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Dynamic Asset Allocation in Portfolios including Alternative Investments

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ABSTRACT
In this paper, the problem of portfolio construction including alternative investments, e.g., hedge funds, is analyzed and solved for an investor having a constant relative risk aversion utility function. The investment opportunities are modelled in a framework of continuous-time stochastic differential equations. In a first step, the general solution for an arbitrary number of risky investment opportunities as well as an arbitrary number of risk factors is presented. The general solution is used to derive the explicit solution for a typical investor. The typical investor, in this context, has three risk-bearing investment opportunities. These are the stock market, the bond market, and the alternative investment universe. The fixed income part is modelled by a short rate model. For the market portfolio, the usual geometric Brownian motion model is used. For the alternative investment, we use a model with the Greek letters α and β as in Sharpe's capital asset pricing model. The resulting optimal asset allocation law is then analyzed for typical values.

KEY WORDS
Investment Models, Stochastic Control, Hedge Funds, Optimization, Portfolio Management

1 Introduction

It is the nature of hedge funds to implement dynamic investment strategies, categorized by their styles. However, even in the realm of hedge funds, studies and presentations are scarcely found which are not arguing in a mean-variance framework. Proper risk management including hedge funds can hardly be achieved by using a mean-variance framework (see, e.g., [1]). In order to account for the dynamic behavior of different investment classes, it is much more reasonable to optimize in a stochastic differential equations framework. Therefore, the problem of portfolio construction including alternative investments is analyzed and solved for an investor having a constant relative risk aversion (CRRA) utility function. We are interested in the general solution with arbitrary risk factors as well as the special case for a portfolio including hedge funds. The general solution for an arbitrary number of risky investment opportunities as well as an arbitrary number of risk factors is presented. For the special case, we consider a typical investor which has three risk-bearing investment opportunities. These are the stock market, the bond market, and the alternative investment universe. Each of the three investment opportunities offers a different risk-return profile. The fixed income part is modelled by a short rate model, i.e., we use the Vasicek model for the short rate and the bond price evolution. The second investment opportunity is a passive fund, regarded as a proxy of the market portfolio. The S&P500 is a popular index, often used as a proxy for the market portfolio. The passive fund is modelled by a geometric Brownian motion. Its drift and diffusion are constant. In the context of hedge funds, it is virtually impossible to ignore the Greek letters α and β since they are used by practitioners and academics alike. Usually, α and β are used in the context of Sharpe's capital asset pricing model (CAPM), see [3]. In this paper we use the same terminology, but however, do not state wether the CAPM holds or not. This is no restriction, because we do not need the assumptions of the CAPM to hold, since we only use its terminology. As we are dealing in a continuous time framework, we use a model resembling to Merton's intertemporal capital asset pricing model (ICAPM), see [4]. The behavior of the optimal asset allocation is analyzed for typical values. The resulting investment strategy is used in a backtest with US data.

2 General solution

The investment opportunities are modelled as stochastic differential equations. From these opportunity sets, we can derive the wealth equation of the investor. The investor is assumed to have a constant relative risk aversion (CRRA) utility function. This kind of problem can be solved by using Bellman's optimality principle.

2.1 Asset price dynamics

In order to model assets traded on an organized exchange, we make the following assumptions:

- Trading is continuous.
- There are no transaction costs, fees, or taxes.
- The investor is a price taker and does not possess enough market power to influence prices.

We consider a market in which $n \geq 1$ risk-bearing investments exist. The asset price processes $(P_1(t), P_2(t), \dots, P_n(t))$ of the risk-bearing investments satisfy the stochastic differential equations

$$\begin{aligned} \frac{dP_i(t)}{P_i(t)} &= \mu_i(t, x(t))\, dt + \sigma_i(t) dZ_P(t)\,, \\ P_i(0) &= p_{i0} > 0\,. \end{aligned}$$

Here, $\mu_i(t, x(t)) \in \mathbb{R}$ is the relative expected instantaneous change in price of P_i per unit time and $\sigma_i(t)\sigma_i(t)^T$ is the instantaneous variance per unit time ($\sigma_i \in \mathbb{R}^{1\times n}$ is the i-th row of the matrix $\sigma(t) \in \mathbb{R}^{n\times n}$). The n-dimensional Brownian motion $dZ_P(t)$ is defined on a fixed, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ with $\mathcal{F}_t$ satisfying the usual conditions. By adding a further, "risk-free" asset or rather a bank account with a short-term interest rate, i.e., with volatility $\sigma_0 \equiv 0$, and instantaneous rate of return $\mu_0(t, x(t))$ referred to as $r(t, x(t))$, we obtain a risk free asset $P_0(t)$ given by the differential equation

$$\begin{aligned} \frac{dP_0(t)}{P_0(t)} &= r(t, x(t))dt\,, \\ P_0(0) &= p_{00} > 0\,. \end{aligned}$$

The drift terms of the risk-bearing and risk-free assets depend on the m-dimensional factor processes $(x_1(t), x_1(t), \dots x_m(t))$. Therefore, the factors affect the mean return of the risk-bearing assets and the interest rate of the risk-free asset. Furthermore, we assume that the drift terms are affine functions of the factor levels, as given by

$$\mu_i(t, x(t)) = G_i(t)x(t) + g_i(t)\,, \tag{1}$$
$$\mu_0(t, x(t)) = r(t, x(t)) = F_0(t)x(t) + f_0(t)\,, \tag{2}$$

where $x(t) = (x_1(t), x_1(t), \dots x_m(t))^T \in \mathbb{R}^m$, $G_i(t), F_0(t) \in \mathbb{R}^{1\times m}$, and $g_i(t), f_0(t) \in \mathbb{R}$.

2.2 Factor dynamics

The factors are modelled as Gaussian stochastic processes obtained by

$$dx(t) = (A(t)x(t) + a(t))dt + \nu(t)dZ_x(t)\,,$$

where $x(0) = x_0$, $A(t) \in \mathbb{R}^{m\times m}$, $a(t) \in \mathbb{R}^m$, $\nu(t) \in \mathbb{R}^{m\times m}$, and $dZ_x(t) \in \mathbb{R}^m$. The correlation matrix between $dZ_P(t)$ and $dZ_x(t)$ is $\overline{\rho}(t) \in \mathbb{R}^{n\times m}$. The m-dimensional Brownian motion $dZ_x(t)$ is defined on a fixed, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ with $\mathcal{F}_t$ satisfying the usual conditions. The factor process $x(t)$ allows us to model variables of either macroeconomic, industry specific, or company specific nature which affects the mean returns of the risk-bearing assets.

2.3 Self-financing portfolio and wealth dynamics

Assuming that an investor's wealth only derives gains form his investments, his wealth dynamics (portfolio dynamics) can be expressed by the following expression:

$$dW(t) = \sum_{i=0}^{n} u_i(t)W(t)\mu_i(t, x(t))\, dt \tag{3}$$
$$+ \sum_{i=0}^{n} u_i(t)W(t)\sigma_i(t)dZ_{P_i}(t) \tag{4}$$
$$W(0) = W_0$$

where $u_i(t)$ denotes the fraction of wealth (portfolio value) invested in the i-th asset at time t. The weights fulfil the constraints $\sum_{i=0}^{n} u_i(t) = 1$ and $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T \in \mathcal{U}$ where $\mathcal{U}$ is a convex set. Assuming that no money flows in to or out of the portfolio, the dynamics of the portfolio are self-financing. For the derivation of the wealth dynamics, the reader may refer to [6, Chapter 5].

Furthermore, we utilize (1) and (2) to obtain the form of the wealth equation we work with in this paper:

$$\begin{aligned} dW(t) &= u(t)^T(F(t)x(t) + f(t))W(t)\, dt \\ &\quad + (F_0(t)x(t) + f_0(t))W(t)\, dt \\ &\quad + W(t)u(t)^T\sigma(t)dZ_P(t)\,, \end{aligned} \tag{5}$$

where we use the following abbreviations

$$\begin{aligned} &G(t) = [G_1^T(t), G_2^T(t), \dots, G_n^T(t)]^T \in \mathbb{R}^{n\times m}, \\ &F(t) = G(t) - eF_0(t) \in \mathbb{R}^{n\times m} \\ &g(t) = (g_1(t), \dots, g_n(t))^T \in \mathbb{R}^{n\times 1}, \\ &f(t) = g(t) - ef_0(t) \in \mathbb{R}^{n\times 1} \\ &\sigma = [\sigma_i^T, \dots, \sigma_n^T] \in \mathbb{R}^{n\times n}, \\ &e = (1, 1, \dots, 1)^T \in \mathbb{R}^{n\times 1}\,. \end{aligned}$$

The term $F(t)x(t)+f(t)$ denotes the excess mean return of the risky investments (i.e., the mean return above the risk-free interest rate) and the term $F_0(t)x(t) + f_0(t)$ denotes the risk-free interest rate.

2.4 Portfolio optimization with CRRA utility

The portfolio choice problem is to maximize the expected power utility defined for the terminal wealth, i.e., $\mathrm{E}[\frac{1}{\gamma}W(T)^\gamma]$. The mean returns of risk-bearing assets and the interest rates of the risk-free bank account are linear affine functions of the factor process. Furthermore, we assume that leveraging, short-selling, and borrowing at the risk-free rates are unrestricted, i.e., $\mathcal{U} = \mathbb{R}^n$. The factor dynamics and the risk-bearing asset dynamics are assumed to be correlated. Mathematically the problem statement is:

$$\begin{aligned}
\max_{u(\cdot)\in\mathbb{R}^n} \quad & \mathrm{E}\Big[\frac{1}{\gamma}W(T)^{\gamma}\Big] \\
\text{s.t.} \quad & \\
dW(t) &= W(t)[F_0(t)x(t)+f_0(t) \\
&\quad + u^T(t)(F(t)x(t)+f(t))]dt \\
&\quad + W(t)u^T(t)\sigma(t)dZ_P\,, \\
dx(t) &= (A(t)x(t)+a(t))dt+\nu(t)dZ_x\,, \\
dZ_P dZ_x &= \overline{\rho}(t)dt\,, \qquad (6)
\end{aligned}$$

with $W(0) = W_0$ and $x(0) = x_0$, T denotes the time horizon and $\gamma < 1$ denotes the coefficient of risk aversion.

2.5 Solution to the portfolio optimization with CRRA utility

The Hamilton-Jacobi-Bellman (HJB) equation for this particular optimal control problem is given by

$$\begin{aligned}
& J_t(\cdot) + \max_{u(\cdot)\in\mathbb{R}^n}\Big[W(t)(F_0(t)x(t)+f_0(t) \\
& +u^T(t)(F(t)x(t)+f(t)))J_W(\cdot) \\
& +(A(t)x(t)+a(t))^T J_x(\cdot) \\
& +\frac{1}{2}W^2(t)u^T(t)\Sigma(t)u(t)J_{WW}(\cdot) \\
& +W(t)u^T(t)\sigma(t)\overline{\rho}(t)\nu^T(t)J_{Wx}(\cdot) \\
& +\frac{1}{2}\mathrm{tr}\{J_{xx}(\cdot)\nu(t)\nu^T(t)\}\Big] = 0\,, \qquad (7)
\end{aligned}$$

where $\Sigma(t) = \sigma(t)\sigma^T(t)$ and with the terminal condition $J(T, W(T), x(T)) = \frac{1}{\gamma}W(T)^{\gamma}$. We have omitted the arguments of $J(\cdot) = J(t, W(t), x(t))$ for writing convenience. The solution to (7) is the optimal feedback controller obtained by

$$\begin{aligned}
u^*(\cdot) &= \frac{1}{1-\gamma}\Sigma^{-1}(t)\Big(F(t)x(t)+f(t) \\
&\quad +\sigma(t)\overline{\rho}(t)\nu^T(t)(K_3(t)x(t)+k_2(t))\Big), \qquad (8)
\end{aligned}$$

where $k_2(t)$ and $K_3(t)$ are the solutions of two ordinary differential equations (ODEs). The first ODE for the vector $k_2(t)$ is given by

$$\begin{aligned}
\dot{k}_2 &+ \gamma F_0^T + K_3\nu\nu^T k_2 + A^T k_2 + K_3 a \\
&- \frac{\gamma}{(\gamma-1)}\Big(F^T\Sigma^{-1}f + F^T\Sigma^{-1}\sigma\overline{\rho}\nu k_2 \\
&+ K_3\nu\overline{\rho}^T\sigma^T\Sigma^{-1}f + K_3\nu\overline{\rho}^T\overline{\rho}\nu^T k_2\Big) = 0 \\
k_2(T) &= 0\,. \qquad (9)
\end{aligned}$$

The second ODE for the matrix $K_3(t)$ is obtained as

$$\begin{aligned}
\dot{K}_3 &+ K_3\nu\nu^T K_3 + K_3 A + A^T K_3 \\
&- \frac{\gamma}{(\gamma-1)}\Big(F^T\Sigma^{-1}F + F^T\Sigma^{-1}\sigma\overline{\rho}\nu^T K_3 \\
&+ K_3\nu\overline{\rho}^T\sigma^T\Sigma^{-1}F + K_3\nu\overline{\rho}^T\overline{\rho}\nu^T K_3\Big) = 0 \\
K_3(T) &= 0\,. \qquad (10)
\end{aligned}$$

The value function for this HJB problem is

$$\begin{aligned}
J(t, W, e) &= \frac{1}{\gamma}W(t)^{\gamma}l(t, x), \\
l(t, x) &= e^{k_1(t)+k_2^T(t)x(t)+\frac{1}{2}x^T(t)K_3(t)x(t)}\,, \qquad (11)
\end{aligned}$$

with terminal conditions $k_1(T) = 0$, $k_2(T) = 0$, and $K_3(T) = 0$. We have omitted the ODE for $k_1(t)$ since the optimal controller does not depend on this scalar. This results in solving $\frac{1}{2}m^2 + \frac{3}{2}m$ ODEs, where m is the dimension of external variables in vector $x(t)$. For a detailed derivation of the HJB equation and its solution the reader may refer to [7]. The conditions for solving this matrix Riccati equations are found in [7] as well.

3 Application with alternative investments

In this section, the general solution is applied to a specific case. We consider an investor, having three risky investment opportunities. Therefore the investor faces three risk exposures, i.e., market risk, interest rate risk, and the risks involved in the alternative investment. Each of the three investment opportunities offers a different risk-return profile. The fixed income part is modelled by a Vasicek short rate model (see [2]). The second investment opportunity is the market portfolio. We use the S&P500 as a proxy for the market portfolio because of its popularity. It is modelled by a geometric Brownian motion. Its drift and diffusion are constant. For the hedge fund we use a model including the Greek letters α and β. As mentioned in the introduction, α and β originate from Sharpe's capital asset pricing model (CAPM). In this paper, however, we only use the terminology of the CAPM but do not need the assumptions of the CAPM. We use a model resembling to Merton's intertemporal capital asset pricing model (ICAPM), see [4], because we are modelling in a continuous-time framework. The authors of [5] use a similar model for hedge funds. As a consequence, the alternative investment does not have constant risk premium. This is also the case for the market portfolio since its drift term is constant and the risk-free rate is not. The investment opportunities are modelled as appropriate stochastic differential equations. The investor's utility function is chosen to have constant relative risk aversion. The problem is solved by using the results of Section 2.5.

3.1 The model

In order to derive the optimal investment strategy, we first need to model the three considered investment opportunities. We use stochastic differential equations (SDEs) to model the dynamic behavior of the assets. The Brownian motions of the SDEs are defined on a fixed, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ with $\mathcal{F}_t$ satisfying the usual conditions. For the fixed income part, we use the short rate model of Vasicek (see [2]). The investor is able to put his money into a bank account. The bank account has an interest equivalent to the short rate. We therefore have the following mean-reverting SDE for the short rate r,

$$\begin{aligned} dr &= \kappa(\theta - r)dt + \sigma_r dZ_r, \\ r(0) &= r_0, \end{aligned} \tag{12}$$

where $\kappa \in \mathbb{R}$, $\theta \in \mathbb{R}$, and $\sigma_r \in \mathbb{R}$ are the constant parameters of the short rate. Given the short rate, we solely need to determine the price of risk λ to determine the dynamics of the bond B with maturity T,

$$\begin{aligned} dB &= B\Big(r + \frac{\lambda\sigma_r}{\kappa}a_T(t)\Big)dt - B\frac{\sigma_r}{\kappa}a_T(t)dZ_r, \\ B(T) &= 1, \end{aligned} \tag{13}$$

where the scalar function $a_T(t)$ is defined as

$$a_T(t) = 1 - e^{-\kappa(T-t)}. \tag{14}$$

The second investment opportunity is a passive fund, regarded as a proxy of the market portfolio. The S&P500 is a popular index used as a proxy for the market portfolio. Therefore, the passive fund S follows the SDE

$$\begin{aligned} dS &= S\mu_S dt + S\sigma_S dZ_S, \\ S(0) &= S_0, \end{aligned} \tag{15}$$

where $\mu_S \in \mathbb{R}$ and $\sigma_S \in \mathbb{R}$ are the constant parameters of the model. The Brownian motion Z_S is assumed to be independent of Z_r. As a last step, the model for the alternative asset remains to be introduced. The price of the alternative asset, denoted by A, evolves as follows:

$$\begin{aligned} dA &= A(r + \beta(\mu_s - r) + \alpha)dt \\ &\quad + A\sigma_A(\rho dZ_s + \sqrt{1-\rho^2}dZ_A), \\ A(0) &= A_0, \end{aligned} \tag{16}$$

where $\beta \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\sigma_A \in \mathbb{R}$, and $\rho \in [0,1]$ are the constant parameters of the model. The Brownian motion Z_A is assumed to be independent of Z_r and Z_S. In this context, $r + \beta(\mu_s - r)$ describes the risk adjusted return of the asset with respect to the market, whereas the α is the outperformance of the alternative asset. The β parameter is defined to be

$$\beta = \frac{\text{cov}(dS/S, dA/A)}{\sigma_S^2} = \frac{\rho\sigma_A}{\sigma_S}, \tag{17}$$

where ρ denotes correlation of the return of the market portfolio and the return of the alternative asset. We introduce a three dimensional control vector u. The three components of u represent the percentage of total wealth invested in the respective investment category. The wealth equation was introduced in Section 2.3,

$$\begin{aligned} dW(t) &= u(t)^T(F(t)x(t) + f(t))W(t)\,dt \\ &\quad + (F_0(t)x(t) + f_0(t))W(t)\,dt \\ &\quad + W(t)u(t)^T\sigma(t)dZ_P(t), \\ W(0) &= W_0. \end{aligned}$$

For the specific case considered, the factor loading matrices are $\nu = \sigma_r$, $A = -\kappa$, $a = \kappa\theta$, $F_0 = 1$, $f_0 = 0$,

$$G = \begin{pmatrix} 1 \\ 0 \\ 1-\beta \end{pmatrix}, \quad g(t) = \begin{pmatrix} \frac{\lambda\sigma_r}{\kappa}a_T(t) \\ \mu_S \\ \beta\mu_S + \alpha \end{pmatrix},$$

$$F = \begin{pmatrix} 0 \\ -1 \\ -\beta \end{pmatrix}, \quad f(t) = \begin{pmatrix} \frac{\lambda\sigma_r}{\kappa}a_T(t) \\ \mu_S \\ \beta\mu_S + \alpha \end{pmatrix}.$$

In our specific case, the wealth equation becomes

$$\begin{aligned} dW &= W\Big[u^T\mu(t,r) + r\Big]dt + Wu^T\sigma(t)dZ, \\ W(0) &= W_0, \end{aligned} \tag{18}$$

where $u \in \mathbb{R}^{3\times 1}$, $dZ = [dZ_r, dZ_s, dZ_A]^T$. The vector $\mu(t,r)$ in equation 18 is defined by

$$\mu(t,r) = Fr + f(t) = \begin{pmatrix} \frac{\lambda\sigma_r}{\kappa}a_T(t) \\ \mu_s - r \\ \beta(\mu_s - r) + \alpha \end{pmatrix}, \tag{19}$$

whereas the matrix $\sigma(t)$ is defined to be

$$\sigma(t) = \begin{bmatrix} -\frac{\sigma_r}{\kappa}a_T(t) & 0 & 0 \\ 0 & \sigma_s & 0 \\ 0 & \sigma_A\rho & \sigma_A\sqrt{1-\rho^2} \end{bmatrix}. \tag{20}$$

We need the matrix $\sigma(t)\sigma^T(t)$ to be invertible and therefore demand that $|\rho| < 1$.

3.2 Asset allocation with CRRA utility

The portfolio choice problem is to maximize the expected power utility defined over terminal wealth, i.e., $\text{E}[\frac{1}{\gamma}W(T)^\gamma]$. Furthermore we assume that leveraging, short-selling, and borrowing at the risk-free rates are unrestricted, i.e., $\mathcal{U} = \mathbb{R}^n$. Mathematically the problem statement is

$$\begin{aligned} \max_{u\in\mathbb{R}^3} & \quad \text{E}\Big\{\frac{1}{\gamma}W^\gamma(T)\Big\} \\ s.t. & \\ dW &= W\Big[u^T\mu(t,r) + r\Big]dt + Wu^T\sigma(t)dZ \\ W(0) &= W_0 \\ dr &= \kappa(\theta - r)dt + \sigma_r dZ_r, \\ r(0) &= r_0, \end{aligned}$$

where T denotes the time horizon and $\gamma < 1$ denotes the coefficient of risk aversion.

3.3 Solution to asset allocation with CRRA utility

The Hamilton-Jacobi-Bellman (HJB) equation of this problem, given $\Sigma(t) = \sigma(t)\sigma(t)^T$ and $e_1 = \overline{\rho}(t) = [1,0,0]^T$, is

$$
\begin{aligned}
J_t(t,x) + \max_{u\in\mathbb{R}^3} \Big[& W[u^T\mu(t,r)+r]J_W \\
& +\kappa(\theta-r)J_r + \frac{1}{2}J_{WW}W^2u^T\Sigma u \\
& +J_{Wr}\sigma_r W u^T\sigma e_1 + \frac{1}{2}J_{rr}\sigma_r^2\Big] \quad = \quad 0. \quad (21)
\end{aligned}
$$

with terminal condition $J(T,W(T)) = \frac{1}{\gamma}W^\gamma(T)$. For the sake of completeness, we give the resulting partial differential equation (PDE) which has to be solved

$$
\begin{aligned}
J_t - \frac{1}{2}\frac{J_W^2}{J_{WW}}\mu^T\Sigma^{-1}\mu + rWJ_W & \\
+\kappa(\theta-r)J_r + \frac{J_W J_{Wr}}{J_{WW}}\sigma_r\lambda & \\
-\frac{1}{2}\frac{J_{Wr}^2}{J_{WW}}\sigma_r^2 + \frac{1}{2}J_{rr}\sigma_r^2 & = 0. \qquad (22)
\end{aligned}
$$

As in the general solution, the value function for the HJB problem is of the form

$$
J(t,W) \;=\; \frac{1}{\gamma}W(t)^\gamma e^{k_1(t)+k_2(t)r(t)+\frac{1}{2}k_3(t)r^2(t)}
$$

with terminal conditions $k_1(T) = k_2(T) = k_3(T) = 0$. Inserting this in the HJB equation (22) serves as a verification of the Ansatz. The two functions $k_2(t)$ and $k_3(t)$ are the solutions of two coupled ordinary differential equations (ODEs). The ODE for $k_3(t)$ is

$$
\begin{aligned}
\dot{k}_3 - 2\kappa k_3 + \frac{\sigma_r^2}{1-\gamma}k_3^2 - h_1 &= 0 \\
k_3(T) &= 0\,. \qquad (23)
\end{aligned}
$$

The only unknown in the ODE for $k_3(t)$ is the constant h_1, which is defined by

$$
h_1 = \frac{\gamma}{(\gamma-1)\sigma_S^2}.
$$

The ODE for k_3 is independent of k_1 and k_2 and can be therefore solved independently. Because of the form of the ODE (23), k_3 can be solved analytically. Its solution is

$$
k_3(t) = \frac{1-\gamma}{\sigma_r^2}\Big(\kappa + \delta\tanh\Big\{(t-T)\delta - \operatorname{atanh}\Big(\frac{\kappa}{\delta}\Big)\Big\}\Big).
$$

The constant δ in the solution of $k_3(t)$ is defined by

$$
\delta = \sqrt{\frac{h_1\sigma_r^2}{1-\gamma} + \kappa^2}.
$$

The only unknown of the system remains to be $k_2(t)$. From the general solution we know that $k_2(t)$ is the solution of an ODE which is dependent of $k_3(t)$. The ODE for $k_2(t)$ is, for our specific case, given by

$$
\begin{aligned}
\dot{k}_2 - \kappa k_2 + \frac{\sigma_r^2}{1-\gamma}k_2k_3 & \\
+k_3(\kappa\theta + \frac{\gamma}{\gamma-1}\sigma_r\lambda) + \gamma + h_2 &= 0 \\
k_2(T) &= 0\,. \quad (24)
\end{aligned}
$$

As in the ODE for $k_3(t)$ (23), there is one constant in the ODE (24) to be defined. The constant h_2 is defined by

$$
h_2 = \frac{\gamma\mu_S}{(\gamma-1)\sigma_S^2}.
$$

Again, we can give an analytical solution, but the form of the solution of (24) is more complicated than for $k_3(t)$.

$$
k_2(t) = l(t)\Big\{C + \int \frac{-k_3(t)h_3 - \gamma + h_2}{l(t)}dt\Big\}
$$

The integral in the solution cannot be analytically simplified any more. The integration constant C has to be chosen such that the terminal condition $k_2(T) = 0$ is met. The function $l(t)$ in the solution for $k_2(t)$ is defined by

$$
l(t) = \sqrt{\frac{k_3(t)\sigma_r^2 - \kappa(1-\gamma)}{\delta(1-\gamma)} - 1}.
$$

The constant h_3, appearing in the solution of $k_2(t)$, is

$$
h_3 = \kappa\theta - \frac{\gamma}{\gamma-1}\sigma_r\lambda.
$$

We can finally state the optimal control law as

$$
\begin{aligned}
u^*(t,r) \;=\; & \frac{1}{1-\gamma}\Sigma(t)^{-1}\Big(\mu(t,r) \\
& -[k_2(t)+k_3(t)r(t)]\frac{\sigma_r^2}{\kappa}a_T(t)e_1\Big). \quad (25)
\end{aligned}
$$

The conditions for existence of a solution are $\sigma_r \neq 0$, $\sigma_S \neq 0$, and $\gamma < 1$.

Since we want to optimize a portfolio with alternative investments, we are especially interested in the third component of u in (25). It reflects the fraction of wealth allocated to the alternative investment, and is given by

$$
u_3^* \;=\; \frac{\alpha}{(1-\gamma)(1-\rho^2)\sigma_A^2}. \qquad (26)
$$

Obviously, u_3^* is independent of the time horizon T, the bond price, and its parameters, as well as the market itself. The amount of capital invested in the alternative investment increases linearly with α. The closer σ_A^2 is to zero, the more is invested in the alternative investment. At first sight, it is counter-intuitive that the larger the absolute value of the correlation ρ, the more is invested in the alternative asset. But if we take a look at u_2^*, the fraction of wealth invested in the market, we observe that, for large absolute values of ρ, the value of u_2^* changes significantly. The allocation rule exploits the correlation property by taking much more extreme positions when large positive or negative correlation is present.

$$
u_2^* \;=\; \frac{1}{(1-\gamma)\sigma_S^2}(\mu_S - r) - u_3^*\frac{\sigma_A}{\sigma_S}\rho. \qquad (27)
$$

The first term of u_2^* is seen to be the well-known Merton solution (see [6, Chapter 4]) whereas the second term depends on the amount of wealth invested in the alternative investment u_3^*. If the correlation ρ is equal to zero, the position in the market is the same as in the standard Merton case. If ρ is positive, the position in the market is reduced in favor of the position in the alternative investment (assuming a positive α). The interesting property lies in the fact that, if the correlation is negative, the optimal weight in the market is larger than in positively correlated case. The lower the correlation the more the downturns of the alternative investment are hedged by the position in the stock market.

One reason to include hedge funds in portfolios is because of their benefits of diversification, i.e., low correlation. In the perfect case ($\rho = 0$), the fraction of wealth invested in the market remains unaffected in terms of the Merton solution. Because of the lack of dependence between the bond market and the stock market and the alternative investment respectively, the amount of wealth invested in the Bond is independent of the characteristics of the market and the alternative investment.

4 Analysis of the control law with US data

In this section, the optimal control vector $u(t)$ is computed for real market conditions. We use US stock market data and the CSFB/Tremont hedge fund index. For the passive fund, i.e., the substitute for the market portfolio, the S&P500 is used. As a proxy for the short rate, we use three month treasury bills, which have interest rates close to the ones payed on a money market account. For the bond portfolio part, the *DATASTREAM USA TOTAL 3-5 YEARS* bond index is used. In order to account for the coupon payments, the total return index data is used which is a suitable approximation for the zero coupon bond. All data sets were obtained from Thomson DATASTREAM (DS). The three month treasury bills and the S&P500 data ranges from 1972 to 2004. The data for the bond index ranges from 1980 to 2004 and the CSFB/Tremont hedge fund index ranges from 1994 to 2004. The data is obtained on a weekly basis except for the CSFB/Tremont hedge fund index which is only available on a monthly basis.

The three month treasury bills are used as a proxy for the short rate for two reasons. The first is because of its long availability (since 1972), which is important for the estimation of the short rate parameters. The second reason is that its stochastic behavior reflects more an SDE type behavior rather than the federal fund rate.

The resulting control vector $u(t)$ crucially depends on the parameters chosen. The parameters used for the market portfolio can be estimated with long time series of data and are therefore reliable long term estimates. This is also the case for the fixed income security.

We discretize the stochastic differential equation for the short rate (12) with the method of Euler (see [8]). This yields

$$r_{t+1} \quad = \quad \kappa\theta\Delta t + (1-\kappa\Delta t)r_t + \sigma_r\sqrt{\Delta t}\xi_r,$$

where Δt is the time increment and ξ_r is a standard normal random variable. The parameters of the short rate are estimated by doing an ordinary least squares estimation on the discrete version of the short rate.

The allocation in the bond and the money market are very high if the maturity of the bond and the end of the investment horizon coincide. The controller is making use of the convergence of the r and B process for $t \to T$. The highly leveraged positions in the bond are, of course, very unrealistic. Therefore, we assume that the bond has a fixed duration, i.e., having a roll-over bond portfolio part in the over all portfolio. This can be achieved by changing the the time-varying function $a_T(t)$, equation (14), to be a function of the duration of the bond portfolio part only. If we denote the duration of the bond by τ, we get

$$a_T(\tau) = 1 - e^{-\kappa\tau}.$$

The duration is estimated from the DS bond index time series. By using a fixed duration for the bond dynamics we rather invest in a bond index than in a specific bond itself. We discretize the stochastic differential equation of the logarithmic bond prices (13) with the method of Euler (see [8]) and get the following series

$$\ln(B_{t+1}) - \ln(B_t) - r_t\Delta t = \Big(\lambda\frac{\sigma_r}{\kappa}a_T(\tau) + \frac{1}{2}\Big(\frac{\sigma_r}{\kappa}a_T(\tau)\Big)^2\Big)\Delta t + \frac{\sigma_r}{\kappa}a_T(\tau)\sqrt{\Delta t}\xi_B,$$

where Δt is the time increment and ξ_B is a standard normal random variable. The duration τ and the price of risk λ of the bond index are estimated by estimating mean and variance of the series above.

The drift and the diffusion of the market portfolio are computed similarly to the bond prices. The prices (15) are transformed with the natural logarithm. The resulting stochastic differential equation is then used in its discrete version, using the method of Euler (see [8]). This gives the relationship

$$\ln(S_{t+1}) - \ln(S_t) = \big(\mu_S - \frac{1}{2}\sigma_S^2\big)\Delta t + \sigma_S\sqrt{\Delta t}\xi_S,$$

where Δt is the time increment and ξ_S is a standard normal random variable.

It is well known that hedge fund indices cannot give a true picture of the hedge fund universe (see [9]). We are well aware of this fact but nevertheless use a hedge fund index. This because many investable hedge fund have emerged recently (e.g., CSFB/Tremont, HFR, MSCI, S&P) and therefore are of practical relevance. The drift and the diffusion of the CSFB/Tremont hedge fund index are computed as for the market portfolio. The correlation is estimated by calculating the correlation of the residuals of $\ln(A)$ and $\ln(S)$. With these estimates at hand, α can be estimated by subtracting $r + \beta(\mu_S - r)$ from the mean of the CSFB/Tremont hedge fund index returns.

Using the time series of the market and the short rate from 1972 to 2004, the bond index from 1980 to 2004, and

Table 1. Typical values for the estimated parameters.

Parameter	value	std. error
κ	0.27	0.05
θ	0.07 p.a.	0.004 p.a.
σ_r	0.02 p.a.	0.0006 p.a.
λ	0.55	0.04
τ	3.77 years	0.4 years
μ_S	0.1 p.a.	0.008 p.a.
σ_S	0.15 p.a.	0.004 p.a.
ρ	0.5	0.07
α	0.04 p.a.	0.005 p.a.
σ_A	0.09 p.a.	0.005 p.a.

the alternative asset from 1994 to 2004, we estimate the parameters of the stochastic processes. Table 1 shows the results.

Note that price of risk λ for the bond has a positive value as in the original Vasicek model although it is often found to be introduced with a negative sign in recent texts. In Figure 1, the asset allocation for an investor with a risk aversion coefficient $\gamma = -5$ is displayed. We see that for the observed parameter values, the weight in the bond is the biggest for every t. This may change by choosing different parameter values. The bigger γ, the more aggressive the investor is allocating his wealth.

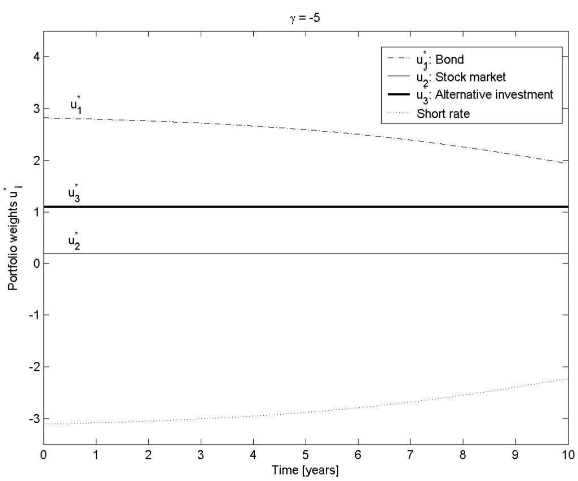

Figure 1. Asset allocation for $\gamma = -5$, $r = 0.03$, and an investment horizon $T = 10$

As already mentioned, the values of $u(t)$ crucially depend on the choice of the parameters. The notoriously secretive and intransparent field of alternative investments does not allow to estimate its parameters in the same quality as for the other securities. Performance numbers are usually published monthly. Hedge funds do not have to register. As a result, various biases such as survivorship bias, selection bias, etc. are present in the available hedge fund databases. As a consequence, there is no index representing the true performance of the hedge fund industry. The task for a reliable estimation of α and β remains rather difficult. A further problem is the question of stability for α and β. Because of the lack of sufficient data, the question of stability is even harder to answer. We therefore analyze the amount of wealth invested in the alternative investment for varying α and ρ (or β, respectively). Figure 2 shows

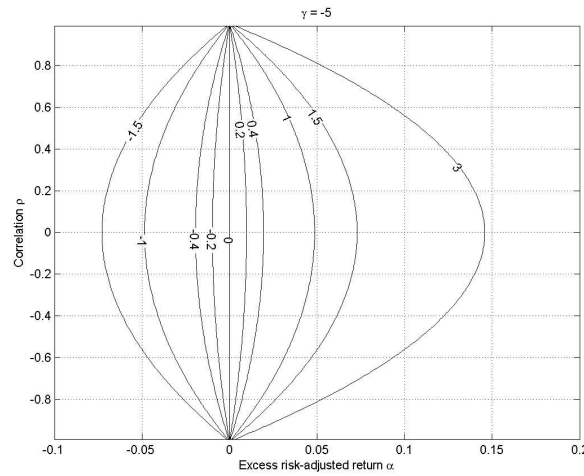

Figure 2. Optimal asset allocation to alternative investment as a function of ρ and α: contour plot of u_3^* for $\gamma = -5$

a contour plot of u_3^* as a function of $\alpha \in [-0.1, 0.2]$ and $\rho \in (-1, 1)$. Note that β can be calculated by equation (17). Again we see that investment opportunities offering no α are not included in the optimal portfolio. The contours of the plot are symmetric with respect to $u_3^* = 0$. By increasing γ, i.e., making the investor less risk averse, the same levels of constant u_3^* as in Figure 2 are moving closer to the $u_3^* = 0$ line. It is significant that a low correlation results in a lower weight of the alternative investment in the portfolio.

5 Back-test of the control law with US data

In this section, the derived investment strategies are tested with data introduced in the last section. In order to have reasonable estimates of all parameters, the investment strategy is implemented starting in January 1997. The portfolio can be adjusted every month, the investment horizon of the investor ends in March 2004. As new observations are available, the model parameters are recalculated using all past data available. The application of the data is done as in [10] and [11]. The investment strategy is always implemented in an out-of-sample manner. Figure 3 shows the results for $\gamma = -5$. The investment strategy outperforms the S&P500 and the CSFB/Tremont hedge fund index by far. This is mostly because of the highly leveraged positions in the bond index. Table 2 shows the key figures of the considered time series. The Sharpe ratio of the mar-

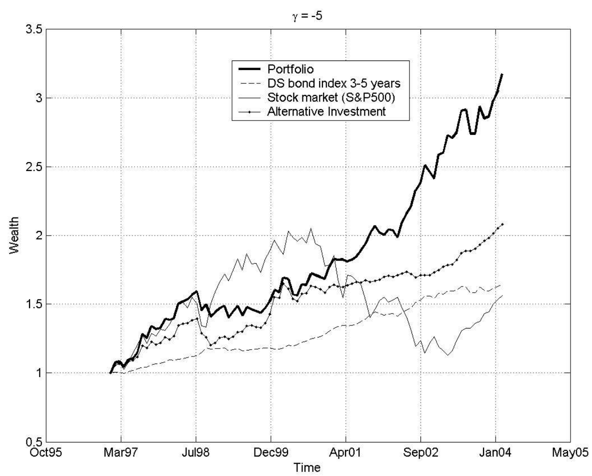

Figure 3. Wealth evolution for $\gamma = -5$

ket is pretty poor compared with the others due to the bear market from 2000 to 2003. It is noteworthy that the Sharpe ratio of our portfolio does not change significantly for different γs. The high Sharpe ratios in Table 2 give evidence that the risk adjusted returns of the investment strategy are superior to the ones of the single assets.

Table 2. Key figures of the time series

	return (p.a.)	volatility (p.a.)	Sharpe ratio
$\gamma = -5$	0.17	0.12	1.06
$\gamma = -10$	0.11	0.07	1.06
$\gamma = -20$	0.08	0.04	1.07
DS 3-5 years	0.07	0.04	0.9
S&P500	0.08	0.17	0.23
CSFB/Tremont	0.11	0.08	0.84

In the phase of July 1998 to December 1999, the investment strategy does not show a good performance. This because of the enormous drop of performance of the hedge fund index in 1998 which causes the controller to significantly reduce its position in the alternative asset. In the beginning of 2001, the controller is starting to take short positions in the market, which is still the case at the end of the considered time period.

6 Conclusions

The solution for a general case with an arbitrary number of risk factors is presented. The general solution is applied to a specific case where the investor has three risky investment opportunities. The investor is able to invest in the stock market, the bond market, the money market, and in the alternative investment universe. The model for the alternative asset resembles Mertons intertemporal capital asset pricing model and includes the parameters α and β. The optimal amount of capital invested in the alternative asset depends only on the parameters of the alternative investment, the correlation to the market portfolio, and on the risk aversion coefficient. The more the market and the alternative asset are correlated, the more the investment strategy makes use of this fact by leveraging the positions in the market and in the alternative asset. If the correlation of alternative asset and the market is zero, the optimal amount of capital invested in the market is the same as if there is no alternative investment opportunity. The resulting investment strategy is tested with US data and the CSFB/Tremont hedge fund index. The strategy outperforms the single assets in terms of absolute returns and in terms of risk adjusted returns. This is mainly because of the highly leveraged positions in the fixed-income security and in the short rate.

Further research may solve the problem with constraints on the asset allocation in order to obtain results for real-world decision making.

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