Convergence to equilibrium for some nonlinear evolution equations with dynamical boundary condition

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In this short note we present some convergence results of solutions to certain equilibrium for some nonlinear evolution equations subject to dynamical boundary condition by using the Łojasiewicz-Simon approach. The work was done in collaboration with S. Zheng and M. Grasselli.

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1 Introduction

The asymptotic behavior of global solutions of nonlinear evolution equations, in particular convergence to a certain equilibrium as time tends to infinity has become one of the main concerns in the field of nonlinear evolution equations since 1980s.

For one space dimension case, significant progresses have been made (see e.g., [6]). However, the situation in higher space dimension case is much more complicated. There are counterexamples (e.g., [7]) showing that even if the nonlinearity of a semilinear parabolic equation belongs to $C^\infty$, the $\omega$-limit set of its bounded global solution could be diffeomorphic to the unit circle $S^1$. Many assumptions have been made to ensure the convergence for bounded global solutions in the literature. In 1983, L. Simon [10] made the breakthrough that if the nonlinearity of a nonlinear parabolic equation is analytic in the dependent variable, then the convergence holds. His idea relies on the extension of a gradient inequality by S. Łojasiewicz for analytic functions defined in $\mathbb{R}^n$ (Ref. [5]) to the infinite-dimensional space. Since then, a lot of work has been done in this direction (see e.g., [1–4, 9, 13] and the references therein). However, most of the previous work is concerned with evolution equations subject to homogeneous boundary conditions. For many nonlinear evolution equations with other type boundary conditions (for instance, the dynamical boundary condition) which are very important from the physical point of view, the framework in the previous literature as well as the Łojasiewicz-Simon inequality which plays a crucial role cannot apply directly.

In this short note we present some results on the study of convergence of global solutions for some nonlinear evolution equations with dynamical boundary condition. Under the basic assumption that the nonlinearity is analytic with respect to the dependent variable, we develop several new Łojasiewicz-Simon type inequalities (with boundary term) which vary from problem to problem (Ref. [11, 12, 14–16]) and obtain the convergence result as well as the estimates for convergence rate.

2 Main Results

1. We consider the Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} = \Delta (-\Delta u - u + u^3), \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad (1)$$

with dynamical boundary condition

$$\sigma_s \Delta ||u - \partial_\nu u + h_s - g_s u = \frac{1}{\Gamma_s} u_t, \quad \partial_\nu (-\Delta u - u + u^3) = 0, \quad (x, t) \in \Gamma \times \mathbb{R}^+, \quad (2)$$

and the initial condition

$$u|_{t=0} = u_0(x). \quad (3)$$

$\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \leq 3$) with smooth boundary $\Gamma$, and $\Gamma_s > 0, \sigma_s > 0, g_s > 0, h_s$ are given constants; $\Delta_\Gamma$ is the Laplace-Beltrami operator on $\Gamma$, and $\nu$ is the outward normal direction to the boundary. The global existence and uniqueness result was proven in [8]. Introduce the Hilbert space $V = C^1(\bar{\Omega})$: $(u, v)_V = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Gamma (\sigma_s \nabla ||u \cdot \nabla v + g_s uv) \, ds$.

Our main result is

Theorem 2.1 (Ref. H. Wu and S. Zheng [15]) For any initial datum $u_0 \in V$, the global solution $u(x, t)$ to problem (1)–(3) converges to an equilibrium $\psi(x)$,

$$\|u - \psi\|_{\mu^1(\Omega)} \leq C(1 + t)^{-\theta/(1-2\theta)}, \quad t \geq 1. \quad (4)$$

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Here, $C \geq 0$, $\psi(x)$ is an equilibrium to problem (1)–(3), i.e., a solution to the following nonlinear boundary value problem:

$$-\Delta \psi - \psi + \psi^3 = \text{Const.}, \quad x \in \Omega, \quad \sigma_s \Delta \psi - \partial_\nu \psi + h_s - g_s \psi |_{\Gamma} = 0, \quad \int_{\Omega} \psi(x) dx = \int_{\Omega} u_0(x) dx,$$  \hspace{1cm} (5)

and $\theta \in (0, \frac{1}{2})$ is a constant depending on $\psi(x)$.

2. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\Gamma$ and $\nu$ is the outward normal direction to the boundary. Suppose that a two-phase stress-free material occupies $\Omega$, for any time $t \geq 0$. Denote by $\theta$ its (relative) temperature and by $\chi$ the order parameter (or phase-field function). We consider the parabolic-hyperbolic phase-field system

$$\begin{cases}
(\theta + \chi_t) - \Delta \theta = 0, & (x, t) \in \Omega \times \mathbb{R}^+, \\
\chi_{tt} + \chi_t - \Delta \chi + \phi(\chi) - \theta = 0, & (x, t) \in \Omega \times \mathbb{R}^+,
\end{cases}$$  \hspace{1cm} (6)

with dynamical boundary condition (for $\chi$)

$$\begin{cases}
\partial_\nu \theta = 0, & (x, t) \in \Gamma \times \mathbb{R}^+, \\
\partial_\nu \chi_t + \chi_t = 0, & (x, t) \in \Gamma \times \mathbb{R}^+,
\end{cases}$$  \hspace{1cm} (7)

and the initial conditions

$$\begin{align*}
\theta|_{t=0} &= \theta_0(x), & \chi|_{t=0} &= \chi_0(x), & \chi_t|_{t=0} &= \chi_1(x), & x \in \Omega.
\end{align*}$$  \hspace{1cm} (8)

Assumptions on the nonlinearity $\phi$ are:

(H1) $\phi(s)$ is analytic in $\mathbb{R}$;
(H2) there exists $c > 0$ such that $|\phi''(s)| \leq c(1 + |s|), \quad \forall s \in \mathbb{R}$;
(H3) $\liminf_{|s| \to \infty} \frac{\phi(s)}{|s|^{\lambda+1}} > -\lambda$, where $\lambda > 0$ such that $\int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} u^2 dS \geq \lambda \int_{\Omega} u^2 dx$.

Define

$$\mathcal{H} = H^1(\Omega) \times L^2(\Omega) \quad \text{and} \quad \mathcal{D} = \{(\chi, u)^{tr} \in H^2(\Omega) \times H^1(\Omega) \mid \partial_\nu \chi + u |\Gamma = 0\}.$$  \hspace{1cm}

Our main result is

**Theorem 2.2** (Ref. H. Wu, M. Grasselli and S. Zheng [14]) Suppose that assumptions (H1)–(H3) are satisfied. Then, for any initial data $(\theta_0, \chi_0, \chi_1)^{tr} \in H^1(\Omega) \times \mathcal{D}$, problem (6)–(8) admits a unique global solution $(\theta, \chi, \chi_t)^{tr}$ such that

$$\theta \in C([0, \infty); H^1(\Omega)), \quad \theta_t \in L^2((0, \infty); L^2(\Omega)), \quad (\chi, \chi_t)^{tr} \in C([0, \infty); \mathcal{D}) \cap C^1([0, \infty); \mathcal{H}).$$  \hspace{1cm} (9)

Moreover, $(\theta, \chi, \chi_t)^{tr}$ converges to an equilibrium $(\theta_\infty, \psi, 0)^{tr}$ as time goes to infinity, i.e., for all $t \geq 0$,

$$\|\theta(t) - \theta_\infty\|_{H^1(\Omega)} + \|\chi(t) - \psi\|_{H^1(\Omega)} + \|\chi_t(t)\|_{L^2(\Omega)} \leq C(1 + e^{-\sigma/(1-2\sigma)}).$$  \hspace{1cm} (10)

Here $C \geq 0$, $(\theta_\infty, \psi)^{tr}$ is a classical solution to the following nonlinear nonlocal elliptic boundary value problem

$$\begin{cases}
-\Delta \psi + \phi(\psi) - \left( -m - \frac{1}{|\Omega|} \int_{\Omega} \psi dx \right) = 0, \\
\partial_\nu \psi + \psi |\Gamma = 0, \\
\theta_\infty = m - \frac{1}{|\Omega|} \int_{\Omega} \psi dx
\end{cases}$$  \hspace{1cm} (11)

where $m = \frac{1}{|\Omega|} \int_{\Omega} (\theta_0 + \chi_0) dx$. $\sigma \in (0, \frac{1}{2})$ is a constant depending on $(\theta_\infty, \psi)^{tr}$.

**Remark 2.3** The key tools to prove the above results are extended Łojasiewicz - Simon inequalities involving boundary term (Ref. [15, Lemma 3.4] and [14, Lemma 4.4]). Moreover, our approach can apply to other nonlinear evolution equations with complex boundaries, for instance, the damped semilinear wave equation with critical growth exponent and dissipative boundary condition [16]; semilinear parabolic equation with dynamical boundary condition [11]; Cahn - Hilliard equation with the Wentzell boundary condition [12].

**References**