Invariant manifolds for nonsmooth systems with sliding mode

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Abstract

Invariant manifolds play an important role in the study of Dynamical Systems, since they help to reduce the essential dynamics to lower dimensional objects. In that way, a bifurcation analysis can easily be performed. In the classical approach, the reduction to invariant manifolds requires smoothness of the system which is typically not given for nonsmooth systems. For that reason, techniques have been developed to extend such a reduction procedure to nonsmooth systems. In the present paper, we present such an approach for systems involving sliding motion. In addition, an analysis of the reduced equation shows that the generation of periodic orbits through nonlinear perturbations which is usually related to Hopf bifurcation follows a different type of bifurcation if nonsmooth elements are present, since generically symmetry is broken by the nonsmooth terms.

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1. Introduction

The use of invariant manifolds, in particular of center manifolds, has been established as a key tool for analyzing the dynamics of high dimensional dynamical systems, since it allows a reduction to systems of low dimension (see [5]). Following that approach, a bifurcation analysis can be carried out with the aim, for example, to study the generation of periodic orbits.

For linear systems, the reduction corresponds to a decomposition into invariant subspaces characterized by the eigenvalues; the notion of invariant manifolds has been developed as the analog for nonlinear systems. To carry out the reduction, appropriate smoothness of the system is required, hence, that approach fails for nonsmooth systems. Nevertheless, it has been possible to define invariant sets for nonsmooth systems as well (see [11,12,16]). For planar systems there is no need for a reduction; Hence, planar piecewise smooth systems have been studied first for example in [8,15,17] to understand the modifications due to nonsmoothness. For piecewise linear systems, they appear as the surface of invariant cones, hence, two-dimensional objects which can be used to describe the dynamics of the full

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systems. In relevant cases, the cones are attractive, but the dynamics on the cone can be asymptotically stable, stable or unstable. The existence of such invariant cones for piecewise linear systems consisting of periodic orbits has already been observed in [1–4]. For smooth systems, these cones become flat, hence, degenerate to planes. The flow within the plane, for example, may change from a stable focus through a center to an unstable focus if an appropriate parameter is varied. In the case of linear systems, this mechanism provides the basis for Hopf bifurcation; if the system is perturbed by higher order terms periodic orbits might occur (see [9,14]).

In [16], this approach has been generalized to piecewise smooth systems. Starting with a piecewise linear system as basic system, it has been shown that the corresponding invariant cones will be deformed to a cone-like surface if higher order terms are added. Using standard techniques such as the Hadamard-Graph-Transform, the existence of such an invariant “manifold” has been established, but in the first approach only for systems without sliding motion. Since sliding motion is a key feature of nonsmooth systems, that setting has to be taken into account. Although, the dimension of the flow is reduced by the sliding part another difficulty arises as the flow within the sliding area is nonlinear even for piecewise linear systems. To set up an appropriate basic system, it helps that its flow is always homogeneous. In addition, situations can be characterized, where the sliding flow is linear. For such special cases the extension of the manifold notion is straightforward.

In the present paper, we will show that the existence of invariant lower dimensional manifolds capturing the essential dynamics can also be obtained, for systems involving sliding. Using the reduced system, bifurcation analysis can be performed by analyzing a one-dimensional Poincaré-map defined on an invariant curve, where a non-trivial fixed point corresponds to an periodic orbit of the original system. At the first glance, this appears as a direct generalization of Hopf bifurcation. A detailed analysis of the Poincaré-map though elucidates some new features. Due to inherent symmetries of smooth systems in the case of Hopf bifurcation, it is possible to eliminate all nonlinear terms of even order so that generically third order terms turn out to be dominating for bifurcation (see for example [9,14]). For nonsmooth systems that kind of symmetry is broken, implying that already quadratic terms determine the bifurcation behavior.

More precisely, in [16], we proved that under certain attractivity and transversality conditions (see Section 3) piecewise nonlinear systems (PWNS) of the form

\[ \dot{\xi} = \begin{cases} A^+ \xi + g_+ (\xi), & n^T \xi > 0, \\ A^- \xi + g_- (\xi), & n^T \xi < 0 \end{cases} \]

(1)

with constant matrices \( A^\pm \) and nonlinear \( C^k \)-parts \( g_\pm (\xi) = o(\| \xi \|), k \geq 1 \), exhibit an invariant cone-like surface if the corresponding piecewise linear system (PWLS)

\[ \dot{\xi} = \begin{cases} A^+ \xi, & n^T \xi > 0, \\ A^- \xi, & n^T \xi < 0 \end{cases} \]

(2)

possesses an invariant cone. The analysis in [16] was done in case of direct crossing, i.e., the dynamics on the cone does not involve sliding motion.

We proceed as follows: In Section 2, we characterize and study the system governing the dynamics in case of sliding mode, for piecewise linear (PWLS) and piecewise nonlinear systems (PWNS). Section 3 is dedicated to review shortly the situation without sliding mode, which is already treated in [16]. Supplementary to the conclusions in [16], we discuss, in Section 4, similar results in presence of sliding motion, which are proven in Section 6. In Section 5, we analyze the generation of periodic solution using the derived results. We compare the mechanism with the well-understood Hopf bifurcation in smooth systems and point out the crucial differences.

2. Sliding motion

2.1. Sliding mode in PWLS

We consider piecewise linear systems (2) in \( \mathbb{R}^N \) consisting of 2 components separated by the hyperplane \( \mathcal{M} = \{ \xi \in \mathbb{R}^N \mid n^T \xi = 0 \} \) with constant matrices \( A^\pm \) and normal vector \( n^T \). Using \( \rho(\xi) = n^T A^+ \xi \cdot n^T A^- \xi \), we define the sliding
mode set \( \mathcal{M}^{t} = \{ \xi \in \mathcal{M} \mid \rho(\xi) < 0 \} \). The flow through \( \xi \in \mathcal{M}^{t} \) is restricted to the separation manifold \( \mathcal{M} \) in forward time, not necessarily in backward time. In this case, the dynamics is governed by Filippov’s extension (see [6,7]):

\[
\dot{\xi} = \frac{1}{n^T[A^- - A^+]} [n^T A^- \xi] A^+ \xi - (n^T A^+ \xi) A^- \xi = f_\delta(\xi). \tag{3}
\]

It is remarkable, that in case of sliding, i.e. \( \rho(\xi) < 0 \), the denominator in (3) does not vanish.

The sliding mode set is further classified as attractive or repulsive:

\[ \mathcal{M}_a := \{ \xi \in \mathcal{M}^t \mid n^T A^+ \xi < 0 \}, \]
\[ \mathcal{M}_r := \{ \xi \in \mathcal{M}^t \mid n^T A^+ \xi > 0 \}. \]

If \( \xi \in \mathcal{M}_a \), the vector field of both systems at \( \xi \) points toward \( \mathcal{M}^t \); hence, the flow cannot leave \( \mathcal{M}^t \) at \( \xi \). The set \( \mathcal{M}_a \) is called attractive sliding area. If \( \xi \in \mathcal{M}_r \), both vector fields are directed away from \( \mathcal{M}^t \) at \( \xi \). Hence, the flow in forward time is not uniquely defined at \( \xi \), and \( \mathcal{M}_r \) is called repulsive. Therefore, we just consider dynamics involving attractive sliding mode.

Finally, we define \( \mathcal{M}_a^0 \) and \( \mathcal{M}_r^0 \) as subsets of \( \mathcal{M} \) where the normal vector \( n^T \) is orthogonal to the vector field of the minus and plus system, respectively. In case of \( \mathcal{M}_a^0 = \mathcal{M}_r^0 \), there is no sliding motion at all, or the whole separation hyperplane consists of sliding mode. In the latter case, the dynamics in (3) is linear, since \( n^T A^\xi = -\kappa \cdot n^T A^- \xi \) for a \( \kappa > 0 \). We consider the case \( \mathcal{M}_a^0 \neq \mathcal{M}_r^0 \). The following theorem gives, in this case, a characterization of the right hand side of the Filippov system (3).

**Theorem 1.** Let \( \mathcal{M}_a^0 \neq \mathcal{M}_r^0 \). The right hand side \( f_\delta(\xi) \) of (3) is linear if and only if there exist vectors \( v_1, v_2 \in \mathbb{R}^N \) with

\[ (A^- - A^+)(I - nn^T) = v_1 v_2^T. \]

In any case, \( f_\delta(\xi) \) is positive homogeneous, i.e., \( f_\delta(\rho\xi) = \rho f_\delta(\xi) \) for all \( \rho > 0 \).

**Proof.** Without loss of generality, we assume \( n = e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^N \) and

\[ e_1^T A^+ = (a^+_{11}, -1, 0, \ldots, 0). \]

Hence, for \( \xi \in \mathcal{M} \), i.e. \( \xi_1 = 0 \), system (3) is equivalent to

\[
\dot{\xi} = A^+ \xi - n^T A^+ \xi (A^- - A^+) \xi \\
= A^+ \xi + \frac{\dot{\xi}_2}{v_2^T \xi} (A^- - A^+) (I - nn^T) \xi
\]

with \( v_2 := n^T [A^- - A^+] \). Because of \( \mathcal{M}_a^0 \neq \mathcal{M}_r^0 \), the vectors \( e_2 \) and \( v_2 \) are linearly independent. Hence, the dynamics is linear if and only if there exists a vector \( v_1 \in \mathbb{R}^N \) with

\[ (A^- - A^+) (I - nn^T) \xi = v_1 v_2^T \xi \]

for all \( \xi \in \mathcal{M} \).

Easy, but important observations about general positive homogeneous systems are formulated in the following lemma:

**Lemma 1.** Consider \( \dot{\xi} = f(\xi) \) with smooth and positive homogeneous right hand side \( f \) (as above), i.e., \( f(\rho\xi) = \rho f(\xi) \) for \( \rho > 0 \), and let \( Y(t) = \partial_\xi \phi(t, \xi) \) be the solution of the variational equation \( \dot{Y} = Df(\phi(t, \xi))Y \) with initial value \( Y(0) = I \). Then, the flow \( \phi(t, \xi) \) of the system maps half-rays into half-rays, i.e.,

\[ \phi(t, \rho \xi) = \rho \phi(t, \xi), \quad \rho > 0. \]

More precisely, we have

\[ \phi(t, \xi) = Y(t) \xi. \]
Proof. Obviously, \( \rho \varphi(t, \xi) \) solves the differential equation with initial value \( \rho \xi \). Hence, \( \varphi(t, \rho \xi) = \rho \varphi(t, \xi) \). Differentiating this equation with respect to \( \rho \), gives

\[
\frac{\partial}{\partial \xi} \varphi(t, \rho \xi) \xi = \varphi(t, \xi),
\]

where the left hand side is independent of \( \rho \) and by definition equal to \( Y(t) \).

Example 1. A dimensionless version of the brake system introduced in [12], in case of simple Coulomb friction characteristic (see plot below for the friction force), is given by

\[
\begin{align*}
\dot{x}_1 + d \dot{x}_1 + cx_1 &= -\bar{\mu}_2 \text{sgn}(\dot{x}_1 - a \dot{x}_2) c_3 x_2, \\
\dot{x}_2 - d \bar{\mu}_1 \dot{x}_1 - c \bar{\mu}_1 x_1 + c_3 x_2 &= 0, \\
\dot{\varphi} + b d \varphi + b c_3 \varphi - d h \bar{\mu}_1 \dot{x}_1 - c h \bar{\mu}_1 x_1 - c a x_2 &= \bar{\mu}_2 \text{sgn}(\dot{x}_1 - a \dot{x}_2) c_3 a x_2,
\end{align*}
\]

where \( \bar{\mu}_1 = \mu_1 k_1, \bar{\mu}_2 = \mu_2 k_2 \) and \( \mu_i \in [0, 1] \).

Transforming via

\[
\begin{align*}
z_1 &= \dot{x}_1 - \varphi, \\
z_2 &= x_1 - \varphi, \\
z_3 &= \dot{x}_1, \\
z_4 &= \dot{x}_2, \\
z_5 &= x_1, \\
z_6 &= x_2,
\end{align*}
\]

and

\[
F(Y) = \begin{cases}
F(0, 0) = 0, \\
F(0, v) = \frac{d}{bc} (b - h \bar{\mu}_1 - 1), \\
F(1, 0) = \frac{1}{b} (b - h \bar{\mu}_1 - 1), \\
F(1, \epsilon_2) = \frac{c_3}{bc} [\bar{\mu}_2 (1 + a) - a],
\end{cases}
\]

we get a system of form (2) with

\[
A^\pm = \begin{pmatrix}
-bd & -bc & 0 & 0 & 0 & a_{16}^+ \\
1 & 0 & a_{23} & a_{24} & a_{25} & \pm a_{26} \\
0 & 0 & -d & 0 & -c & \pm a_{36} \\
0 & 0 & d \bar{\mu}_1 & 0 & c \bar{\mu}_1 & -c_3 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},
\]

where \( a_{23} = \frac{d^2 - c}{bc} (b - h \bar{\mu}_1 - 1), a_{24} = -\frac{c}{bc} [\bar{\mu}_2 (1 + a) - a], a_{25} = \frac{d}{bc} (b - h \bar{\mu}_1 - 1), a_{16}^+ = 0, a_{16}^- = -2c_3 \bar{\mu}_2 (1 + a), a_{26} = \frac{d^2}{bc} c_3 \bar{\mu}_2 (b - h \bar{\mu}_1 - 1) \) and \( a_{36} = -c_3 \bar{\mu}_2 \). In this case, the vector defining the separation manifold is given by \( n = e_1 \). Obviously, we have \( A^+ - A^- = v_1 e_6^1 \), thus, the corresponding Filippov system (3) is linear.
2.2. Sliding mode in PWNS

In a next step, we introduce nonlinearities \( g_{\pm} \), i.e. we consider systems of form (1). Using \( f_{\pm}(\xi) = A^\pm \xi + g_{\pm}(\xi) \) and \( \rho(\xi) = n^T f_{\pm}(\xi) \cdot n^T f_{\pm}(\xi) \), we define the sliding mode set as in Section 2.1. The dynamics within \( \mathcal{M}^3 \) is then given by (see again [6,7]):

\[
\dot{\xi} = \frac{1}{n^T [f_- (\xi) - f_+ (\xi)]} \left[ n^T f_- (\xi) \cdot f_+ (\xi) - n^T f_+ (\xi) \cdot f_- (\xi) \right].
\]

(4)

**Lemma 2.** Let \( f_{\pm}(\xi) = A^\pm \xi + g_{\pm}(\xi) \) with nonlinear \( C^k \)-parts \( g_{\pm}(\xi) = o(\|\xi\|) \) for \( k = 1 \) and \( g_{\pm}(\xi) = \mathcal{O}(\|\xi\|^2) \) for \( k > 1 \). Then (4) can be written as

\[
\dot{\xi} = \frac{1}{n^T [A^- - A^+]} \left[ (n^T A^- \xi) A^+ \xi - (n^T A^+ \xi) A^- \xi \right] + g(\xi),
\]

(5)

where the perturbation \( g \) is a \( C^k \)-function with the same “smallness” properties as \( g_{\pm} \).

**Proof.** The numerator is given by

\[
n^T f_- (\xi) \cdot f_+ (\xi) - n^T f_+ (\xi) \cdot f_- (\xi) = n^T A^- \xi \cdot A^+ \xi - n^T A^+ \xi \cdot A^- \xi + n^T g_- (\xi) \cdot A^+ \xi + n^T A^- \xi \cdot g_+ (\xi) + n^T g_+ (\xi) \cdot A^- \xi - n^T g_- (\xi) \cdot g_+ (\xi) - n^T g_+ (\xi) \cdot g_- (\xi),
\]

whereas the denominator can be written as

\[
n^T [f_- (\xi) - f_+ (\xi)] = n^T [A^- - A^+] \xi + n^T [g_- (\xi) - g_+ (\xi)]
\]

\[
= n^T [A^- - A^+] \xi \cdot (1 + \kappa(\xi))
\]

with \( \kappa(\xi) = n^T [g_- (\xi) - g_+ (\xi)]/n^T [A^- - A^+] \xi \). Taylor expansion of \((1 + \kappa(\xi))^{-1}\) and \( \kappa(0) = 0 \) give the result.

**Remark 1.** In case of \( g_{\pm}(\xi) = 0 \) for \( \xi \in \mathcal{M} \), the nonlinearity \( g(\xi) \) of Lemma 2 vanishes.

**Example 2.** Introducing the friction characteristic

\[
\mu_2(v) = \text{sgn}(v) \left[ \alpha + \frac{\beta}{1 + \gamma/|v|} + \delta v^2 \right],
\]

as proposed in [12], for the brake system of Example 1, we end up with a system of form (1), where the nonlinearities \( g_{\pm} \) vanish on the separation manifold \( \mathcal{M} \).

\[
F(v)
\]

3. Invariant cones without sliding mode revisited

Considering piecewise linear systems of form (2), we define the direct crossing set \( \mathcal{M}^c = \{ \xi \in \mathcal{M} \mid \rho(\xi) = n^T A^+ \xi \cdot n^T A^- \xi > 0 \} \). Furthermore, this set can be partitioned into two subsets:

\[
\mathcal{M}^c_- := \{ \xi \in \mathcal{M}^c \mid n^T A^+ \xi < 0 \},
\]

\[
\mathcal{M}^c_+ := \{ \xi \in \mathcal{M}^c \mid n^T A^+ \xi > 0 \}.
\]
For any initial value $\xi \in \mathcal{M}^c \cup \mathcal{M}_0^0 \cup \mathcal{M}_0^1$, the trajectory given by $\varphi(t, \xi) = e^{A^{-1}t}\xi$ enters $\mathbb{R}^N_-$ = \{ $\xi \in \mathbb{R}^N \mid n^T\xi < 0$ \} immediately in forward time if at least one of the quantities $n^TA^\pm\xi$ has negative sign. Assume that $\varphi(t, \xi)$ reaches $\mathcal{M}$ again for the first time $t_-(\xi)$ at $\eta \in \mathcal{M}$; hence, there exists the intersection time $t_-(\xi) = \min\{t > 0 \mid n^Te^{A^{-1}t}\xi = 0\}$. We know $t_-$ is constant on half-rays (see [11]), i.e., $t_-(\rho\xi) = t_-(\xi)$, $0 < \rho < \infty$.

The time $t_+(\eta)$ can be defined in the same way, where similar results hold for $t_+$.

For initial data $\xi$ and $\eta$ for which the intersection times exist, we define the Poincaré-maps $P_-(\xi) := e^{-(\xi)A^{-1}}\xi$ and $P_+(\eta) := e^{+(\eta)A^{-1}}\eta$. Due to the linearity of each subsystem in (2) the flows of these systems map half-rays into half-rays. Hence, the Poincaré-maps transform half-rays of the separating hyperplane $\mathcal{M}$ into half-rays of $\mathcal{M}$. If both $P_-$ and $P_+$ are well-defined so that $P_+(P_-(\xi))$ exists, we can study the behavior of the combined map $P = P_+ \circ P_-$. If there exists $\tilde{\xi} \in \mathcal{M}$ such that

$$P(\tilde{\xi}) = \bar{\lambda}\tilde{\xi},$$

(6)

for some $\bar{\lambda} > 0$, then the same holds for the half-ray $\{\rho\tilde{\xi} \mid \rho > 0\}$. In that way, an invariant cone is generated by the flow of (2).

**Remark 2.** In case of smooth system, such cones are actually planes. In general, we call a cone trivial, if it is a plane.

The “eigenvalue” parameter $\bar{\lambda}$ determines the dynamics on the cone: if $\bar{\lambda} < 1$ resp. $\bar{\lambda} > 1$ the flow on the cone spirals in resp. out; for $\bar{\lambda} = 1$, the cone is foliated by periodic orbits of (2). The “eigenvalue” $\bar{\lambda}$ of $P$ is an eigenvalue of the linear operator $DP$ evaluated at $\tilde{\xi}$ as well.

Attractivity of the cone is determined by the remaining eigenvalues $\lambda_i$, $i = 1, \ldots, N - 2$, of $DP(\tilde{\xi})$:

$$|\lambda_i| \leq \gamma < \min\{1, \bar{\lambda}\}.$$  

(7)

A second crucial property of a cone given by $\tilde{\xi}$ (see (6)) is the transversality:

$$\rho(\tilde{\xi}) > 0, \; \rho(P_-(\tilde{\xi})) > 0,$$

(8)

i.e., both $\tilde{\xi}$ and $P_-(\tilde{\xi})$ are within the crossing set $\mathcal{M}^c$.

The existence of cone-like invariant manifolds in presence of small perturbations is proved in [16]:

**Theorem 2.** Let a cone of the PWLS (2) be given by $\tilde{\xi}$ as in (6). Assume that the attractivity condition (7) and the transversality condition (8) hold. Then, the PWNS (1) exhibits a cone-like invariant manifold locally at the origin, which is tangential to the cone at $\tilde{\xi} = 0$.

More precisely, there exists a sufficiently small $\delta$ and a $C^1$-function $h : [0, \delta) \rightarrow \mathcal{M}$ satisfying $h(0) = 0$ and $h'(0) = \tilde{\xi}$ such that $\{h(u) \mid 0 \leq u < \delta\}$ is locally invariant and attractive under the Poincaré-map $P$ of system (1). For $k = 2$, the function $h$ is $C^k$ in case of $\bar{\lambda} \geq 1$ and $C^{\min(k, 0)}$ in case of $\bar{\lambda} < 1$ and $\gamma < \bar{\lambda}$.

**Remark 3.** The statement of the theorem still remains true if the map in Eq. (6) consists of several minus and plus parts as long as (7) holds and all transitions are transversal in the sense of (8).

**Example 3.** The linear brake system of Example 1 possesses a trivial cone given by the invariant plane span\{e_1, e_2\} corresponding to the eigenvalues $\alpha \pm \iota\omega = -\frac{bd}{2} \pm \sqrt{\left(\frac{bd}{2}\right)^2 - bc}$ independently of the parameters $\mu_i$, $i = 1, 2$. In the physically meaningful case $0 < bd \ll bc$, this eigenvalues are complex and $\tilde{\xi}$ can be chosen as $e_2$. The dynamics on the cone spirals in because of $\alpha < 0$ and $\bar{\lambda} = e^{\omega_2/\iota\omega}$, whereas the cone itself is not attracting using the parameters as in [12]. For $\mu_2 = 0$, there exists another trivial cone corresponding to the eigenvalue $\pm \iota\omega = \pm \sqrt{\gamma_3}$ which is attractive and becomes non-trivial for $\mu_2 \neq 0$. Fig. 1 shows branches of this cone for different values of $\mu_1$ (left plot) and bifurcation curves separating cones with and without sliding parts (right plot).

4. **Invariant cones with sliding mode**

Consider a vector $\tilde{\xi} \in \mathcal{M}$ with

$$n^TA^c\tilde{\xi} = 0,$$

$$n^TA^s\tilde{\xi} < 0,$$

(9)
i.e., \( \tilde{\xi} \in \mathcal{M}^0_+ \) and the flow \( \varphi(t, \tilde{\xi}) \) will leave the separation manifold \( \mathcal{M} \) tangentially, entering \( \mathbb{R}^N \) in forward time.

**Remark 4.** The situation \( n^T A^+ \tilde{\xi} = 0 \) and \( n^T A^- \tilde{\xi} > 0 \) can easily be transformed to condition (9): Transforming linearly the coordinates of system (2) by \( T = I - 2mn^T \) and defining \( \tilde{A}^- = TA^+T \) and \( \tilde{A}^+ = TA^-T \), we get condition (9) for a system of form (2), where \( A^\pm \) is replaced by \( \tilde{A}^\pm \). In this sense, the situation of (9) can be seen as the most general one.

We assume \( (P_- \circ (P_+ \circ P_-)^k)(\tilde{\xi}) = \tilde{\eta} \) or \( (P_+ \circ P_-)^k+1(\tilde{\xi}) = \tilde{\eta} \) for some integer \( k \geq 0 \) and \( \tilde{\eta} \in \mathcal{M}^s_- \), i.e., after a certain time involving just one dynamics or both the solution will end up in the area of stable sliding motion, and the entering is non-tangential:

\[
\rho(\tilde{\eta}) < 0.
\]

Furthermore, we assume that the sliding-flow \( \varphi_s(t, \cdot) \) maps \( \tilde{\eta} \) at time

\[
t_s(\tilde{\eta}) = \inf\{t > 0 \mid \rho(\varphi_s(t, \tilde{\eta})) = 0\}
\]

back onto the half-ray defined by \( \tilde{\xi} \), i.e., \( P_s(\tilde{\eta}) = \varphi_s(t_s(\tilde{\eta}), \tilde{\eta}) = \tilde{\lambda} \tilde{\xi}, \tilde{\lambda} > 0 \). Then, the vector \( \tilde{\xi} \) generates an invariant cone including sliding motion. Let \( \tilde{\xi} \) be the composition of the involved Poincaré-maps. An invariant cone with sliding motion is then generated by an “eigenvector” \( \tilde{\xi} \in \mathcal{M}^0_+ \) of the nonlinear eigenvalue problem

\[
P(\tilde{\xi}) = \tilde{\lambda} \tilde{\xi}
\]

with some real positive “eigenvalue” \( \tilde{\lambda} \). Again, the value \( \tilde{\lambda} > 0 \) determines the dynamics on the cone, as in case of invariant cones without sliding mode.

**Remark 5.** In the 3-dimensional case, the subspaces \( \mathcal{M}^0_- \) and \( \mathcal{M}^0_+ \) always have dimension 1. Therefore, there exists at the most one half-ray of vectors, which fulfill assumption (9) and at the most one half-ray as discussed in Remark 4. Hence, there exist at the most two cones including sliding motion which are not completely restricted to the sliding region. Within the sliding region there might be other invariant half-rays, but at the most three (see [12]). The following example illustrates that situation (see also the left plots of Figs. 2 and 3). Additionally, we have \( k = 0 \), i.e., there are just two cases \( P_-(\tilde{\xi}) = \tilde{\eta} \) and \( (P_+ \circ P_-)(\tilde{\xi}) = \tilde{\eta} \).

**Example 4.** We consider the linear 3-dimensional system \( \dot{\xi} = A^\pm \xi, \ n = e_1 \) with

\[
A^- = \begin{pmatrix}
a_{11} & -1 & 0.8 \\
a_{21} & 0 & 6 \\
a_{31} & 0 & 2
\end{pmatrix}, \quad A^+ = \begin{pmatrix}
a_{11}^+ & -1 & -0.8 \\
a_{21}^+ & 0 & -6 \\
a_{31}^+ & 0 & -2
\end{pmatrix}
\]
where the first columns are determined by $A^{\pm}v^{\pm} = \lambda v^{\pm}$ with $v^{-} = (-\epsilon, \delta^{-}, 1)^T$, $v^{+} = (\epsilon, -\delta^{+}, 1)^T$. This system can possess two invariant cones including sliding mode simultaneously, see left plots of Figs. 2 and 3. This phenomenon, in case of 3 dimensions, is just found in systems where the sliding set contains a 1-dimensional repulsive invariant (with respect to the flow $\varphi^{t}$) subspace (see right plot of Fig. 2).

We assume that all possible crossings of the separation hyperplane $M$ take place in a transversal manner. More precisely, we assume

$$\rho(\xi_{j}) > 0$$

for all $\xi_{j}$ defined by $(P_{+} \circ P_{-})^{j}(\hat{\xi}) = \xi_{2j}$, $j = 1, \ldots, k$, and $(P_{-} \circ (P_{+} \circ P_{-})^{j})(\hat{\xi}) = \xi_{2j+1}$, $j = 0, \ldots, k - 1$ or $j = 0, \ldots, k$. Additionally, the exit of the sliding motion area, the crossing of $M_{-}^{0}$, is assumed to be transversal: The vector field of (3) reduces for $\xi = \hat{\xi}$, in case of condition (9), simply to $A^{-}\hat{\xi}$. Therefore, we would like to have

$$v^{T}A^{-}\hat{\xi} > 0, \quad v^{T} = -\frac{n^{T}A^{-}(I - nn^{T})}{\|n^{T}A^{-}(I - nn^{T})\|},$$

for the normal vector $v \in \mathbb{R}^N \cap M$ of the subspace $M_{-}^{0}$. We call a cone transversal if all transversality conditions (10), (12) and (13) are fulfilled.

**Remark 6.** Condition (9) together with (13) is called 1st-order exit condition.
So far, we just have defined invariant cones including one piece of sliding mode. More precisely, the Poincaré-map \(P\) is a composition of just one map due to the sliding flow and possibly several maps corresponding to the minus and plus systems. The definition is easily generalized to invariant cones including several parts of sliding motion (see right plot of Fig. 3 for the 3D case).

We emphasize that \(P\) is defined on a subset of the \((N - 2)\)-dimensional subspace \(\mathcal{M}^0\). Due to the homogeneity of the Poincaré-map (see Lemma 1), we find

\[
DP(\bar{\xi})\bar{\xi} = \bar{\lambda}\bar{\xi},
\]

similarly to [16]. Hence, \(\bar{\lambda}\) is an eigenvalue of \(DP(\bar{\xi})\) with eigenvector \(\bar{\xi}\). For the remaining \(N - 3\) eigenvalues \(\lambda_i, i = 1, \ldots, N - 3\), we finally assume the attractivity condition

\[
|\lambda_i| \leq \gamma < \min\{1, \bar{\lambda}\}.
\]

We are now able to formulate the main result:

**Theorem 3.** Assume conditions (9)–(14) hold true, i.e., there exists a transversal and attractive cone including sliding mode. Then, the PWNS (1) exhibits a cone-like invariant manifold including sliding mode locally at the origin, which is tangential to the cone at \(\xi = 0\). More precisely, as in the case without sliding mode (see Theorem 2), there exists a sufficiently small \(\delta\) and a \(C^1\)-function \(h : [0, \delta) \rightarrow \mathcal{M}\) satisfying \(h(0) = 0\) and \(h'(0) = \bar{\xi}\) such that

\[
\{h(u) \mid 0 \leq u < \delta\}
\]

is locally invariant and attractive under the Poincaré-map \(P\) composed of corresponding Poincaré-maps of system (1) and (5). For \(k = 2\), the function \(h\) is \(C^2\) in case of \(\bar{\lambda} \geq 1\) and \(C^{\min(2, \delta)}\) in case of \(\bar{\lambda} < 1\) and \(\gamma < \bar{\lambda}\).

**Remark 7.** In the 3-dimensional case, there is no attractivity condition. Therefore, just the transversality conditions guarantee the existence of the invariant set of Theorem 3.

**Example 5.** Adding the nonlinearities \(\rho \pm g(\xi)\) with

\[
g(\xi) = \begin{pmatrix}
\xi_2\xi_3/(1 + \xi_3^2) \\
\xi_1\xi_3 + \xi_2\xi_3 \\
\xi_1\xi_2 + \xi_2\xi_3
\end{pmatrix}
\]

to the system of Example 4, we find the two functions \(h_1, h_2\) of Theorem 3 explicitly:

\[
h_1(\xi_3) = \xi_3 \begin{pmatrix}
0 \\
0.8 \\
1
\end{pmatrix} + \xi_3^2 \begin{pmatrix}
0 \\
0.8\rho^-/(1 + \xi_3^2 - \rho^-\xi_3) \\
0
\end{pmatrix}, \quad \xi_3 \geq 0,
\]

and the same expression for \(h_2(\xi)\) where 0.8 is replaced by \(-0.8\) (see Fig. 4).

5. **Generalized Hopf bifurcation in PWNS**

The reduction of the dynamics to an invariant surface respectively an invariant curve allows a straightforward bifurcation analysis.

If there is a robust invariant cone (in the sense of Theorem 3) consisting of periodic orbits for the corresponding piecewise linear system, then generically a perturbation by higher order terms will give rise to a periodic orbit of the perturbed system, bifurcating from the trivial stationary solution \(\xi = 0\). For smooth systems, this behavior corresponds to the standard Hopf bifurcation. For the evaluation of standard Hopf bifurcation, the existence of bifurcating periodic orbits is generically determined by the cubic terms, since all terms of even order, and in particular quadratic terms, can be eliminated by transformations. Going along with that mechanism, the standard normal form for Hopf bifurcation is derived (see for example [9,14]).

For nonsmooth systems these transformations do not work. In fact, it turns out that already generically the existence of bifurcating periodic orbits is determined by the quadratic terms. We consider an attractive and transversal cone given...
by $\tilde{\xi}(\tilde{\lambda}) \in \mathcal{M}$ with $\tilde{\xi}(\tilde{\lambda})^T\tilde{\xi}(\tilde{\lambda}) = 1$ and $\tilde{\lambda} \approx 1$. By Theorem 2, Remark 3 and Theorem 3, in case of sliding, respectively, the existence of a $C^2$-function $h : [0, \delta) \rightarrow \mathcal{M}$, $h(u) = u\tilde{\xi}(\tilde{\lambda}) + u^2\xi(\tilde{\lambda}) + o(u^2)$ with $\xi(\tilde{\lambda}) \in \mathcal{M}$ and

$$P(h(u)) = h(\tilde{u})$$

is guaranteed if $\gamma < \tilde{\lambda}^2$. We emphasize that the latter inequality holds true, in case of $\gamma < \tilde{\lambda}$ (see the attractivity conditions (7) respectively (14)), as long as $\tilde{\lambda}$ is sufficiently close to 1. Without restriction, we assume $\xi(\tilde{\lambda})^T\xi(\tilde{\lambda}) = 0$ (otherwise we write $\xi = \rho \tilde{\xi} + \xi$ with $\tilde{\xi}^T\tilde{\xi} = 0$ and use a different parametrization $\tilde{u} = u + \rho u^2$). Depending on $\xi$, we compute (see below)

$$P(h(u)) = u\tilde{\lambda}\tilde{\xi}(\tilde{\lambda}) + u^2\tilde{\xi}(\tilde{\lambda}) + o(u^2).$$

If $\tilde{\xi}(\tilde{\lambda})^T\tilde{\xi}(\tilde{\lambda}) \neq 0$, there is a nontrivial fixed point $u^* > 0$ for $\tilde{\lambda} > 1$ or $\tilde{\lambda} < 1$:

$$u^* = \frac{1 - \tilde{\lambda}}{\tilde{\xi}(\tilde{\lambda})^T\tilde{\xi}(\tilde{\lambda})} + o(u^*).$$

In the case of smooth systems, it holds that $\tilde{\xi}(1)^T\tilde{\xi}(1) = 0$, so that higher order terms are required to determine $u^*$. For nonsmooth systems generically $\tilde{\xi}(\tilde{\lambda})^T\tilde{\xi}(\tilde{\lambda})$ can be calculated using $\tilde{\xi}(1)^T\tilde{\xi}(1)$ which typically does not vanish. To see this, we derive further below explicit expressions for $\xi(\tilde{\lambda})$ and $\tilde{\xi}(\tilde{\lambda})$, which will be used to evaluate $\tilde{\xi}(\tilde{\lambda})^T\tilde{\xi}(\tilde{\lambda})$.

We recall that $\tilde{\xi}(1) \in \mathcal{M}$ and $A^+ \tilde{\xi}(1) \notin \mathcal{M}$. Without restriction (see the following Lemma) we may therefore assume

$$\tilde{\xi}(1)^T A^- \tilde{\xi}(1) = 0.\tag{17}$$

**Lemma 3.** Let a transformation matrix $T$ be given with $n^T T = n^T$. Then any cone $\tilde{\xi}$ of system (2) corresponds to a cone $T\tilde{\xi}$ of the transformed system

$$\dot{\tilde{\xi}} = \begin{cases} \tilde{\lambda}^+ \tilde{\xi}, & n^T\tilde{\xi} > 0, \\ \tilde{\lambda}^- \tilde{\xi}, & n^T\tilde{\xi} < 0, \end{cases}$$

$\tilde{\lambda}^\pm = T^{-1} A^\pm T$, and vice versa. The dynamics-parameter $\tilde{\lambda}$ (see (6) and (11)) of the corresponding cones is equal.

**Proof.** Let $\tilde{\varphi}(t, \tilde{\xi})$ be the flow of the transformed system and $\varphi(t, \xi)$ of system (2). For each subsystem (minus, plus, and sliding (see (3))) we have

$$\tilde{\varphi}(t, T\tilde{\xi}) = T\varphi(t, \xi).$$

Therefore, we see

$$\tilde{P}(T\tilde{\xi}) = TP(\tilde{\xi}) = \tilde{\lambda} T\tilde{\xi},$$

where $\tilde{P}$ denotes the Poincaré-map of the transformed system.
Thorem 4. Let an attractive and transversal cone $\bar{\xi}(\bar{\lambda}) \in \mathcal{M}$ with $\bar{\xi}(\bar{\lambda})^T \bar{\xi}(\bar{\lambda}) = 1$ and $\bar{\lambda} \approx 1$ be given, which fulfills the assumptions of Theorem 2 respectively Theorem 3. Furthermore, we assume that (17) hold true and we use $\bar{\xi} = \bar{\xi}(1)$. Then there exists $\beta \in \mathcal{M}$ with

$$\bar{\xi}(\bar{\lambda})^T \bar{\xi}(\bar{\lambda}) = \bar{\xi}^T \beta + \mathcal{O}(1 - \bar{\lambda})$$

which just depend on the quadratic parts of the nonlinearities. If further $\bar{\xi}^T \beta \neq 0$, then

$$u_* = \frac{1 - \bar{\lambda}}{\bar{\xi}^T \beta} + o(1 - \bar{\lambda}).$$

Remark 8. The typical situation of Theorem 4 is the following. The PWLS, more precisely the corresponding matrices, depends smoothly on a parameter $\mu$. For $\mu = \mu_0$ there is a transversal cone given by $\bar{\xi}_0 = \bar{\xi}(\mu_0), \bar{\xi}_0^T \bar{\xi}_0 = 1$. Assuming that $\bar{\lambda}(\mu_0)$ is a simple eigenvalue of the Jacobian $DP(\bar{\xi}_0)$ of the corresponding Poincaré-map, the existence of transversal invariant cones $\bar{\xi}(\mu)$ with $\bar{\xi}(\mu)^T \bar{\xi}(\mu) = 1$ and dynamics-parameter $\bar{\lambda}(\mu)$ for $\mu$ sufficiently close to $\mu_0$ can be guaranteed. For attractive cones (see condition (7) and (14)), this assumption holds true. The typical situation of Theorem 4 is then $\bar{\lambda}(\mu_0) = 1$ and $\frac{\partial}{\partial \mu} \bar{\lambda}(\mu_0) \neq 0$.

5.1. The case without sliding mode

To simplify matters, we consider the situation described in Section 3. The ideas can be extended easily to the case of Remark 3, where the cone not just consists of two different parts (one minus and one plus part).

First, we define the so-called monodromy matrix $M$ corresponding to the cone given by $(P_+ \circ P_-)(\bar{\xi}) = \bar{\lambda} \bar{\xi}$ as

$$M = J^+ e^{t A^+} J^- e^{t A^-},$$

where $t_{\pm}$ denote the intersection times corresponding to $\bar{\xi}$ and $\bar{\eta} = e^{t A^-} \bar{\xi}$, and $J^\pm$ are the so-called jump matrices which transform the vector fields on $M$ into each other. More precisely, we define

$$J^+ = I + \frac{(A^- - A^+) \bar{\xi}}{n^T A^+ \bar{\xi}} n^T,$$

$$J^- = I - \frac{(A^- - A^+) \bar{\eta}}{n^T A^- \bar{\eta}} n^T.$$

By definition, the monodromy matrix has two eigenvalues equal to $\bar{\lambda}$ with linear independent eigenvectors $\bar{\xi}$ and $A^- \bar{\xi}$. The remaining eigenvalues are exactly the remaining eigenvalues of $DP(\bar{\xi})$ (see the discussion corresponding to condition (7)).

Second, we compute $\mathcal{P}_-(\xi)$ with $\xi = u \bar{\xi} + u^2 \bar{\xi} + o(u^2)$, omitting the minus indices. Using the variation-of-constants formula, we find

$$\mathcal{P}(\xi) = e^{t A} \bar{\xi} + \int_0^t e^{(t-t') A} g(\xi(s)) ds,$$

where $t$ is the intersection time defined by

$$0 = n^T e^{t A} \bar{\xi} + n^T \int_0^t e^{(t-s) A} g(\xi(s)) ds,$$

and $\bar{\xi}(s) = A \bar{\xi}(s) + g(\xi(s))$, $\bar{\xi}(0) = \bar{\xi}$. Let $t$ denote the intersection time of the corresponding linear system defined by $0 = n^T e^{t A} \bar{\xi}$. We now replace the nonlinear intersection time $t$ by $\tau$. Simple computations, using the transversality condition (8), give

$$\tau - t = -\frac{1}{n^T e^{t A} A^+ \bar{\xi}} n^T \int_0^t e^{(t-s) A} g(\xi(s)) ds + \mathcal{O}(\|\xi\|^2).$$
and, thus,

\[ \mathcal{P}(\xi) = e^{tA} \xi + e^{tA} A \xi (t - t) + \int_0^t e^{(t-s)A} g(\xi(s))ds + \mathcal{O}(\|\xi\|^3) \]

\[ = e^{tA} \xi + \left[ I - \frac{e^{tA} A \xi}{n^T e^{tA} A \xi} \right] \int_0^t e^{(t-s)A} g(\xi(s))ds + \mathcal{O}(\|\xi\|^3). \]

Making use of \( \xi = u\bar{\xi} + u^2\zeta + o(u^2) \) and \( \bar{\eta} = e^{tA} \bar{\xi} \), we get

\[ t - \bar{t} = -u \frac{n^T e^{tA} \xi}{n^T A \bar{\eta}} + o(u), \]

and, thus,

\[ \mathcal{P}(\xi) = e^{tA} \xi + e^{tA} A \xi (t - \bar{t}) + \left[ I - \frac{e^{tA} A \xi}{n^T e^{tA} A \xi} \right] \int_0^t e^{(t-s)A} g(\xi(s))ds + o(u^2) \]

\[ = u\bar{\eta} + u^2 \zeta + o(u^2) \]

with

\[ \zeta = \left[ I - \frac{A \bar{\eta} n^T}{n^T A \bar{\eta}} \right] e^{tA} \left( \zeta + \int_0^t e^{-sA} g_2(e^{sA} \bar{\xi})ds \right), \]

where \( g_2 \) denotes the quadratic part of the \( C^2 \)-nonlinearity \( g(\xi) = g_2(\xi) + o(\|\xi\|^2) \).

**Lemma 4.** Let a transversal cone be given by \((P_+ \circ P_-)(\bar{\xi}) = \lambda \bar{\xi} \) with intersection times \( \bar{t}_\pm \). For \( \xi = u\bar{\xi} + u^2\zeta + o(u^2) \), we get

\[ \mathcal{P}(\xi) = (P_+ \circ P_-)(\bar{\xi}) = u\lambda \bar{\xi} + u^2 \zeta + o(u^2), \]

where \( \zeta \) depends linearly on \( \zeta \). More precisely, we have

\[ \zeta = (M - A^- \bar{\xi}^T) \zeta + b \]

with \( b = B^+ b_+ + b_- \) and

\[ b_- = B^- \int_0^{\bar{t}_-} e^{-sA^-} g_2(\epsilon^{A^-} \bar{\xi})ds, \quad B^- = \left[ I - \frac{A^- \bar{\eta} n^T}{n^T A^- \bar{\eta}} \right] e^{tA^-}, \]

\[ b_+ = B^+ \int_0^{\bar{t}_+} e^{-sA^+} g_2(\epsilon^{A^+} \bar{\eta})ds, \quad B^+ = \left[ I - \frac{A^+ \bar{\xi} n^T}{n^T A^+ \bar{\xi}} \right] e^{tA^+}, \]

\[ c^T = \frac{n^T e^{tA^+} J^+ e^{-tA^-} A^-}{n^T A^+ \bar{\xi}}. \]

**Proof.** We know \( \mathcal{P}_-(\xi) = u\bar{\eta} + u^2 \zeta + o(u^2) \) with \( \bar{\xi} = B^- \zeta + b_- \). Applying the Poincaré-map of the plus system, similarly, we get \( \mathcal{P}_+(\mathcal{P}_-(\xi)) = u\lambda \bar{\xi} + u^2 \zeta + o(u^2) \) with

\[ \zeta = B^+ \zeta + b_+ \]

\[ = B^+ B^- \zeta + B^+ b_- + b_+. \]

The equations \( B^+ B^- = B^+ J^- e^{tA^-} + J^+ e^{tA^+} J^- e^{tA^-} - A^- \bar{\xi} \cdot \frac{n^T e^{tA^+} J^- e^{tA^-}}{n^T A^+ \bar{\xi}} \) complete the proof.

**Remark 9.** In case of smooth systems \((A^\pm = A \text{ and } g^\pm = g)\), it is well known that the quadratic terms do not guarantee the existence of one branch of periodic orbits, which bifurcates at \( \lambda = 1 \). More precisely, we have \( \bar{\xi}(1)^T \bar{\xi}(1) = \bar{\xi} \beta = 0 \)
for the corresponding expression of Theorem 4. This can be seen easily: Assuming without loss of generality $n = e_1$, $\bar{\xi} = e_2$ and

$$A = \begin{pmatrix} \alpha & -\omega & 0 \\ \omega & \alpha & 0 \\ 0 & 0 & \bar{\Lambda} \end{pmatrix}, \quad \bar{\Lambda} \in \mathbb{R}^{N-2,N-2},$$

we conclude $c^T \xi = 0$, $\bar{\xi}^T M \bar{\xi} = 0$, and, thus $\bar{\xi}^T \dot{\bar{\xi}} = \bar{\xi}^T b$ for $c$ and $\bar{\xi}$ as in Lemma 4. Using $B^± = B$ and the quadratic structure of $g_2$, we get

$$\bar{\xi}^T b = \bar{\xi}^T B (B + I) \int_0^\bar{t} e^{-sA} g_2 (e^{sA^±} \bar{\xi}) ds = 0,$$

for $\alpha = 0$. We see that the existence of one branch of periodic solution at $\bar{\lambda} = 1$, i.e., $\alpha = 0$, cannot be guaranteed considering just the quadratic terms. Considering the cubic and even higher order terms, will lead to the important concept of Lyapunov coefficients (see for example [13]).

**Example 6.** We consider the nonsmooth system $\dot{\xi} = A^±\xi$, $n = e_1$ with

$$A^- = \begin{pmatrix} -0.5 & -2 & 0.5\kappa \\ 1 & 1.5 & 0.5 \\ 0 & 0 & \mu_- \end{pmatrix}, \quad A^+ = \begin{pmatrix} -0.5 & -1 & 0 \\ 1 & -0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix},$$

and $\kappa = -0.5 - \mu_-$ (see the example in [16]). For $\mu_-^0 \approx -1.06039$, the dynamics on the cone changes from spiraling in to spiraling out. More precisely, we have $\bar{\lambda}(\mu_-^0) = 1$ and $\frac{\partial}{\partial \mu_-} \bar{\lambda}(\mu_-^0) \neq 0$ (see left plot of Fig. 5). Introducing the nonlinearities

$$g_-(\xi) = -0.01 \begin{pmatrix} \xi_3^2 \\ 0 \\ 0 \end{pmatrix}, \quad g_+'(\xi) = 0.1 \begin{pmatrix} 0 \\ 0 \\ \xi_1^2 + \xi_2^2 \end{pmatrix},$$

we observe that a stable periodic orbit is born at $\mu_-^0$ (see Fig. 6). More precisely, we have at the bifurcation point $b^T \approx (0, 0.111 e^{-2}, 1.728 e^{-2})$, $\bar{\xi}^T \zeta \approx (0, 3.331 e^{-3}, 0.187 e^{-3})$, $\bar{\xi}^T \bar{\xi} \approx (0, 0.160 e^{-2}, 3.106 e^{-2})$ and, most important, $\bar{\xi}^T \zeta \approx -3.092 e^{-2} < 0$ (see right hand side of Fig. 5).

5.2. The case with sliding mode

Due to the homogeneity of system (3), the situation of cones including sliding mode is quite similar to the case in Section 5.1. Again, we just study a simple case which includes all interesting features. More precisely, we assume
Fig. 6. Left: Branch of fixed points $\mu^*$ of the piecewise nonlinear system of example 6. For $\mu_\sim = -0.0551$, the periodic orbit loses its stability. The red curve shows the euclidian norm of the predicted fixed points via formula (16). At $\mu_\sim = 0.5$ the cone of the PWLS loses its transversality. Right: Periodic orbit at $\mu_\sim = -0.1$.

$$(P_\sim \circ P_-)(\bar{\xi}) = \bar{\lambda} \bar{\xi} \text{ for } \bar{\xi} \in \mathcal{M}_\sim$$

such that condition (9) and the transversality conditions (10), (12) and (13) hold true. Here $P_\sim$ denotes again the Poincaré-map corresponding to system (3). The monodromy matrix is now given by

$$M = \frac{\partial}{\partial s} \varphi(s, \bar{\eta}) J_s e^{J_s - A^-} = Y(\bar{i}_s) J_s e^{J_s - A^-},$$

where $\bar{i}_-$ and $\bar{\eta} = \bar{\lambda}^{J_s - A^-} \bar{\xi}$ are defined as in the previous section, $\bar{i}_s$ denotes the intersection time of the sliding system with $v^T \varphi(s, \bar{i}_s, \bar{\eta}) = \bar{\lambda} \cdot v^T \bar{\xi} = 0$ (see (9) and (13)), and $Y(t)$ is the solution of the variational equation corresponding to $\bar{\eta}$ (see Lemma 1). The jump matrix is then defined by

$$J^s = I - A^- \bar{\eta} - f_\sim(\bar{i}, \bar{\eta}) n^T,$$

where $f_\sim(\bar{i})$ is the right hand side of (3). Again, the monodromy matrix has the eigenvalue $\bar{\lambda}$ with corresponding linear independent eigenvectors $\bar{\xi}$ and $A^- \bar{\xi}$. In contrast to the situation in Section 5.1, we know, that $M$ is singular, i.e., one eigenvalue is equal to 0, due to the singularity of the jump matrix $J^s$. The remaining $N - 3$ eigenvalues are again the remaining eigenvalues of $DP(\bar{\xi})$ (see the discussion concerning condition (14)).

As in Section 5.1, we compute $\mathcal{P}(\bar{\xi})$ for $\bar{\xi} = u \bar{\xi} + u^2 \bar{\xi} + o(u^2)$.

**Lemma 5.** Let a transversal cone be given by $(P_\sim \circ P_-)(\bar{\xi}) = \bar{\lambda} \bar{\xi}$ with intersection times $\bar{i}_-$ and $\bar{i}_s$. For $\xi = u \bar{\xi} + u^2 \bar{\xi} + o(u^2)$ we get

$$\mathcal{P}(\bar{\xi}) = (P_\sim \circ P_-)(\bar{\xi}) = u \bar{\lambda} \bar{\xi} + u^2 \bar{\xi} + o(u^2),$$

with $b = B^b b_+ + b_-$ and

$$b_+ = B^- \int_{\bar{i}_-}^{\bar{i}_s} e^{-s A^-} g_2(e^{s A^-} \bar{\xi}) ds, \quad B^- = \left[ I - \frac{A^- \bar{\eta} n^T}{n^T A^- \bar{\eta}} \right] e^{J_s A^-},$$

$$b_- = B^- \int_{\bar{i}_-}^{\bar{i}_s} Y(s)^{-1} g_2(Y(s) \bar{\eta}) ds, \quad B^- = \left[ I - \frac{A^- \bar{\xi} v^T}{v^T A^- \bar{\xi}} \right] Y(\bar{i}_s),$$

$$c^s = \frac{v^T Y(\bar{i}_s) J_s e^{J_s A^-}}{v^T A^- \bar{\xi}}.$$
Proof. From the previous section we know
\[ \mathcal{P}_-(\xi) = \eta = u\bar{\eta} + u^2\bar{\xi} + o(u^2), \quad \bar{\xi} = B^{-1}\bar{\zeta} + b_-, \]
for \( \xi = u\bar{\xi} + u^2\bar{\zeta} + o(u^2) \). The main ingredient in computing \( \mathcal{P}_s(\eta) \), where \( \mathcal{P}_s \) denotes the Poincaré-map of system (5), is the application of the nonlinear variation-of-constants formula to systems (3) and (5) (see for example Theorem 14.5 in [10]):
\[ \mathcal{P}(\eta) = \varphi(\tau, \eta) + \int_0^\tau \frac{\partial}{\partial s} \varphi(\tau - s, \eta(s))g(\eta(s))ds, \]
(omitting the sliding index), where \( \eta(s) \) denotes the solution of (5) with initial value \( \eta \). The intersection time \( \tau \) is defined by
\[ 0 = v^T\varphi(\tau, \eta) + v^T\int_0^\tau \frac{\partial}{\partial s} \varphi(\tau - s, \eta(s))g(\eta(s))ds. \]
The next steps of the proof are similar to the steps of the proof of Lemma 4. We compute
\[ \tau - t = -\frac{1}{v^T f(\varphi(\tau, \eta))} v^T \int_0^\tau \frac{\partial}{\partial s} \varphi(\tau - s, \eta(s))g(\eta(s))ds + \mathcal{O}(\|\eta\|^2), \]
and therefore,
\[ \mathcal{P}(\eta) = \varphi(t, \eta) + \left[I - \frac{f(\varphi(t, \eta))v^T}{v^T f(\varphi(t, \eta))}\right] \int_0^t \frac{\partial}{\partial s} \varphi(t - s, \eta(s))g(\eta(s))ds + \mathcal{O}(\|\eta\|^2). \]
With
\[ t - \bar{t} = -u \frac{v^T f(\varphi(t, \eta))}{v^T f(\varphi(t, \eta))} + o(u) \]
\[ = -u \frac{v^T Y t \bar{\zeta}}{v^T f(\varphi(t, \eta))} + o(u) \]
for \( \eta = u\bar{\eta} + u^2\bar{\xi} + o(u^2) \) we find
\[ \mathcal{P}(\eta) = u\varphi(\bar{t}, \bar{\eta}) + u^2 \left[I - \frac{f(\varphi(\bar{t}, \bar{\eta}))v^T}{v^T f(\varphi(\bar{t}, \bar{\eta}))}\right] \left( Y \bar{\zeta} + \int_0^\bar{t} \frac{\partial}{\partial s} \varphi(\bar{t} - s, \eta(s))g(\eta(s))ds \right) + o(u^2). \]
Using the identities \( \varphi(\bar{t}, \bar{\eta}) = \bar{\lambda} \bar{\xi} \) and \( \frac{\partial}{\partial s} \varphi(\bar{t} - s, \eta(s)) = Y(\bar{t})Y(s)^{-1} \) and the homogeneity of \( f \), we get
\[ \mathcal{P}(\eta) = u\bar{\lambda} \bar{\xi} + u^2 (B^2 \bar{\zeta} + b) + o(u^2). \]
Therefore, \( \bar{\zeta} = B^2 \bar{\zeta} + b = B^2 B^{-1} \bar{\zeta} + B^2 b_- + b = B^2 B^{-1} \bar{\zeta} + b \). The statement of the lemma follows now from \( Y(\bar{t})f(\bar{\eta}) = f(\varphi(\bar{t}, \bar{\eta})) = \bar{\lambda} f(\bar{\xi}) = \bar{\lambda} A^{-1} \bar{\xi} \) and \( B^2 A^{-1} \bar{\xi} = 0 \).

5.3. Proof of Theorem 4

Again, we neglect the dependency on \( \bar{\lambda} \). First, we compute \( \zeta \), using the invariance condition (15).

Lemma 6. The vector \( \zeta \) with \( \zeta^T \bar{\xi} = 0 \) and \( \zeta \in \mathcal{M} \) is uniquely defined by
\[ (M - \bar{\lambda}^2 I)(M - \bar{\lambda} I)\zeta = -(M - \bar{\lambda} I)b, \]
as long as \( \bar{\lambda} \) is sufficiently close to 1.

Proof. Equation (15) implies
\[ \ddot{u} = u\ddot{\lambda} + u^2 \ddot{\xi} + o(u^2). \]
Inserting this expression into (15) and comparing the coefficients of \( u^2 \), yields
\[
(I - \tilde{\xi}\tilde{\xi}^T) \tilde{\zeta} = \tilde{\lambda}^2 \zeta.
\]

In order to derive the equation of the lemma, we use the identity \((M - \tilde{\lambda} I)(I - \tilde{\xi}\tilde{\xi}^T) = M - \tilde{\lambda} I\) and the formula of \( \tilde{\zeta} \) of Lemma 4 respectively Lemma 5:
\[
(M - \tilde{\lambda} I)(I - \tilde{\xi}\tilde{\xi}^T)\tilde{\zeta} = (M - \tilde{\lambda} I)\tilde{\zeta} = (M - \tilde{\lambda} I)(M\zeta + b) = (M - \tilde{\lambda} I)\tilde{\lambda}^2 \zeta,
\]
recalling, that \( A^{-\tilde{\xi}} \) is an eigenvector of the monodromy matrix to the eigenvalue \( \tilde{\lambda} \). Sorting the terms of the last equation with respect to \( \zeta \) and \( b \), gives the desired result. In order to prove the uniqueness, we decompose \( \zeta = \zeta_k + P\zeta \) with \( \zeta_k \in \ker(M - \tilde{\lambda} I) \) and the orthogonal projection \( P \) onto \( \ker(M - \tilde{\lambda} I)^{-1} \). This decomposition depends smoothly on \( \tilde{\lambda} \), because of the attractivity of the cone (condition (7) and (14)). Using \( \tilde{\xi}^T\zeta = 0 \) and \( n^T\zeta = 0 \), we compute
\[
\zeta_k = -\frac{n^TP\zeta}{n^TA^{-\tilde{\xi}}(I - \tilde{\xi}\tilde{\xi}^T)A^{-\tilde{\xi}}}. 
\]

Decomposing \( b \) similar to \( \zeta \), we find for the equation, which is already proved,
\[
P(M - \tilde{\lambda}^2 I)\zeta = (PMP - \tilde{\lambda}^2 I)P\zeta = -Pb.
\]

For \( \tilde{\lambda} \) sufficiently close to 1, the matrix \( PMP - \tilde{\lambda}^2 I \) is regular, which proves the lemma.

In a second step, we find
\[
\zeta_k = -\frac{n^TP\zeta}{n^TA^{-\tilde{\xi}}}A^{-\tilde{\xi}}, 
\]
assuming (17), and therefore,
\[
\tilde{\xi}\tilde{\xi} = \tilde{\xi}^TM\zeta + \tilde{\xi}^Tb = \tilde{\xi}^TP\zeta + \tilde{\xi}^Tb = -\tilde{\xi}^TM(PMP - I)^{-1}Pb + \tilde{\xi}^Tb = \tilde{\xi}^T(I - M(PMP - I)^{-1})Pb.
\]

Defining \( \beta = (I - M(PMP - I)^{-1})b(1) \), completes the proof of Theorem 4.

6. Proof of Theorem 3

Similar to [16], we decompose the Poincaré-map of the PWLS corresponding to Eq. (11) by
\[
P(\xi) = DP(\xi)\xi + Q(\xi)
\]
for \( \xi \in \mathcal{M}^0 \), which are in a \( \varepsilon \)-sector \( S_\varepsilon(\tilde{\xi}) = \{ \rho\xi \in \mathcal{M}^0_+ | \rho > 0, \| \xi - \tilde{\xi} \| \leq \varepsilon \} \) of \( \tilde{\xi} \). For \( \varepsilon \) sufficiently small, conditions (9), (10), (12), and (13) guarantee that \( P \) is well-defined on \( S_\varepsilon(\tilde{\xi}) \).

Using the properties of \( P \), we immediately obtain (as in [16]) the following summarizing Lemma.

Lemma 7. The nonlinearity \( Q \) fulfills

(i) \( Q(\tilde{\xi}) = 0, DQ(\tilde{\xi}) = 0 \),
(ii) \( Q(\rho\xi) = \rho Q(\xi) \) and \( DQ(\rho\xi) = DQ(\xi) \) for \( \xi \in S_\varepsilon(\tilde{\xi}), \rho > 0 \),
(iii) \( D^{j+1}(\rho\xi)|_{\xi^j} = 0 \) for \( \xi \in S_\varepsilon(\tilde{\xi}), \rho > 0 \) and \( j \geq 1 \).

The last identity of the lemma is remarkable, especially, in view of \( D^{j+1}(\rho\xi) = \frac{1}{\rho^j} D^{j+1}(\xi) \), which holds true for \( \xi \in S_\varepsilon(\tilde{\xi}), \rho > 0 \) and \( j \geq 1 \).

We use the usual techniques of “cut-off and scale” to globalize the situation of system (1) in the following sense: Depending on the scaling parameter \( \delta \), we find a constant \( o(1) \) with \( \| g_{\pm} \| + \| Dg_{\pm} \| \leq o(1), \delta \to 0 \) and \( \text{supp} g_{\pm} \subset |\xi| \in

In $\mathbb{R}^n$ if $||\xi|| \leq \delta$. In case of $k \geq 2$, without loss of generality, we assume $g_{\pm} = O(||\xi||^2)$. A global invariant manifold of the cut and scaled system gives a local invariant manifold of system (1).

**Lemma 10.** Without loss of generality, we assume $n^T g_-(\xi) = 0$ for $\xi \in \mathcal{M}^0_-$, i.e., within the separation manifold $\mathcal{M}$ the vector-field $f_-(\xi) = A^- \xi + g_-(\xi)$ is still tangential to $\mathcal{M}$ in $\mathcal{M}^0_-$ even in the presence of perturbations.

Let $\tau_-$, $\tau_+$ and possibly $\tau_s$ be the intersection times of the cut and scaled system (1). The existence of these functions for initial values close to the cone is guaranteed by the Implicit Function Theorem, due to the transversality conditions (10), (12), (13), and the hypothesis on the perturbations $g_{\pm}$. Similar to [16], we find:

**Lemma 8.** Let $\tau$ be one of the intersection times $\tau_-$, $\tau_s$, and possibly $\tau_+$ and let $t$ be the corresponding intersection time of system (2). Then, $\tau$ is a $C^k$-function with

\[
\tau(\xi) - t(\xi) = o(1), \\
\tau'(\xi) - t'(\xi) = o(||\xi||^{-1}),
\]

for $\delta \to 0$ or $||\xi|| \to 0$. In case of $k \geq 2$ we have

\[
\tau^{(j)}(\xi) - t^{(j)}(\xi) = O(||\xi||^{-j}), \quad 0 \leq j \leq 2,
\]

for $\delta \to 0$ or $||\xi|| \to 0$.

**Proof.** The statements about $\tau_-$ and, similar, about $\tau_+$ are already proved in [16].

First, we explain how the existence of $\tau(\eta) = \tau_s(\eta)$, $\eta \in S_s(\tilde{\eta}) = \{\rho \eta \in \mathcal{M} | \rho > 0, \ ||\eta - \tilde{\eta}|| \leq \epsilon\}$, for $\epsilon$ and $\delta$ sufficiently small can be proved. If the existence is established, the differentiability as a local property is given by local arguments for every $\eta \in S_s(\tilde{\eta})$. In particular, the intersection times will be $C^k$ if the perturbations are $C^k$.

Let $\epsilon$ be sufficiently small, such that $t(\eta) = t_s(\eta)$ is defined on $S_s(\tilde{\eta})$ and all transversality conditions remain fulfilled. The existence of $\tau(\eta)$ for $\eta$ with $||\eta - \tilde{\eta}|| \leq \epsilon$ and $\epsilon, \delta$ sufficiently small can easily be established, using, for example, the Implicit Function Theorem. To prove the existence of $\tau(\rho \eta)$ for $\rho > 0$, we apply the Contraction-Mapping Theorem to the operator

\[
T : I \to \mathbb{R}, \quad t \mapsto t - \frac{1}{\rho |\beta|} F(t, \rho \eta)
\]

on the interval $I = \{t \in \mathbb{R} | |t - t(\eta)| \leq \delta_1\}$, where

\[
F(t, \eta) = v^T \varphi(t, \eta) + \int_0^t \frac{\partial}{\partial s} \varphi(t, s) g(\eta(s)) ds
\]

and $\beta := v^T \varphi(t, \eta), \ \eta = v^T A^- \xi > 0$. The function $\eta(s)$ is the solution of (5) with initial value $\eta$. The other statements are proved similar as in [16], using the homogeneity of the sliding-flow, i.e., $\varphi(t, \rho \eta) = \rho \varphi(t, \eta)$.

The Poincaré-map $\mathcal{P}$ is defined on the sector $S_s(\tilde{\xi})$ as long as $\epsilon$ and $\delta$ are sufficiently small. We decompose the Poincaré-map of system, using the Poincaré-map of the PWLS:

\[
\mathcal{P}(\xi) = P(\xi) + R(\xi), \quad R(\xi) := \mathcal{P}(\xi) - P(\xi).
\]

Due to the compact support of the perturbations $g_{\pm}$ depending on $\delta$, we know that $R(\xi) = 0$ for $||\xi|| \geq \text{const} \cdot \delta, \text{const}$ sufficiently large.

**Lemma 9.** The remaining term $R$ is $C^k$ in $S_s(\tilde{\xi})$. Furthermore, there is a constant $K_\delta$ independent of $\epsilon$ with $K_\delta \to 0$ for $\delta \to 0$ and

\[
||R(\xi)|| + ||D R(\xi)|| \leq K_\delta, \quad \xi \in S_s(\tilde{\xi}).
\]

Let $\xi \in S_s(\tilde{\xi})$ with $||\xi|| = 1$. In case of $k = 2$, we get

\[
D^2 R(\rho \xi) = O(1), \quad 0 < \rho < \infty.
\]
Proof. The Lemma can be proved similar to [16], using the homogeneity of the sliding-flow.

Using Lemma 8 and 9, the proof of Theorem 3 can be established in exactly the same way as in [16].

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References