ddot{v}_c - \ddot{v}_d) \right. \\ &\quad \left. + K_i D^{\mu-1} (v_c - v_d) \right]. \end{aligned} \quad (28)$$

Substituting (9) and (25) in (28) yields,

$$\begin{aligned} \dot{V} &= S \left[ K_p D^\mu (v_c - v_d) + K_i D^{\mu-1} (v_c - v_d) + \right. \\ &\quad \left. K_d D^{\mu-1} \left( -\frac{1}{LC} v_c - \frac{1}{RC} \dot{v}_c + \frac{V_g}{2LC} [1 - \text{sgn}(S)] - \ddot{v}_d \right) \right]. \end{aligned} \quad (29)$$

The same as previous theory assuming  $K_p = \frac{K_d}{RC}$ ,  $K_i = \frac{K_d}{LC}$  and  $v_d$  as a constant value, (29) can be rewritten as,

$$\dot{V} \simeq \frac{K_d V_g}{2LC} S \left[ D^{\mu-1} \left( 1 - 2\frac{v_d}{V_g} - \text{sgn}(S) \right) \right]. \quad (30)$$

Consider following functions,

$$f(S, v_d) = 1 - 2\frac{v_d}{V_g} - \text{sgn}(S), \quad (31)$$

where  $\text{sgn}(S) = \begin{cases} 1, & S \geq 0 \\ -1, & S < 0 \end{cases}$ . It is obvious that  $f(S, v_d)$ , always has a constant value, hence

$$f(S, v_d) = \begin{cases} -2\frac{v_d}{V_g}, & S \geq 0 \\ 2 - 2\frac{v_d}{V_g}, & S < 0 \end{cases}. \quad (32)$$

Using definition (1), fractional integral of (32) can be achieved as:

$$g(S, v_d) = D^{\mu-1} f(S, v_d) = \begin{cases} -2\frac{v_d}{V_g} \frac{t^{1-\mu}}{\Gamma(2-\mu)}, & S \geq 0 \\ \left( 2 - 2\frac{v_d}{V_g} \right) \frac{t^{1-\mu}}{\Gamma(2-\mu)}, & S < 0 \end{cases} \quad (33)$$

From (32) and (33), it is obvious that,

$$\text{sgn}(f) = \text{sgn}(g), \quad \forall t > 0. \quad (34)$$

Relation (33) for  $V_g = 20$ ,  $0 < v_d \leq V_g$  and  $t = 0.01s$  is depicted in Fig.(3).

From (34) and Fig. (3),

$$\begin{cases} g < 0, & S \geq 0 \\ g > 0, & S < 0 \end{cases} \quad (35)$$

Therefore,

$$\dot{V} = S\dot{S} = \frac{K_d V_g}{2LC} S.g < 0. \quad (36)$$

where derivative of Lyapunov function is always negative if  $0 < v_d \leq V_g$  and the sliding condition is satisfied.

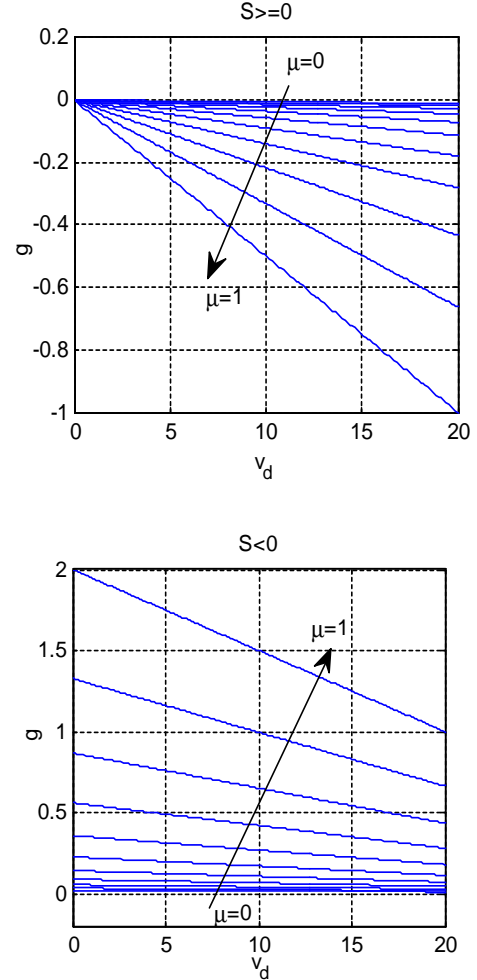


Fig. 3. Numerical result of  $g$  vs  $v_d$

## 5. SIMULATION RESULTS

The selected values for the converter parameter are  $L = 3.24mH$ ,  $C = 48\mu F$ ,  $R = 117\Omega$ ,  $v_d = 8$  and  $V_g = 20$  for a switching frequency of  $2kHz$ . The controller parameter i.e.  $K_d$  is set to 0.1. The simulation results for each three surfaces are shown in Figs.(4) and (5), respectively. In Fig.(4) the output response for different surfaces through PD, PID and  $F_r$ PID are compared (i.e. Theorem 1-3). Control actions for each surfaces are shown in this figure. As it can be seen, although the simulation result for PD has better transient response, it has not enough energy to achieved the desired value and it has steady state error. For the  $F_r$ PID surface the output response is much more faster than PID surface and both achieve the desired values. Testing different values for fractional parameter  $\mu$  found that the best results are around 0.68 (see Figs.(6) and (7)). Phase portrait of the system for each surfaces are shown in Fig.(5). From this figure, it is also obvious that the fractional surface has better response and will achieve the desired value faster than the other. In order to find the better value for fractional parameter, it is simulated for different values i.e. 0.5, 0.6, 0.7, 0.8, 0.9 and the results are shown in Figs. (6) and (7). From both output response and phase portrait, it is obvious that the controller has better response around  $\mu = 0.7$ .

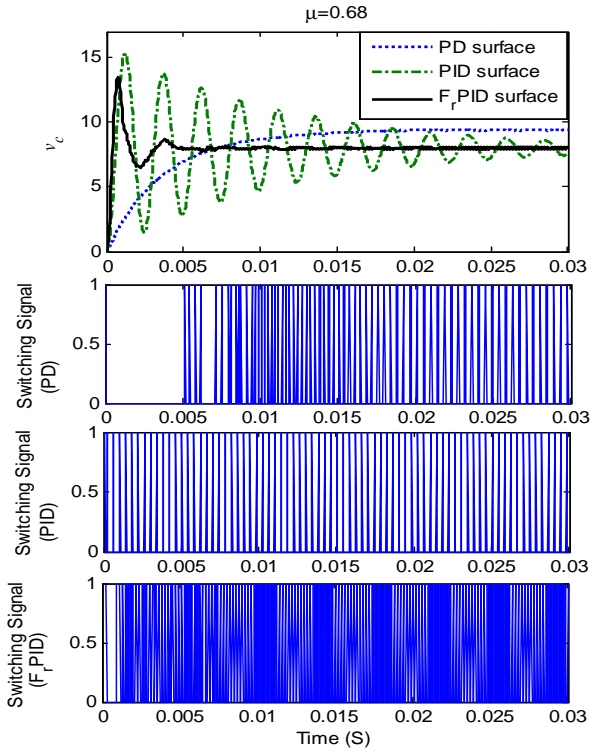


Fig. 4. Comparing three theories: output response

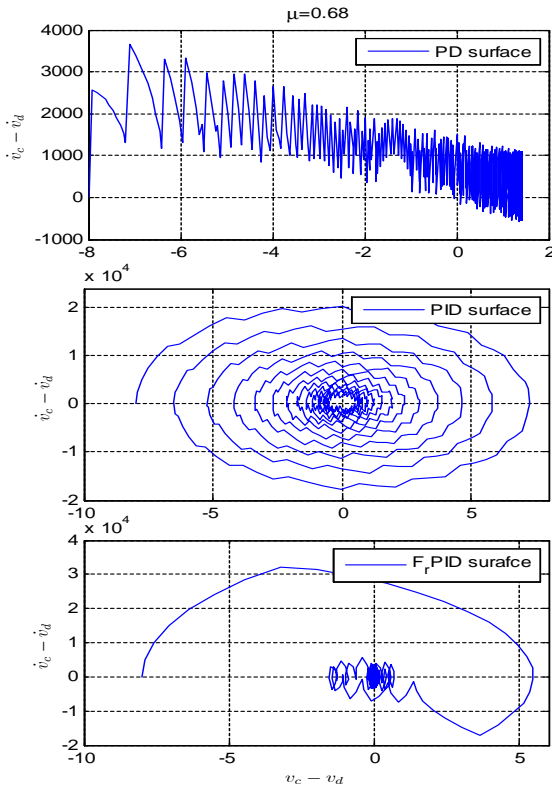


Fig. 5. Comparing three theories: phase portrait

In order to show the performance and robustness of the controller, the output measurements includes additive Gaussian noise with variance of 0.1. As it can be seen from Fig.(8) the results are robust to the noise which show the efficiency of the control and robustness of SMC.

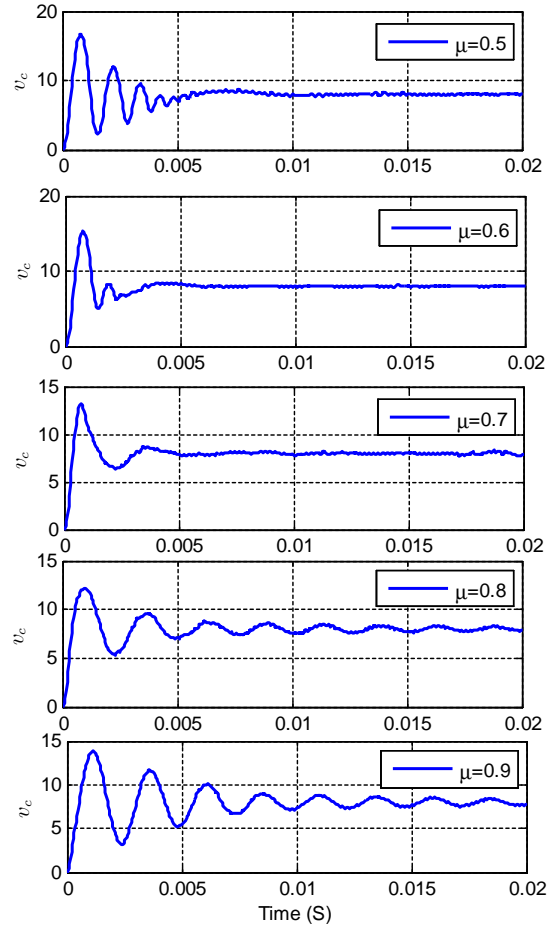


Fig. 6. Output response through  $F_r$ -PID surface for different values of  $\mu$ .

## 6. CONCLUSION

A sliding mode control is designed through this methodology to produce Boolean control actions, directly which avoid the usability of the pulse-width modulation generally used to control power converters. Three different surfaces i.e. integer order PD and PID fractional order PID are considered to design the switching control law. It is shown that the fractional order surface produces better results than integer one and the response was much more faster. It is also shown that the fractional controller show better response around  $\mu = 0.68$  for DC-DC buck converter. The simulation results have been carried out using Matlab/SIMULINK software and based G-L definition of fractional order derivative. In this work a special kind of fractional PID i.e.  $P^{\mu-1}I^{\mu-2}D^{\mu}$  as a sliding surface has been used which can be extended for  $PI^{\lambda}D^{\mu}$  surfaces in the future works. In addition this method can be easily applied in other power converter such as DC-DC boost converter and DC-DC buck-boost converter or other switching systems. The practical application is in progress.

## REFERENCES

- Ahn, H.S. Chen, Y. and Podlubny, I. (2007), Robust stability test of a class of linear time-invariant interval fractional-order system using Lyapunov inequality, *Applied Mathematics and Computation*, **187**, 27-34.

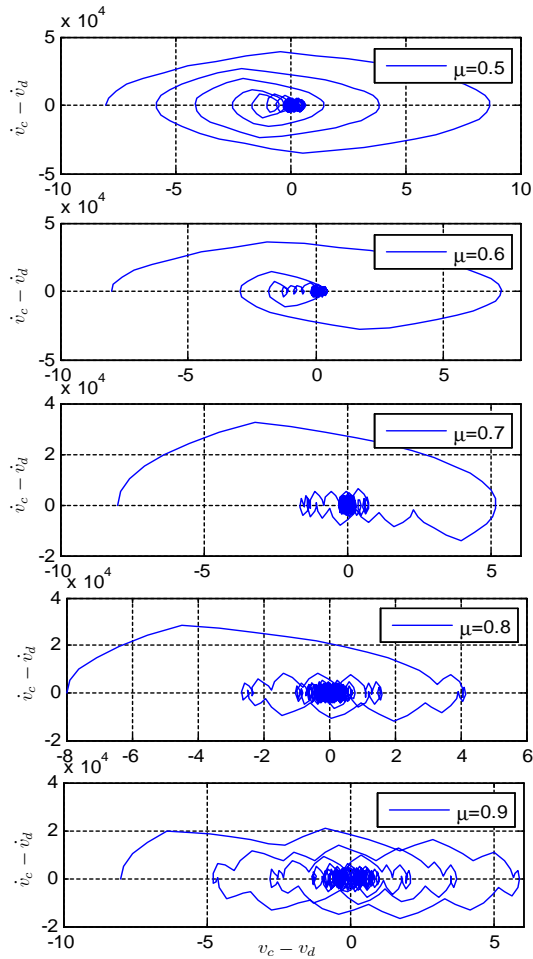


Fig. 7. Phase portriate of the system through  $F_r$ PID surface for different values of  $\mu$ .

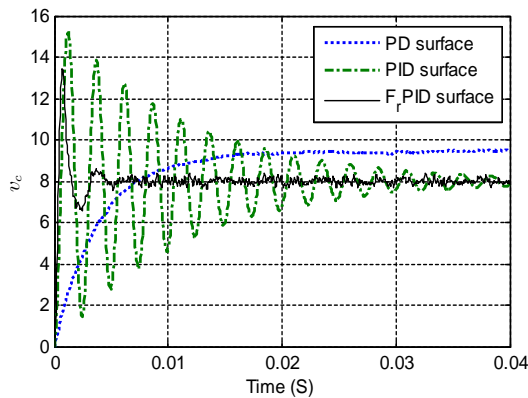


Fig. 8. Simulation results through output noise

Calderón, A.J. Vinagre, B.M. and Feliu, V. (2006), Fractional order control strategies for power electronic buck converters, *Signal Processing*, **86**, 2803-2819.

Feliu, V. Pérez, R. and Rodríguez, L. (2007), Fractional robust control of main irrigation canals with variable dynamic parameters, *Control Engineering Practice*, **15**, 673-686.

Hilfer, R. (2001), Applications of fractional calculus in physics, *World Scientific*, New Jersey.

Podlubny, I. (1999), Fractional Differential Equations, *Academic Press*, New York.

Richard, P.Y. Cormerais, H. and Buisson, J. (2006), A generic design methodology for sliding mode control of switched systems, *Nonlinear Analysis*, **65**, 1751-1772.

Valério, D. and Costa, J. (2006), Tuning of fractional PID controllers with Ziegler-Nichols-type rules, *Signal Processing*, **86**, 2771-2784.