A Note on Novel \((G'/G)\)-expansion Method in Nonlinear Physics

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Abstract:
The exact solutions of nonlinear evolution equations (NLEEs) play a critical role to make known the internal mechanism of complex physical phenomena. In this article, we construct the traveling wave solutions of the (1+1)-dimensional KdV equation and the Benjamin-Ono equation by means of the novel \((G'/G)\)-expansion method. Abundant traveling wave solutions with arbitrary parameters are successfully obtained by this method and the wave solutions are expressed in terms of the hyperbolic, trigonometric and rational functions. It is shown that the novel \((G'/G)\)-expansion method is a powerful and concise mathematical tool for solving nonlinear partial differential equations.

Keywords:
Novel \((G'/G)\)-expansion method; the (1+1)-dimensional KdV equation; Benjamin-Ono equation; nonlinear differential equation; traveling wave solutions

1. INTRODUCTION

It is well known that nonlinear partial differential equations (NLPDEs) are widely used to describe complex phenomena in various fields of science, especially in physics, in fluid mechanics, solid-state physics, biophysics, chemical kinematics, geochemistry, electricity, propagation of shallow water waves, plasma physics, high-energy physics, condensed matter physics, quantum mechanics, optical fibers, elastic media and so on. The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory. The available exact solution of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions. To this end, many powerful methods seeking exact solutions to the nonlinear differential equations have been proposed, such as the homogeneous balance method [1], the tanh-function method [2], the extended tanh-function method [3, 4], the Exp-function method [5, 6], the sine-cosine method [7], the modified Exp-function method [8], the generalized Riccati equation [9], the Jacobi elliptic function expansion method [10, 11], the Hirota’s bilinear method [12], the Miura transformation [13], the \((G'/G)\)-expansion method [14–18], the novel \((G'/G)\)-expansion method [19, 20], the modified simple equation method [21, 22], the improved \((G'/G)\)-expansion method [23], the inverse scattering transform [24], the Jacobi elliptic function expansion method [25, 26], the new
A Note on Novel \((G'/G)\)-expansion Method in Nonlinear Physics

generalized \((G'/G)\)-expansion method \([27–31]\), the Adomian decomposition method \([32, 33]\), the method of bifurcation of planar dynamical systems \([34, 35]\), the wave of translation method \([36]\), the ansatz method \([37, 38]\), the Cole-Hopf transformation \([39]\) and so on.

The objective of this article is to apply the novel \((G'/G)\) expansion method to construct the exact solutions for nonlinear evolution equations in mathematical physics via the \((1+1)\)-dimensional KdV equation and the Benjamin-Ono equation.

The article is prepared as follows: In Section 2, the novel \((G'/G)\) expansion method is discussed. In Section 3, we apply this method to the nonlinear evolution equations pointed out above; in Section 4, physical explanations; in Section 5 conclusions are given.

2. THE NOVEL \((G'/G)\)-EXPANSION METHOD

Suppose the nonlinear evolution equation is of the form

\[
P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0,
\]

where \(P\) is a polynomial in \(u(x,t)\) and its partial derivatives wherein the highest order partial derivatives and the nonlinear terms are concerned. The main steps of the method are as follows:

**Step 1:** Combining the real variables \(x\) and \(t\) by a complex variable \(\xi\), we suppose that

\[
u(x, t) = v(\xi), \quad \xi = x \pm V t,
\]

where \(V\) is the speed of the traveling wave. Eq. (2) transforms Eq. (1) into an ODE for \(v = v(\xi)\):

\[
Q(v, v', v'', v''', \cdots) = 0,
\]

where \(Q\) is a function of \(v(\xi)\) and its derivatives wherein prime stands for derivative with respect to \(\xi\).

**Step 2:** Assume the solution of Eq. (3) can be expressed in powers \(\psi(\xi)\):

\[
u(\xi) = \sum_{j=-N}^{N} a_j (\psi(\xi))^j
\]

where

\[
\psi(\xi) = (d + \Phi(\xi))
\]

and \(\Phi(\xi) = \frac{G'(\xi)}{G(\xi)}\).

Herein \(a_{-N}\) or \(a_{N}\) may be zero, but both of them could not be zero simultaneously. \(a_j (j = 0, \pm 1, \pm 2, \cdots, \pm N)\) and \(d\) are constants to be determined later and \(G = G(\xi)\) satisfies the second order nonlinear ODE:

\[
GG'' = \lambda G G' + \mu G^2 + \nu (G')^2
\]

where prime denotes the derivative with respect \(\xi\); \(\lambda, \mu,\) and \(\nu\) are real parameters.

The Cole-Hopf transformation \(\Phi(\xi) = \ln(G(\xi))\) reduces the Eq. (6) into Riccati equation:

\[
\Phi'(\xi) = \mu + \lambda \Phi(\xi) + (\nu - 1) \Phi^2(\xi)
\]
Eq. (7) has individual twenty five solutions (see Zhu, [40] for details).

**Step 3:** The value of the positive integer $N$ can be determined by balancing the highest order linear terms with the nonlinear terms of the highest order come out in Eq. (3). If the degree of $u(\xi)$ is $D[u(\xi)] = n$, then the degree of the other expressions will be as follows:

$$D\left[\frac{d^p u(\xi)}{d\xi^p}\right] = n + p, \quad D[u^p\left(\frac{d^q u(\xi)}{d\xi^q}\right)^j] = np + s(n + q).$$

**Step 4:** Substitute Eq. (4) including Eqs. (5) and (6) into Eq. (3), we obtain polynomials in $\left(d + \frac{G'(\xi)}{\xi}\right)^j$ and $\left(d + \frac{G'(\xi)}{\xi}\right)^{-j}$, $(j = 0, 1, 2, \cdots, N)$. Collect each coefficient of the resulted polynomials to zero, yields an over-determined set of algebraic equations for $\alpha_j$ $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$, $d$ and $V$.

**Step 5:** Suppose the value of the constants can be obtained by solving the algebraic equations obtained in Step 4. Substituting the values of the constants together with the solutions of Eq. (6), we will obtain new and comprehensive exact traveling wave solutions of the nonlinear evolution equation (1).

**Remark 2.1.**
It is remarkable to survey that if we replace $\lambda$ by $-\lambda$ and $\mu$ by $-\mu$ and put $v = 0$ in Eq. (6), then the novel $(G'/G)$-expansion method coincide with Akbar et al.’s [18] generalized and improved $(G'/G)$-expansion method. On the other hand, if we put $d = 0$ in Eq. (5) and $v = 0$ in Eq. (6) then the method is identical to the improved $(G'/G)$-expansion method presented by Zhang et al. [16]. Again if we set $d = 0$, $v = 0$ and negative the exponents of $(G'/G)$ are zero in Eq. (4), then the method turn out into the basic $(G'/G)$-expansion method introduced by Wang et al. [14]. Finally, if we put $v = 0$ in Eq. (6) and $\alpha_j (j = 1, 2, 3, \cdots, N)$ are functions of $x$ and $t$ instead of constants then the method is transformed into the generalized the $(G'/G)$-expansion method developed by Zhang et al. [17]. Thus the methods presented in the Ref. [14, 16–18] are only special cases of the novel $(G'/G)$-expansion method.

### 3. APPLICATION OF THE METHOD

In this section, we will exert the novel $(G'/G)$-expansion method discussed in Section 2 to solve two NPDEs.

#### 3.1 The (1+1)-dimensional KdV equation

In this section, we will exert the novel $(G'/G)$-expansion method to obtain some new and more general exact traveling wave solutions of the (1+1)-dimensional KdV equation [41].

Let us consider the (1+1)-dimensional KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0 \quad (8)$$

Bring to bear the traveling wave transformation $\xi = x - Vt$, Eq. (8) is changing into the following ODE:

$$-Vu' + 6uu' + u'' = 0 \quad (9)$$

Integrating Eq. (9), we obtain

$$C - Vu + 3u^2 + u'' = 0 \quad (10)$$
where \( C \) is a constant of integration. Substituting Eq. (4) into Eq. (10) and balancing the highest order derivative \( u^n \) with the nonlinear term of the highest order \( u^2 \), we obtain \( N = 2 \).

Therefore, the solution of Eq. (10) takes the form,

\[
    u(\xi) = \alpha_{-2} (\psi(\xi))^{-2} + \alpha_{-1} (\psi(\xi))^{-1} + \alpha_0 + \alpha_1 (\psi(\xi)) + \alpha_2 (\psi(\xi))^2.
\]  

(11)

Substituting Eq. (11) into Eq. (10), the left hand side is transformed into polynomials of \( (d + \frac{G'(\xi)}{G(\xi)})^j \) and \( (d + \frac{G'(\xi)}{G(\xi)})^{-j}, \) \((j = 0, 1, 2, \cdots, N)\). Equating the coefficients of like power of these polynomials to zero, we obtain an over-determine set of algebraic equations (for simplicity we leave out to display the equations) for \( \alpha_0, \alpha_1, \alpha_2, \alpha_{-1}, \alpha_{-2}, d, C \) and \( V \). Solving the over-determined set of algebraic equations by using the symbolic computation software, such as Maple, we obtain

**Set 1:**

\[
    C = 72\lambda d^3 v^2 - 28\lambda^2 d^2 v + 14\lambda^2 d^2 v^2 - 48\mu d^2 v^2 - 2\lambda^3 d v
    - 24v^3 \lambda + 16\mu v^3 d^2 + 12\alpha_0 v^2 d^2 - 24\alpha_0 v d^2 + 12\alpha_0 \lambda d
    + 8\alpha_0 \mu v + 2\lambda^2 v \mu - 72\lambda d^3 v + 48\mu d^2 v^2 - 16\mu d d + 24\lambda d^3
    + 14\lambda^2 d^2 + 72\lambda^2 d^4 + 48\lambda^4 d^4 - 16\mu d + \alpha_0 L^2 - 8\alpha_0 \mu
    + 12\alpha_0 d^2 - 16\mu \lambda v^2 d + 32\mu \lambda v d - 12\alpha_0 \lambda d v + 4\mu^2
    + 12\mu^3 d^4 - 48\lambda^3 d^4 + 2\lambda^3 d + 3\alpha_0^2 - 2\mu \lambda^2 - 8\mu^2 v
    + 4\mu^2 v^2 + 12d^4,
\]

\[
    V = (12v^2 d^2 - 12\lambda v d + 8\mu v - 24v d^2 + 12d \lambda
    + 12d^2 - 8\mu + \lambda^2 + 6\alpha_0),
\]

\[
    \alpha_1 = (4v^2 d - 8v d - 2\lambda v + 2\lambda + 4d), d = d,
\]

\[
    \alpha_0 = (-2v^2 - 2 + 4v), \alpha_{-2} = 0,
\]

\[
    \alpha_{-1} = 0, \alpha_0 = \alpha_0.
\]

(12)

where \( \alpha_0, d, \lambda, \mu \) and \( v \) are arbitrary constants.

**Set 2:**

\[
    C = 72\lambda d^3 v^2 - 28\lambda^2 d^2 v + 14\lambda^2 d^2 v^2 - 48\mu d^2 v^2 - 2\lambda^3 d v
    - 24v^3 \lambda + 16\mu v^3 d^2 + 12\alpha_0 v^2 d^2 - 24\alpha_0 d^2 + 12\alpha_0 \lambda d
    + 8\alpha_0 \mu v + 2\lambda^2 v \mu - 72\lambda d^3 v + 48\mu d^2 v^2 - 16\mu d d + 24\lambda d^3
    + 14\lambda^2 d^2 + 72\lambda^2 d^4 + 48\lambda^4 d^4 - 16\mu d + \alpha_0 L^2 - 8\alpha_0 \mu
    + 12\alpha_0 d^2 - 16\mu \lambda v^2 d + 32\mu \lambda v d - 12\alpha_0 \lambda d v + 4\mu^2
    + 12\mu^3 d^4 - 48\lambda^3 d^4 + 2\lambda^3 d + 3\alpha_0^2 - 2\mu \lambda^2
    - 8\mu^2 v + 4\mu^2 v^2 + 12d^4,
\]

\[
    V = (12v^2 d^2 - 12\lambda v d + 8\mu v - 24v d^2 + 12d \lambda
    + 12d^2 - 8\mu + \lambda^2 + 6\alpha_0),
\]

\[
    \alpha_{-2} = (-4\lambda d^3 - 2\lambda^2 d^2 - 2\mu d^2 + 4\lambda d v - 2\mu^2 - 2d^2 - 4\mu v d^2 + 4v d^4 + 4d^2 + 4\mu \lambda d),
\]

\[
    \alpha_{-1} = (2\lambda^2 d - 6\lambda d^2 v - 8v d^3 + 4v d d - 2\mu \lambda - 4\mu d + 6\lambda d^2 + 4d^3 + 4v^2 d^3),
\]

\[
    d = d, \alpha_0 = \alpha_0, \alpha_1 = 0, \alpha_2 = 0.
\]

\[
    d = 0, \alpha_1 = 0, \alpha_0 = \alpha_0, d = d.
\]

(13)
where $\alpha_0$, $d$, $\lambda$, $\mu$ and $v$ are arbitrary constants.

Set 3:

\[ C = (-16\mu^2 v^2 + 32\mu^2 v + 8\lambda^2 v \mu + 8\alpha_0 \mu v - 8\mu \lambda^2 - \lambda^4 - 16\mu^2 - 8\alpha_0 \mu + 3\alpha_0^2 - 2\alpha_0 \lambda^2), \]

\[ V = (8\mu v - 2\lambda^2 + 6\alpha_0 - 8\mu), d = \frac{\lambda}{2(v - 1)}, \alpha_2 = (-2v^2 - 2 + 4v), \alpha_1 = 0, \alpha_0 = \alpha_0, \]

\[ \alpha_{-2} = -\left\{ \frac{16\mu^2 v^2 - 32\mu^2 v - 8\lambda^2 v \mu + \lambda^4 + 8\mu \lambda^2 + 16\mu^2}{8(v - 1)^2} \right\}, \alpha_{-1} = 0. \quad (14) \]

where $\alpha_0$, $\lambda$, $\mu$ and $v$ are arbitrary constants.

Substituting (12)-(14) into solution (11), we obtain

\[ u_1(x, t) = \alpha_0 + (4v^2 d - 8vd - 2\lambda v + 2\lambda + 4d) \times (d + (G'/G)) \]

\[ + (-2v^2 - 2 + 4v) \times (d + (G'/G))^2 \]

\[ (15) \]

where \[ \xi = x - (12v^2 d^2 - 12\lambda vd + 8\mu v - 24vd^2 + 12d\lambda + 12d^2 - 8\mu + \lambda^2 + 6\alpha_0)t, \]

where $\alpha_0$, $\lambda$, $\mu$ and $v$ are arbitrary constants.

\[ u_2(x, t) = \alpha_0 + (4d^3 - 2\mu \lambda + 2d\lambda^2 - 4d\mu + 6d^2 \lambda + 4d^3 v^2 - 8d^3 v - 6d^2 \lambda v + 4d^2 \mu v) \times (d + (G'/G))^{-1} \]

\[ + (-4\lambda d^3 - 2\lambda^2 d^2 - 2v^2 d^4 + 4\lambda d^3 v - 2\mu^2 - 2d^4 - 4\mu vd^2 + 4vd^4 + 4\mu d^2 + 4\mu \lambda d) \times (d + (G'/G))^{-2} \]

\[ (16) \]

where \[ \xi = x - (12v^2 d^2 - 12\lambda vd + 8\mu v - 24vd^2 + 12d\lambda + 12d^2 - 8\mu + \lambda^2 + 6\alpha_0)t, \]

and $d$, $\lambda$, $\mu$ and $v$ are arbitrary constants.

\[ u_3(x, t) = \alpha_0 + (-2v^2 - 2 + 4v) (4v - 2v^2 - 2) \times \left( \frac{\mu}{(2v - 1)} + (G'/G) \right)^2 \]

\[ - \left\{ \frac{16\mu^2 v^2 - 32\mu^2 v - 8\lambda^2 v \mu + \lambda^4 + 8\mu \lambda^2 + 16\mu^2}{8(v - 1)^2} \right\} \times \left( \frac{\mu}{(2v - 1)} + (G'/G) \right)^{-2} \]

\[ (17) \]

where \[ \xi = x - (8\mu v - 2\lambda^2 + 6\alpha_0 - 8\mu)t, \]

and $d$, $\lambda$, $\mu$ and $v$ are arbitrary constants.

Substituting the solutions $G(x)$ of the Eq. (6) into Eq. (15) and simplifying, we obtain the following solutions:

When $\Omega = \lambda^2 - 4\mu v + 4\mu > 0$ and $\lambda (v - 1) \neq 0$ (or $\mu (v - 1) \neq 0$),

\[ u_{11}(x, t) = (-2v^2 - 2 + 4v) \times \left\{ d - \frac{1}{2v - 1} \left( \lambda + \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right\}^2 \]

\[ + (4v^2 d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d - \frac{1}{2v - 1} \left( \lambda + \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right\} + \alpha_0. \]

\[ (18) \]
\[\xi = x - (12u^2d^2 - 12\lambda ud + 8\mu v - 24ud^2 + 12d\lambda + 12d^2 - 8\mu + \lambda^2 + 6\alpha_0) t,\]

and \(\alpha_0, \lambda, \mu\) and \(v\) are arbitrary constants.

\[u_{12}(x) = (-2v^2 - 2 + 4v) \times \left\{d - \frac{1}{2\sqrt{\lambda^2 + \lambda}} \left(\lambda + \sqrt{\Omega} \cot h(\frac{1}{2}\sqrt{\Omega} \xi)\right)\right\}^2 + (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{d - \frac{1}{2\sqrt{\lambda^2 + \lambda}} \left(\lambda + \sqrt{\Omega} \cot h(\frac{1}{2}\sqrt{\Omega} \xi)\right)\right\} + \alpha_0.\]  

(19)

\[u_{13}(x) = \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{d - \frac{1}{2\sqrt{\lambda^2 + \lambda}} \left(\lambda + \sqrt{\Omega} \left(\tanh(\sqrt{\Omega} \xi) \pm \sec h(\sqrt{\Omega} \xi)\right)\right)\right\}^2 + (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{d - \frac{1}{2\sqrt{\lambda^2 + \lambda}} \left(\lambda + \sqrt{\Omega} \left(\tanh(\sqrt{\Omega} \xi) \pm i \sec h(\sqrt{\Omega} \xi)\right)\right)\right\}.\]  

(20)

\[u_{14}(x) = \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{d - \frac{1}{2\sqrt{\lambda^2 + \lambda}} \left(\lambda + \sqrt{\Omega} \left(\coth(\sqrt{\Omega} \xi) \pm \csc h(\sqrt{\Omega} \xi)\right)\right)\right\}^2 + (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{d - \frac{1}{2\sqrt{\lambda^2 + \lambda}} \left(\lambda + \sqrt{\Omega} \left(\coth(\sqrt{\Omega} \xi) \pm \csc h(\sqrt{\Omega} \xi)\right)\right)\right\}.\]

(21)

where \(A\) and \(B\) are real constants.

\[u_{15}(x,t) = \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{d + \frac{1}{2}\sqrt{\Omega} \left(\tanh(\frac{1}{2}\sqrt{\Omega} \xi) \pm \coth(\frac{1}{2}\sqrt{\Omega} \xi)\right)\right\}^2 + (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{d + \frac{1}{2}\sqrt{\Omega} \left(\tanh(\frac{1}{2}\sqrt{\Omega} \xi) \pm \coth(\frac{1}{2}\sqrt{\Omega} \xi)\right)\right\}.\]  

(22)

\[u_{16}(x,t) = \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{d + \frac{1}{2}\sqrt{\Omega} \left(\tanh(\frac{1}{2}\sqrt{\Omega} \xi) \pm \coth(\frac{1}{2}\sqrt{\Omega} \xi)\right)\right\}^2 + (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{d + \frac{1}{2}\sqrt{\Omega} \left(\tanh(\frac{1}{2}\sqrt{\Omega} \xi) \pm \coth(\frac{1}{2}\sqrt{\Omega} \xi)\right)\right\}.\]

(23)

\[u_{17}(x,t) = \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{d + \frac{1}{2}\sqrt{\Omega} \left(\tanh(\frac{1}{2}\sqrt{\Omega} \xi) \pm \coth(\frac{1}{2}\sqrt{\Omega} \xi)\right)\right\}^2 + (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{d + \frac{1}{2}\sqrt{\Omega} \left(\tanh(\frac{1}{2}\sqrt{\Omega} \xi) \pm \coth(\frac{1}{2}\sqrt{\Omega} \xi)\right)\right\}.\]

(24)
When $\Omega = \lambda^2 - 4 \mu v + 4 \mu < 0$ and $\lambda (v - 1) \neq 0$ (or $\mu (v - 1) \neq 0$),

\[
\begin{align*}
    u_{12}(x,t) &= (-2v^2 - 2 + 4v) \times \left\{ d + \frac{1}{2(v-1)} \left( -\lambda + \sqrt{-\Omega} \tan\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d + \frac{1}{2(v-1)} \left( -\lambda + \sqrt{-\Omega} \tan\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right) \right\} + \alpha_0, \\
\end{align*}
\]

(29)

\[
\begin{align*}
    u_{13}(x,t) &= (-2v^2 - 2 + 4v) \times \left\{ d + \frac{1}{2(v-1)} \left( -\lambda + \sqrt{-\Omega} \cot\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d + \frac{1}{2(v-1)} \left( -\lambda + \sqrt{-\Omega} \cot\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right) \right\} + \alpha_0. \\
\end{align*}
\]

(30)

\[
\begin{align*}
    u_{14}(x,t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d + \frac{1}{2(v-1)} \left( -\lambda + \sqrt{-\Omega} \tan\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \sec\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d + \frac{1}{2(v-1)} \left( -\lambda + \sqrt{-\Omega} \tan\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \sec\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right) \right\}. \\
\end{align*}
\]

(31)

\[
\begin{align*}
    u_{15}(x,t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d + \frac{1}{2(v-1)} \left( -\lambda + \sqrt{-\Omega} \cot\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \csc\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d + \frac{1}{2(v-1)} \left( -\lambda + \sqrt{-\Omega} \cot\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \csc\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right) \right\}. \\
\end{align*}
\]

(32)

\[
\begin{align*}
    u_{16}(x,t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d + \frac{1}{2(v-1)} \left\{ -2\lambda + \sqrt{-\Omega} \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right\} \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d + \frac{1}{2(v-1)} \left\{ -2\lambda + \sqrt{-\Omega} \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right\} \right\}. \\
\end{align*}
\]

(33)

\[
\begin{align*}
    u_{17}(x,t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d + \frac{1}{2(v-1)} \left\{ -\lambda + \sqrt{-\Omega} \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right\} \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d + \frac{1}{2(v-1)} \left\{ -\lambda + \sqrt{-\Omega} \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right\} \right\}. \\
\end{align*}
\]

(34)

\[
\begin{align*}
    u_{18}(x,t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d + \frac{1}{2(v-1)} \left\{ -\lambda + \sqrt{-\Omega} \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right\} \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d + \frac{1}{2(v-1)} \left\{ -\lambda + \sqrt{-\Omega} \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \right\} \right\}. \\
\end{align*}
\]

(35)

where $A$ and $B$ are arbitrary constants such that $A^2 - B^2 > 0$.

\[
\begin{align*}
    u_{19}(x,t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d - \frac{2\mu \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)}{\sqrt{-\Omega} \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) + \lambda \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)} \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d - \frac{2\mu \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)}{\sqrt{-\Omega} \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) + \lambda \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)} \right\}. \\
\end{align*}
\]

(36)

\[
\begin{align*}
    u_{20}(x,t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d + \frac{2\mu \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)}{\sqrt{-\Omega} \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) - \lambda \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)} \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d + \frac{2\mu \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)}{\sqrt{-\Omega} \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) - \lambda \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)} \right\}. \\
\end{align*}
\]

(37)

\[
\begin{align*}
    u_{21}(x,t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d - \frac{2\mu \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)}{\sqrt{-\Omega} \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \lambda \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \sqrt{-\Omega} \xi} \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d - \frac{2\mu \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)}{\sqrt{-\Omega} \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \lambda \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \sqrt{-\Omega} \xi} \right\}. \\
\end{align*}
\]

(38)

\[
\begin{align*}
    u_{22}(x,t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d + \frac{2\mu \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)}{\sqrt{-\Omega} \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \lambda \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \sqrt{-\Omega} \xi} \right\}^2 \\
    &+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d + \frac{2\mu \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right)}{\sqrt{-\Omega} \cos\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \lambda \sin\left( \frac{\lambda}{2} \sqrt{-\Omega} \xi \right) \pm \sqrt{-\Omega} \xi} \right\}. \\
\end{align*}
\]

(39)
When \( \mu = 0 \) and \( \lambda (v - 1) \neq 0 \),

\[
\begin{align*}
\text{Substituting the solutions} & \quad G(\xi) \text{ of the Eq. (6)} \text{ into Eq. (16) and simplifying, we obtain the following solutions:} \\
\text{When } & \quad \Omega = \lambda^2 - 4 \mu v + 4 \mu > 0 \text{ and } \lambda (v - 1) \neq 0 (\text{or } \mu (v - 1) \neq 0), \\
\text{Then } & \quad u_{123}(x, t) = \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d - \frac{\lambda}{(v-1) [(k + \cosh(\lambda \xi)) - \sinh(\lambda \xi)]} \right\}^2 \\
& \quad + (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d - \frac{\lambda}{(v-1) [(k + \cosh(\lambda \xi)) - \sinh(\lambda \xi)]} \right\}.
\end{align*}
\]

\[ u_{124}(x, t) = \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ d - \frac{\lambda}{(v-1) [(k + \cosh(\lambda \xi)) + \sinh(\lambda \xi)]} \right\}^2 \\
+ (4v^2d - 8vd - 2\lambda v + 2\lambda + 4d) \times \left\{ d - \frac{\lambda}{(v-1) [(k + \cosh(\lambda \xi)) + \sinh(\lambda \xi)]} \right\}, \tag{41}
\]

where \( k \) is an arbitrary constant.

When \( (v - 1) \neq 0 \) and \( \lambda = \mu = 0 \), the solution of Eq. (8) is,

\[
\begin{align*}
\text{Similarly, we can write down the other families of exact solutions of Eq. (8) which are omitted for convenience.}
\end{align*}
\]
When $\Omega = \lambda^2 - 4\mu v + 4\mu < 0$ and $\lambda (v - 1) \neq 0$ (or $\mu (v - 1) \neq 0$),

\[ u_{212}(x, t) = (-4\lambda d^3 - 2\lambda^2 d^2 - 2v^2 d^4 + 4\lambda d^3 v - 2\mu^2 - 2d^4 - 4\mu v d^2 + 4v d^4 + 4\mu d^2 + 4\mu \lambda d) \times \left\{ d + \frac{1}{4(v - 1)} \left( -\lambda + \sqrt{-\Omega} \tan\left(\frac{1}{2} \sqrt{-\Omega} \xi\right) \right) \right\}^{-2} + (2\lambda^2 d - 6\lambda d^2 v - 8v d^3 + 4\mu v d - 2\mu \lambda - 4\mu d + 6\lambda d^2 + 4d^3 + 4v^2 d^3) \times \left\{ d + \frac{1}{4(v - 1)} \left( -\lambda + \sqrt{-\Omega} \tan\left(\frac{1}{2} \sqrt{-\Omega} \xi\right) \right) \right\}^{-1} + \alpha_0, \]

\[ u_{213}(x, t) = (-4\lambda d^3 - 2\lambda^2 d^2 - 2v^2 d^4 + 4\lambda d^3 v - 2\mu^2 - 2d^4 - 4\mu v d^2 + 4v d^4 + 4\mu d^2 + 4\mu \lambda d) \times \left\{ d + \frac{1}{4(v - 1)} \left( -\lambda + \sqrt{-\Omega} \cot\left(\frac{1}{2} \sqrt{-\Omega} \xi\right) \right) \right\}^{-2} + (2\lambda^2 d - 6\lambda d^2 v - 8v d^3 + 4\mu v d - 2\mu \lambda - 4\mu d + 6\lambda d^2 + 4d^3 + 4v^2 d^3) \times \left\{ d + \frac{1}{4(v - 1)} \left( -\lambda + \sqrt{-\Omega} \cot\left(\frac{1}{2} \sqrt{-\Omega} \xi\right) \right) \right\}^{-1} + \alpha_0. \]

When $(v - 1) \neq 0$ and $\lambda = \mu = 0$, the solution of Eq. (8) is,

\[ u_{222}(x, t) = (-4\lambda d^3 - 2\lambda^2 d^2 - 2v^2 d^4 + 4\lambda d^3 v - 2\mu^2 - 2d^4 - 4\mu v d^2 + 4v d^4 + 4\mu d^2 + 4\mu \lambda d) \times \left\{ d + \frac{1}{4(v - 1)} \left( -\lambda + \sqrt{-\Omega} \tan\left(\frac{1}{2} \sqrt{-\Omega} \xi\right) \right) \right\}^{-2} + (2\lambda^2 d - 6\lambda d^2 v - 8v d^3 + 4\mu v d - 2\mu \lambda - 4\mu d + 6\lambda d^2 + 4d^3 + 4v^2 d^3) \times \left\{ d + \frac{1}{4(v - 1)} \left( -\lambda + \sqrt{-\Omega} \tan\left(\frac{1}{2} \sqrt{-\Omega} \xi\right) \right) \right\}^{-1} + \alpha_0. \]

where $c_1$ is an arbitrary constant.

We can write down the other families of exact solutions of Eq. (8) which are omitted for convenience. Finally, substituting the solutions $G(\xi)$ of the Eq. (6) into Eq. (17) and simplifying, we obtain the following solutions:

When $\Omega = \lambda^2 - 4\mu v + 4\mu > 0$ and $\lambda (v - 1) \neq 0$ (or $\mu (v - 1) \neq 0$),

\[ u_{31}(x, t) = (-2v^2 - 2 + 4v) \times \left\{ \frac{1}{4(v - 1)} \left( \sqrt{\Omega} \tanh\left(\frac{1}{2} \sqrt{\Omega} \xi\right) \right) \right\}^2 - \left\{ \frac{16\mu v^2 - 32\mu v - 8\lambda^2 \mu + \lambda^4 - 8\mu \lambda^2 + 16\mu^2}{8(v - 1)^2} \right\} \times \left\{ \frac{1}{4(v - 1)} \left( \sqrt{\Omega} \tanh\left(\frac{1}{2} \sqrt{\Omega} \xi\right) \right) \right\}^{-2} + \alpha_0. \]

where $\xi = x - (8\mu v - 2\lambda^2 + 6\alpha_0 - 8\mu)t$, and $\alpha_0$, $\lambda$, $\mu$ and $v$ are arbitrary constants.

\[ u_{32}(x, t) = (-2v^2 - 2 + 4v) \times \left\{ \frac{1}{4(v - 1)} \left( \sqrt{\Omega} \cot h\left(\frac{1}{2} \sqrt{\Omega} \xi\right) \right) \right\}^2 - \left\{ \frac{16\mu v^2 - 32\mu v - 8\lambda^2 \mu + \lambda^4 - 8\mu \lambda^2 + 16\mu^2}{8(v - 1)^2} \right\} \times \left\{ \frac{1}{4(v - 1)} \left( \sqrt{\Omega} \cot h\left(\frac{1}{2} \sqrt{\Omega} \xi\right) \right) \right\}^{-2} + \alpha_0. \]

\[ u_{33}(x, t) = (-2v^2 - 2 + 4v) \times \left\{ \frac{1}{4(v - 1)} \left( \sqrt{\Omega} \left( \tanh\left(\sqrt{\Omega} \xi\right) \pm i \sec h\left(\sqrt{\Omega} \xi\right) \right) \right) \right\}^2 - \left\{ \frac{16\mu v^2 - 32\mu v - 8\lambda^2 \mu + \lambda^4 - 8\mu \lambda^2 + 16\mu^2}{8(v - 1)^2} \right\} \times \left\{ \frac{1}{4(v - 1)} \left( \sqrt{\Omega} \left( \tanh\left(\sqrt{\Omega} \xi\right) \pm i \sec h\left(\sqrt{\Omega} \xi\right) \right) \right) \right\}^{-2} + \alpha_0. \]
For simplicity we omitted others families of exact solutions.

When \( \Omega = \lambda^2 - 4 \mu v + 4 \mu < 0 \) and \( \lambda (v - 1) \neq 0 \) (or \( \mu (v - 1) \neq 0 \)),

\[
\begin{align*}
    u_{3,12}(x, t) &= (-2v^2 - 2 + 4v) \times \left\{ \frac{1}{2(v-1)} \left( \sqrt{-\Omega} \tan \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^2 \\
    &- \left\{ \frac{16\mu^2v^2 - 32\mu^2v - 32\mu^2v + 8\mu^2 + 8\mu^2 + 16\mu^2}{8(v-1)^2} \right\} \times \left\{ \frac{1}{2(v-1)} \left( \sqrt{-\Omega} \tan \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^{-2} + \alpha_0,
\end{align*}
\]

(53)

\[
\begin{align*}
    u_{3,13}(x, t) &= (-2v^2 - 2 + 4v) \times \left\{ \frac{1}{2(v-1)} \left( \sqrt{-\Omega} \cot \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^2 \\
    &- \left\{ \frac{16\mu^2v^2 - 32\mu^2v - 32\mu^2v + 8\mu^2 + 8\mu^2 + 16\mu^2}{8(v-1)^2} \right\} \times \left\{ \frac{1}{2(v-1)} \left( \sqrt{-\Omega} \cot \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^{-2} + \alpha_0.
\end{align*}
\]

(54)

\[
\begin{align*}
    u_{3,14}(x, t) &= \alpha_0 + (-2v^2 - 2 + 4v) \times \left\{ \frac{1}{2(v-1)} \left( \sqrt{-\Omega} \tan \left( \frac{1}{2} \sqrt{-\Omega} \xi \pm \sec \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^2 \\
    &- \left\{ \frac{16\mu^2v^2 - 32\mu^2v - 32\mu^2v + 8\mu^2 + 8\mu^2 + 16\mu^2}{8(v-1)^2} \right\} \times \left\{ \frac{1}{2(v-1)} \left( \sqrt{-\Omega} \tan \left( \frac{1}{2} \sqrt{-\Omega} \xi \pm \sec \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^{-2}.
\end{align*}
\]

(55)

When \((v - 1) \neq 0\) and \(\lambda = \mu = 0\), the solution of Eq. (8) is,

\[
\begin{align*}
    u_{3,15}(x, t) &= (-2v^2 - 2 + 4v) \times \left\{ \frac{1}{2(v-1)} \left( \frac{\lambda}{\xi + c_1} \right) \right\}^2 \\
    &- \left\{ \frac{16\mu^2v^2 - 32\mu^2v - 32\mu^2v + 8\mu^2 + 8\mu^2 + 16\mu^2}{8(v-1)^2} \right\} \times \left\{ \frac{1}{2(v-1)} \left( \frac{\lambda}{(v-1) \xi + c_1} \right) \right\}^{-2} + \alpha_0.
\end{align*}
\]

(56)

where \(c_1\) is an arbitrary constant.

The other families of exact solutions of Eq. (8) are omitted for convenience.

### 3.2 The Banjamin-Ono equation

Consider the Banjamin-Ono equation [42],

\[
    u_t + hu_{xx} + uu_x = 0.
\]

(57)

Making use of the traveling wave transformation \(\xi = x - vt\), Eq. (57) is converted into the following ODE:

\[
    -v u'' + hu'' + \frac{1}{2} u^2 = 0.
\]

(58)

Integrating Eq. (58), we obtain

\[
    C - Vu + hu' + \frac{1}{2} u^2 = 0,
\]

(59)

where \(C\) is a constant of integration. Considering the homogeneous balance between the highest order derivative \(u'\) and the nonlinear term of the highest order \(u^2\) appearing in Eq. (59), we obtain \(N = 1\).

Therefore, the solution of Eq. (57) takes the form,

\[
    u(\xi) = \alpha_{-1} \left( \psi(\xi) \right)^{-1} + \alpha_0 + \alpha_1 \left( \psi(\xi) \right).
\]

(60)

Therefore, executing the similar track of action which is described in sub-section 3.1, we obtain an over-determined set of algebraic equations (for minimalism we leave out to display the equations) for
\(\alpha_0, \alpha_1, \alpha_{-1}, C, d, \) and \(V.\) Solving the set of over-determined algebraic equations using the symbolic computation software, we obtain

**Set 1:**
\[\alpha_{-1} = 0, \alpha_0 = \alpha_0, \alpha_1 = -2h(v-1), d = d, V = h\lambda - 2h\upsilon d + 2hd + \alpha_0\] and
\[C = 2h^2d^2(v^2 - 2v + 1) - 2h^2\lambda d(v - 1) - 2\alpha_0hd(v - 1) + \alpha_0h\lambda + \frac{1}{2}\alpha_0^2 + 2h^2\mu(v - 1) \quad (61)\]

**Set 2:** \(\alpha_1 = 0, \alpha_{-1} = 2hd^2(v - 1) + 2h\mu - 2h\lambda d, \alpha_0 = \alpha_0, V = 2hd(v - 1) - h\lambda + \alpha_0, d = d,\)
\[C = 2h^2d^2(v^2 - 2v + 1) - 2h^2\lambda d(v - 1) + 2\alpha_0hd(v - 1) - \alpha_0h\lambda + \frac{1}{2}\alpha_0^2 + 2h^2\mu(v - 1) \quad (62)\]

**Set 3:** \(C = (8h^2\mu(v - 1) - 2h^2\lambda^2 + \frac{1}{2}\alpha_0^2), \alpha_1 = -2h(v - 1),\)
\[\alpha_0 = \alpha_0, V = \alpha_0, \alpha_{-1} = \frac{h(4\mu v - 4\mu - \lambda^2)}{2(v - 1)}, d = \frac{\lambda}{2(v - 1)}, \quad (63)\]

For Set 1, inserting Eq. (61) and the values of \(\psi(\xi)\) into Eq. (57) and simplifying, we obtain the following:

When \(\Omega = \lambda^2 - 4\mu \upsilon + 4\mu > 0\) and \(\lambda (v - 1) \neq 0\) (or \(\mu (v - 1) \neq 0),\)
\[u_{11}(x, t) = \alpha_0 - 2h(v - 1) \times \left\{ d - \frac{1}{2(v - 1)} \left( \lambda + \sqrt{\Omega} \tanh(\frac{1}{2}\sqrt{\Omega} \xi) \right) \right\} \]
where \(\xi = x - (h\lambda - 2h\upsilon d + 2hd + \alpha_0)t,\) and \(h, d, \alpha_0, \lambda, \mu\) and \(v\) are arbitrary constants.
\[u_{12}(x, t) = \alpha_0 - 2h(v - 1) \times \left\{ d - \frac{1}{2(v - 1)} \left( \lambda + \sqrt{\Omega} \cot h(\frac{1}{2}\sqrt{\Omega} \xi) \right) \right\} \]
\[u_{13}(x, t) = \alpha_0 - 2h(v - 1) \times \left[ d - \frac{1}{2(v - 1)} \left\{ \lambda + \sqrt{\Omega} \left( \tanh(\sqrt{\Omega} \xi) \pm i \sec h(\sqrt{\Omega} \xi) \right) \right\} \right] \]

The other families of exact solutions of Eq. (19) are omitted for convenience.

When \(\Omega = \lambda^2 - 4\mu \upsilon + 4\mu < 0\) and \(\lambda (v - 1) \neq 0\) (or \(\mu (v - 1) \neq 0),\)
\[u_{112}(x, t) = \alpha_0 - 2h(v - 1) \times \left\{ d + \frac{1}{2(v - 1)} \left( -\lambda + \sqrt{-\Omega} \tan(\frac{1}{2}\sqrt{-\Omega} \xi) \right) \right\} \]
\[u_{113}(x, t) = \alpha_0 - 2h(v - 1) \times \left\{ d - \frac{1}{2(v - 1)} \left( \lambda + \sqrt{-\Omega} \cot (\frac{1}{2}\sqrt{-\Omega} \xi) \right) \right\} \]

The other families of exact solutions of Eq. (19) are omitted for convenience.

When \(\mu = 0\) and \(\lambda (v - 1) \neq 0,\)
\[u_{123}(x, t) = \alpha_0 - 2h(v - 1) \times \left\{ d - \frac{\lambda k}{(v - 1)(\cosh(\lambda \xi) - \sinh(\lambda \xi))} \right\} \]
where \(k\) is an arbitrary constant.
When \((v-1) \neq 0\) and \(\lambda = \mu = 0\), the solution of Eq. (19) is,

\[
u_{125}(x, t) = \alpha_0 - 2h(v-1) \times \left\{ d - \frac{1}{(v-1) \xi + c_1} \right\}
\]

where \(c_1\) is an arbitrary constant.

The other families of exact solutions of Eq. (57) are omitted for convenience.

Again for Set 2, substituting Eq. (62) and the values of \(\psi(\xi)\) into Eq. (57) and simplifying, we obtain the following:

When \(\Omega = \lambda^2 - 4 \mu v + 4 \mu > 0\) and \(\lambda (v-1) \neq 0\) (or \(\mu (v-1) \neq 0\)),

\[
u_{12}(x, t) = \{2hd^2(v-1) + 2h\mu + 2h\lambda d\} \times \left\{ d - \frac{1}{2(v-1)} \left( \lambda + \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right\}^{-1} + \alpha_0
\]

where \(\xi = x - \{2hd(v-1) - h\lambda + \alpha_0\} t\), and \(d, \alpha_0, \lambda, \mu\) and \(v\) are arbitrary constants.

\[
u_{22}(x, t) = \{2hd^2(v-1) + 2h\mu - 2h\lambda d\} \times \left\{ d - \frac{1}{2(v-1)} \left( \lambda + \sqrt{\Omega} \cot \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right\}^{-1} + \alpha_0
\]

When \(\Omega = \lambda^2 - 4 \mu v + 4 \mu < 0\) and \(\lambda (v-1) \neq 0\) (or \(\mu (v-1) \neq 0\)),

\[
u_{122}(x, t) = \{2hd^2(v-1) - 2h\mu - 2h\lambda d\} \times \left\{ d + \frac{1}{2(v-1)} \left( -\lambda + \sqrt{-\Omega} \tan \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^{-1} + \alpha_0
\]

\[
u_{223}(x, t) = \{2hd^2(v-1) - 2h\mu + 2h\lambda d\} \times \left\{ d - \frac{1}{2(v-1)} \left( \lambda + \sqrt{-\Omega} \cot \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^{-1} + \alpha_0
\]

When \((v-1) \neq 0\) and \(\lambda = \mu = 0\), the solution of Eq. (19) is,

\[
u_{225}(x, t) = \{2hd^2(v-1) + 2h\mu - 2h\lambda d\} \times \left\{ d - \frac{1}{(v-1) \xi + c_1} \right\}^{-1} + \alpha_0,
\]

where \(c_1\) is an arbitrary constant.

The other families of exact solutions of Eq. (19) are omitted for convenience.

Finally, for Set 3, substituting Eq. (63) and the values of \(\psi(\xi)\) into Eq. (57) and simplifying, we obtain the following:

When \(\Omega = \lambda^2 - 4 \mu v - 4 \mu > 0\) and \(\lambda (v-1) \neq 0\) (or \(\mu (v-1) \neq 0\)),

\[
u_{31}(x, t) = h(4\mu v - 4\mu - \lambda^2) \times \left\{ \lambda - \left( \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right\}^{-1} + \alpha_0 - h \left\{ \lambda - \left( \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right\}.
\]

where \(\xi = x - \alpha_0 t\), and \(\alpha_0, \lambda, \mu\) and \(v\) are arbitrary constants.

\[
u_{32}(x, t) = h(4\mu v - 4\mu - \lambda^2) \times \left\{ \lambda - \left( \sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right\}^{-1} + \alpha_0 - h \left\{ \lambda - \left( \sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right\}
\]

When \(\Omega = \lambda^2 - 4 \mu v - 4 \mu < 0\) and (or),

\[
u_{312}(x, t) = h(4\mu v - 4\mu - \lambda^2) \times \left\{ \left( \sqrt{-\Omega} \tanh \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^{-1} + \alpha_0 - h \left\{ \left( \sqrt{-\Omega} \tanh \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\},
\]

\[
u_{313}(x, t) = -h(4\mu v - 4\mu - \lambda^2) \times \left\{ \left( \sqrt{-\Omega} \tanh \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}^{-1} + \alpha_0 + h \left\{ \left( \sqrt{-\Omega} \tanh \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right\}.
\]

When \((v-1) \neq 0\) and \(\lambda = \mu = 0\), the solution of Eq. (19) is,

\[
u_{325}(x, t) = \frac{h(4\mu v - 4\mu - \lambda^2)}{2(v-1)} \times \left\{ \frac{\lambda}{2(v-1)} - \frac{1}{(v-1) \xi + c_1} \right\}^{-1} + \alpha_0 - 2h(v-1) \left\{ \frac{\lambda}{2(v-1)} - \frac{1}{(v-1) \xi + c_1} \right\}
\]

where \(c_1\) is an arbitrary constant.

The other families of exact solutions of Eq. (19) are omitted for convenience.
4. PHYSICAL EXPLANATION

In this section, we will put forth the physical explanation and the graphical representation of determined travelling wave solutions of the (1+1)-dimensional KdV equation.

4.1 Explanation

1. Solutions (18), (20), (44) and (45) are Cuspon of the (1+1)-dimensional KdV equation (8). Cuspons are other forms of solitons where solution exhibits cusps at their crests. Unlike peakons where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspon diverge. It is important to note that the soliton solution $u(x,t)$, along with its derivatives, tends to zero as
A Note on Novel \((G'/G)\)-expansion Method in Nonlinear Physics

Figure 3. Bell-shape sec\(h^2\) solitary traveling wave solution.

Figure 4. Singular soliton.

\[|x| \to \infty.\] The assumption is that cuspon can be represented as \(u(x, t) = e^{-|x-a|^2}, n > 1.\) It can easily be shown that \(u_x = 0\) at the cusp, and \(u_{xx}, u_{xxx} \ldots \to 0\) to characterize the soliton property. Fig. 1 below shows the cuspon, obtain from solution (18) of the (1+1)-dimensional KdV equation (8) (only shows the shape of solution of Eq. (18) with \(V = 1, \lambda = 1, \mu = -1, v = 2, d = 1, \alpha = 1\) and \(-10 \leq x, t \leq 10\). The shape of figure of solutions (20), (44) and (45) are similar to the figure of solution (18) and so the figure of these solutions is omitted for convenience.

2. Solutions (23), (26)-(28) and (52) are singular cuspon. Fig. 2 shows the shape of singular cuspon of (23) with \(V = 1, \lambda = 1, \mu = -1, v = 2, A = 1, B = 1, d = 1, \alpha = 1\) and \(-10 \leq x, t \leq 10\). The shape of figure of solutions (26)-(28) and (52) are similar to the figure of solution (23) and so the figure of these solutions is omitted for convenience.

3. Solutions (40) and (41) are bell shaped soliton solutions. Fig. 3 shows bell shaped soliton of (40) with \(V = 1, \lambda = 1, \mu = 0, v = -2, k = 1, d = 1, \alpha = 1\) and \(-10 \leq x, t \leq 10\).

4. Solutions (19), (50)-(51) and (21) are singular soliton solutions. Fig. 4 shows the shape of singular soliton of (19) with \(V = 1, \lambda = 1, \mu = -1, v = 2, d = 1, \alpha = 1\) and \(-10 \leq x, t \leq 10\).
5. The fig. 5 represents the silhouette of (22) is singular kink solution with \( V = 1, \lambda = 1, \mu = -1, \nu = 2, d = 1, \alpha = 1 \) and \(-10 \leq x, t \leq 10\).

6. Solutions (24)-(25), (42)-(43), (49) and (56) are soliton solutions. Fig. 6 shows the shape of soliton of (24) with \( V = 1, \lambda = 1, \mu = -1, \nu = 2, A = 1, B = 1, d = 1, \alpha = 1 \) and \(-10 \leq x, t \leq 10\).

7. Solutions of equations (29), (31), (34)-(39), (46)-(48) and (55) represent the exact periodic traveling wave solutions. Periodic solutions are traveling wave solutions that are periodic such as \( \cos(x - t) \). Fig. 7 below shows the periodic solution of Eq. (47). Graph of periodic solution of Eq. (47), for \( V = 1, \lambda = 1, \mu = 1, \nu = 2, d = 1, \alpha = 1 \) with \(-1 \leq x, t \leq 1\). For convenience the figure is omitted.

8. Solutions of equations (30), (32)-(33), (53) and (54) are the exact singular periodic traveling wave solutions. The fig. 8 shows the shape of (30) with \( \lambda = 1, \mu = 1, \nu = 2, d = 2, \alpha = 1 \) with \(-1 \leq x, t \leq 1\).
4.2 Graphical representations of the solutions

The graphical illustrations of the solutions are depicted in the figures 1 to 8 with the aid of commercial software Maple.

5. CONCLUSION

The novel \((G'/G)\)-expansion method has been urbanized and practical to some NLEEs in mathematical physics by means of the \((1+1)\)-dimensional KdV equation and the Banjamin-Ono equation. The performance of the results is consistent, easy and obtained in terms of hyperbolic, trigonometric and rational functions including solitary and periodic solutions. It is noteworthy that our solutions are more general and contain further arbitrary constants and the arbitrary constants imply that these solutions have rich local structures. It is important to notice that the basic \((G'/G)\)-expansion method, the improved \((G'/G)\)-expansion and the generalized and improved \((G'/G)\)-expansion method are only special case of the novel \((G'/G)\)-expansion method and thus the novel \((G'/G)\)-expansion method would be a powerful mathematical tool for solving NLEEs.
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References

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