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ON THE NUMBER OF HAMILTONIAN CYCLES OF $P_4 \times P_n$

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In this paper, we characterize the hamiltonian cycles of the cartesian product $P_4 \times P_n$. From the characterization, we derive a formula expressing the number $H_4(n)$ of hamiltonian cycles of $P_4 \times P_n$ in terms of binomial coefficients. A recurrence relation for $H_4(n)$ is also obtained, from which we determine the explicit and asymptotic values of $H_4(n)$.

1. DEFINITIONS AND NOTATION

Let P_n denote the path on n vertices, and let $H_m(n)$ denote the number of hamiltonian cycles of the cartesian product $P_m \times P_n$. Obviously, $H_2(n) = 1$. It is easy to see that $H_3(2n + 1) = 0$ and $H_3(2n + 2) = 2^n$. Generally, $H_m(n) = H_n(m)$ and $H_m(n) = 0$ if both m and n are odd.

The main purpose of this paper is to derive an explicit formula for the number $H_4(n)$. All definitions concerning graphs are as in Harary[1].

The cartesian product $P_4 \times P_n$ can be embedded in a cartesian plane in the obvious way, with $3(n - 1)$ interior quadrilateral and an exterior face the boundary of which is a cycle of length $2n + 4$. These $3(n - 1)$ interior quadrilaterals form a rectangular array with 3 rows and $n - 1$ columns. Figure 1 shows the cartesian product $P_4 \times P_{12}$, with a hamiltonian cycle drawn in bold lines.

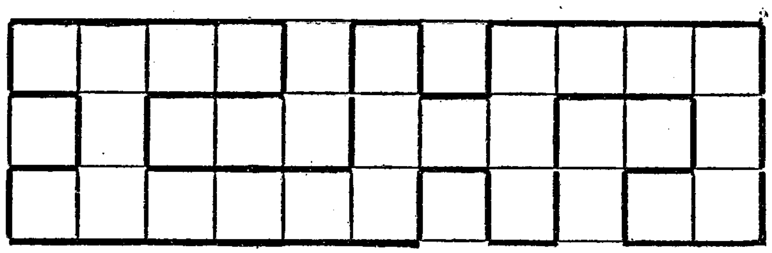

FIG. 1. A hamiltonian cycle of $P_4 \times P_{12}$.

Given any hamiltonian cycle C, label those quadrilaterals lying inside C with 1 and those outside C, along with the exterior face, with 0. Figure 2 shows the labels of the hamiltonian cycle in Fig. 1.

1	1	1	1	0	1	0	1	1	1	1
0	1	0	0	0	1	1	1	0	0	1
1	1	1	1	1	1	0	1	0	1	1

FIG. 2. The labeling of a hamiltonian cycle.

Number the rows 1, 2, 3 from bottom up, and the columns $1, 2, \ldots, n-1$ from left to right. Let b_{ij} denote the label in row i and column j, and let the binary expansion $x_j = b_{3j}\, b_{2j}\, b_{1j}$ be the octal digit formed by the labels in column j. Then, corresponding to C there is a $(n-1)$-string of octal digits $x_1\, x_2 \ldots x_{n-1}$ which we shall call the *signature* of C. Conversely, every hamiltonian cycle C can be recovered from its signature : an edge e of $P_4 \times P_n$ is an edge of C if and only if the faces on the two sides of e are labeled differently. For example, the hamiltonian cycle displayed in Figure 1 can be uniquely identified by the string 57551727457.

2. CHARACTERIZATION OF SIGNATURES

In this section, we shall characterize signatures of hamiltonian cycles.

Theorem 1. Let $\alpha = x_1\, x_2 \ldots x_{n-1}$ be the signature of a hamiltonian cycle C of $P_4 \times P_n$. Then

(a) $x_1 \in \{5, 7\}$ and $x_{n-1} \in \{5, 7\}$;

(b) $x_j \notin \{0, 3, 6\}$ for $1 \leqslant j \leqslant n-1$;

(c) for $1 \leqslant j < n-1$, $x_j \neq x_{j+1}$ unless $x_j = x_{j+1} = 5$, also $\{x_j, x_{j+1}\}$ cannot be both from $\{1, 2, 4, 6\}$;

(d) for $1 < j < n-1$, if $x_j = 2$ then $x_{j-1} = x_{j+1} = 7$;

(e) there exists $i \leqslant k$ such that $x_i = x_k = 7$ and $x_j = 5$ for $1 \leqslant j < i$ and $k < j \leqslant n-1$;

(f) if $x_i = x_k = 7$ for some $1 \leqslant i < k \leqslant n-1$ and $x_j \neq 7$ for $i < j < k$, then there exists a unique j, where $i < j < k$, such that $x_j \in \{1, 2, 4\}$;

(g) if for some $1 \leqslant i < k \leqslant n-1$, we have $\{x_i, x_k\} \subset \{1, 2, 4\}$ and $x_j \notin \{1, 2, 4\}$ for $i < j < k$, then there exists a unique j, where $i < j < k$, such that $x_j = 7$.

Proof : (a) Otherwise, C does not pass through one of the corner vertices of $P_4 \times P_n$; in which case C cannot be hamiltonian, a contradiction.

(b) Certainly $x_i \neq 0$ for $1 \leqslant i \leqslant n-1$, because C must be connected. On the other hand, it is easy to see that in the signature of any hamiltonian cycle, the digits 3 and 6 are adjacent, if they ever exist. So the existence of $x_j \in \{3, 6\}$ implies that $x_{n-1} \in \{3, 6\}$, contradicting (a).

(c) and (d) can be verified easily, again using connectedness of C.

(e) It suffices to consider $x_1 = 5$. It follows from (b), (d) and the connectedness of C that $x_2 \in \{5, 7\}$. We are done if $x_2 = 7$, otherwise repeat the argument on $x_2 = 5$. So it remains to show the existence of $1 \leqslant i \leqslant n-1$ such that $x_i = 7$. If it is false, then $\alpha = 55 \ldots 5$, and C is not connected.

(f) If $k = i + 2$, then $x_{i+1} = 2$. So we may assume $k > i + 2$, which allows us to conclude from (b) and (d) that $x_j \in \{1, 4, 5\}$ for $i < j < k$. Now C is disconnected if all digits are 5 or if there are two digits from $\{1, 4\}$ between x_i and x_k.

(g) It is clear that $x_j \in \{5, 7\}$ for $i < j < k$. Since C is connected, there is at least one $i < j < k$ with $x_j = 7$. Its uniqueness follows from (c) and (f).

Theorem 2. Any string of octal digits $\alpha = x_1 x_2 \ldots x_{n-1}$ satisfying conditions (a) through (g) of Theorem 1 is the signature of a hamiltonian cycle of $P_4 \times P_n$.

Proof : Induct on $n > 1$. It is trivial if $n = 2$ or $n = 3$. Assume it is true up to $n - 1$ where $n > 3$. From condition (a), we have $x_1 \in \{5, 7\}$. If $x_1 = 5$, then $x_2 \in \{5, 7\}$ because of (e). Then $\beta = x_2 x_3 \ldots x_{n-1}$ satisfies conditions (a) through (g), so it is the signature of a hamiltonian cycle of $P_4 \times P_{n-1}$. If $x_1 = 7$, it follows from conditions (b) and (c) that $x_2 \in \{1, 2, 4, 5\}$. Then as a consequence of (b), (c) and (e), we have $x_3 \in \{5, 7\}$, and $\gamma = x_3 x_4 \ldots x_{n-1}$ is the signature of a hamiltonian cycle of $P_4 \times P_{n-2}$. In either case, $\alpha = 5\beta$ or $\alpha = 7x_2\gamma$ is the signature of a hamiltonian cycle of $P_4 \times P_n$.

3. Counting Hamiltonian Cycles of $P_4 \times P_n$

Now we are going to derive a closed form for $H_4(n)$.

Theorem 3. For $n \geqslant 1$,

$$H_4(n) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{k=1}^{\lceil (n-1-2i)/2 \rceil} 2^{k-1} \binom{n-1-2i}{2k-1} \binom{i+k-1}{k-1},$$

where $\lfloor x \rfloor$ ($\lceil x \rceil$) is the greatest (least) integer not greater (less) than x.

Proof : Consider the signature $\alpha = x_1 x_2 \ldots x_m$ of a hamiltonian cycle in which $x_i \neq 2$ for $1 \leqslant i \leqslant m$. If there are k appearances of the digit 7 in α, then it

Substituting $y = z - 2/3$ transforms (2) into $z^3 - 4z/3 + 232/27$. According to Cardan's formula (see, for example, Uspensky[2], p. 85), it has a zero of $2(\mu + \nu)/3$, where

$$\mu = \left(\frac{-29 + 3\sqrt{93}}{2} \right)^{1/3} \quad \text{and} \quad \nu = \left(\frac{-29 - 3\sqrt{93}}{2} \right)^{1/3}.$$

We have, therefore, as a solution to (2), $y = 2(\mu + \nu - 1)/3$. Define

$$K = \sqrt{y+3} = \sqrt{\frac{2(\mu+\nu)+7}{3}}$$

so that $K^2 = y + 3$ and (1) can now be rewritten as

$$\left(x^2 - x + \frac{K^2 - 3}{2} \right)^2 = K^2 \left(x - \frac{K^2 - 1}{2K^2} \right)^2$$

which reduces to the product of two quadratic equations

$$x^2 - (1 \pm K)\, x - 1 + \frac{(K^2 - 1)(K \pm 1)}{2K} = 0.$$

Let G and H be the discriminants of these two quadratic equations:

$$G = 4 - (1 + K)^2 \left(1 - \frac{2}{K} \right) \text{ and } H = 4 - (1 - K)^2 \left(1 + \frac{2}{K} \right).$$

Then, $F(x) = 0$ has four distinct roots

$$\frac{(1 + K) \pm \sqrt{G}}{2} \text{ and } \frac{(1 - K) \pm \sqrt{H}}{2}. \qquad \ldots(3)$$

Theorem 7. The exact value of $H(n)$ is

$$H(n) = \sum_{i=1}^{4} \frac{\alpha_i}{F'(\alpha_i)} \alpha_i^n,$$

where α_i are the zeros of $F(x)$ as given in (3).

Proof : Let α_i be the four distinct zeros of $F(x)$. It is a routine exercise to derive the generating function of $H(n)$:

$$\sum_{n=0}^{\infty} H(n)\, x^n = \frac{x^2}{1 - 2x - 2x^2 + 2x^3 - x^4} = \frac{x^2}{\prod_{i=1}^{4} (1 - \alpha_i x)}$$

$$= \sum_{i=1}^{4} \frac{A_i}{1 - \alpha_i x} = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{4} A_i \alpha_i^n \right) x^n.$$

Proof : The initial conditions can be verified by inspection. For $n \geqslant 3$, we count the number of signatures $x_1 x_2 \ldots x_{n-1}$ of hamiltonian cycles of $P_4 \times P_n$ with $x_1 = 7$.

First of all, note that there is only one cycle with signature 755...5 and $h(n-2)$ cycles whose signatures satisfy $x_1 = 7$ and $x_2 = 2$. In all other cases, there exists $2 \leqslant k \leqslant n-2$, such that $x_k \in \{1, 4\}$, $x_i = 5$ for $2 \leqslant i < k$ and $x_{k+1} \in \{5, 7\}$. Then $x_{k+1} x_{k+2} \ldots x_{n-1}$ is the signature of a hamiltonian cycle of $P_4 \times P_{n-k}$. Conversely, given k, where $2 \leqslant k \leqslant n-2$, and the signature $x_{k+1} x_{k+2} \ldots x_{n-1}$ of a hamiltonian cycle of $P_4 \times P_{n-k}$ we can construct a hamiltonian cycle in $P_4 \times P_n$ with signature $7x_2 x_3 \ldots x_k x_{k+1} \ldots x_{n-1}$ where $x_k \in \{1, 4\}$ and $x_i = 5$ for $2 \leqslant i < k$. Hence, there are $2H(n-k)$ such signatures. The recurrence relation is now established.

Theorem 6. The numbei $H(n)$ of hamiltonian cycles of $P_4 \times P_n$ satisfies the initial conditions $H(1) = 0$, $H(2) = 1$, $H(3) = 2$, $H(4) = 6$ and the recurrence relation

$$H(n) = 2H(n-1) + 2H(n-2) - 2H(n-3) + H(n-4) \quad \text{for } n \geqslant 5.$$

Proof : Using Lemma 5, for $n \geqslant 5$, we have

$$\begin{aligned} h(n) &= h(n-2) + 2H(n-2) - h(n-3) + 1 + h(n-3) \\ &\quad + 2 \sum_{k=2}^{n-3} H(k) \\ &= h(n-2) + 2H(n-2) - h(n-3) + h(n-1). \end{aligned}$$

Now, substitution from Lemma 4 leads to the desired recurrence relation.

To calculate the exact value of $H(n)$, we have to solve $F(x) = 0$, where

$$F(x) = x^4 - 2x^3 - 2x^2 + 2x - 1$$

is the characteristic equation of $H(n)$. Here, we outline the algebraic solution due to Ferrari (see, for example, Uspensky[2], pp. 94-95). First of all, write $F(x) = 0$ as

$$x^4 - 2x^3 = 2x^2 - 2x + 1.$$

Add $x^2 + (x^2 - x)y + y^2/4$ to both sides of the equation to obtain

$$\left(x^2 - x + \frac{y}{2} \right)^2 = (y+3)x^2 - (y+2)x + \frac{y^2+4}{4}. \qquad \ldots(1)$$

We shall choose y such that the right hand side of (1) is also a complete square. We need $(y+2)^2 = (y+3)(y^2+4)$. That is, we have to solve the cubic equation

$$y^3 + 2y^2 + 8 = 0. \qquad \ldots(2)$$

follows from Theorem 1 (f) that there are $k-1$ occurrences of digits from {1, 4}, one between each pair of successive 7's. Thus, there are $2^{k-1}\binom{m}{2k-1}$ choices for α.

Now the signature $\beta = x_1 x_2 \ldots x_{n-1}$ of a hamiltonian cycle of $P_4 \times P_n$ can have i appearances of the digit 2 in it, and every occurrence of 2 implies an occurrence of the substring 727 in β, according to Theorem 1 (d). Replacing 727 in β by 7 reduces the length of β by two. So β can be transformed into the signature $y_1 y_2 \ldots y_{n-1-2i}$ of a hamiltonian cycle in which $y_j \neq 2$ for $1 \leqslant j \leqslant n-1-2i$. Conversely, i successive replacements of 7 by 727 transforms the signature $\gamma = z_1 z_2 \ldots z_{n-1-2i}$ to a signature $w_1 w_2 \ldots w_{n-1}$ with exactly i digits 2. There are $2^{k-1}\binom{n-1-2i}{2k-1}$ choices of γ with exactly k occurrences of 7. Among these k occurrences of 7, there are $\binom{i+k-1}{k-1}$ ways to apply i replacements of 7 by 727. Therefore, there are

$$\sum_{k=1}^{\lceil (n-1-2i)/2 \rceil} 2^{k-1}\binom{n-1-2i}{2k-1}\binom{i+k-1}{k-1}$$

choices for β. The proof is now completed since $0 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor$.

Although Theorem 3 expresses $H_4(n)$ in closed form, it is still rather difficult to compute $H_4(n)$. We next derive a linear recurrence relation for $H_4(n)$, from which its exact and asymptotic values can be derived. For brevity, let $H(n)$ denote $H_4(n)$.

Lemma 4. The number $H(n)$ of hamiltonian cycles of $P_4 \times P_n$ satisfies the initial condition $H(1) = 0$ and the recurrence relation

$$H(n) = H(n-1) + h(n) \quad \text{for } n \geqslant 2$$

where $h(n)$ is the number of hamiltonian cycles of $P_4 \times P_n$ with signature beginning with the digit 7.

PROOF : From Theorems 1 and 2, if $x_1 \neq 7$ then $x_1 = 5$ and $x_2 x_3 \ldots x_{n-1}$ is the signature of a hamiltonian cycle of $P_4 \times P_{n-1}$. Conversely, given the signature $x_2 x_3 \ldots x_{n-1}$ of any hamiltonian cycle of $P_4 \times P_{n-1}$, there exists a hamiltonian cycle of $P_4 \times P_n$ with signature $5\, x_2 x_3 \ldots x_{n-1}$. This lemma is now proved.

Lemma 5. The function $h(n)$ satisfies the initial conditions $h(1) = 0$, $h(2) = 1$, $h(3) = 1$, and the recurrence relation

$$h(n) = 1 + h(n-2) + 2\sum_{k=2}^{n-2} H(k) \quad \text{for } n \geqslant 4.$$

for some constants A_i, $1 \leqslant i \leqslant 4$. Obviously, $\alpha_1^{-2} = A_1 \prod_{i=2}^{4} (1 - \alpha_i \alpha_1^{-1})$; or equivalently

$$\alpha_1 = A_1 \prod_{i=2}^{4} (\alpha_1 - \alpha_i) = A_1 F'(\alpha_1).$$

Values of A_2, A_3 and A_4 can be determined similarly.

Numerically, we have

$$\alpha_1 = 2.539,\ \alpha_2 = 1.276 \text{ and } \alpha_3, \alpha_4 = 0.369 \pm 0.416i.$$

Since $|\alpha_3| = |\alpha_4| < 1$, we have

Theorem 8. Let α_1 and α_2 be the real zeros of $F(x)$. Then, asymptotically

$$H(n) \sim \frac{\alpha_1}{F'(\alpha_1)} + \frac{\alpha_2}{F'(\alpha_2)} \approx 0{\cdot}136\,(2.539)^n + 0.116\,(-1.276)^n.$$

REFERENCES

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2. J. V. Uspensky, *Theory of Equations*, McGraw-Hill, New York, 1948.