Flocking of networked mechanical systems on directed topologies: a new perspective

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Flocking of networked mechanical systems on directed topologies: a new perspective

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In this paper, we investigate the flocking problem for multiple uncertain mechanical systems interacting on directed topologies. We propose an integral-sliding adaptive control to realise the objective of flocking, which gives rise to a cascade closed-loop system, and also propose a new notion – iBIBO (integral-bounded-input bounded-output) stability to characterise a new input–output property of a special class of dynamical systems, which is then used for the stability analysis of the closed-loop system. With the proposed iBIBO analysis tool and Lyapunov-like approach, we show that the proposed adaptive controller ensures the position and velocity consensus of the mechanical systems if and only if the damping is greater than certain lower bound for a given stiffness, and that the velocities of the mechanical systems converge to the weighted average of their initial values and the positions of the mechanical systems converge to the initial-states-dependent ramp trajectory. The performance of the proposed adaptive control scheme is shown by numerical simulation results.

Keywords: networked mechanical systems; iBIBO stability; flocking; adaptive control; uncertainties

1. Introduction

There are a large number of synchronised phenomena in our physical world (see Reynolds, 1987 for some examples), one frequently observed behaviour of which is flocking. The computer generation of the flocking behaviour dates back to the pioneering work in Reynolds (1987), which introduces the following three heuristic rules (also known as Reynolds rules, which are quoted below) to produce flocking:

1. flock centring: attempt to stay close to nearby flockmates;
2. velocity matching: attempt to match velocity with nearby flockmates;
3. collision avoidance: avoid collisions with nearby flockmates.

During the past 10 years, many control schemes have been proposed to solve the flocking problem for networked double-integrator agents (Abdessameud & Tayebi, 2010; Cao & Ren, 2010; Lee & Spong, 2007; Li & Spong, 2010; Olfati-Saber, 2006; Ren, 2008; Ren & Atkins, 2007; Shi, Wang, & Chu, 2009; Su, Wang, & Lin, 2009; Tanner, Jadbabaie, & Pappas, 2007; Yu, Chen, & Cao, 2010). These control schemes can generally be grouped, in accordance with the interaction topologies among the agents, into two classes. The first class of the schemes (e.g., Abdessameud & Tayebi, 2010; Cao & Ren, 2010; Olfati-Saber, 2006; Shi et al., 2009; Su et al., 2009; Tanner et al., 2007) achieves flocking of multiple agents on undirected topologies, where the stability and convergence of the closed-loop system is usually demonstrated by constructing various potential functions. The second class of flocking controllers (e.g., Lee & Spong, 2007; Ren & Atkins, 2007; Yu, Chen, & Cao, 2010; Li & Spong, 2010) achieves the flocking objective under the more general directed topologies, and the major challenge of their stability analysis is the asymmetry of the Laplacian associated with a directed topology. To break through this challenge, passive decomposition approach (Lee & Spong, 2007; Li & Spong, 2010) and Laplacian-eigenvalue-based analysis (Ren & Atkins, 2007; Yu et al., 2010) are proposed, where the work in Lee and Spong (2007) and Li and Spong (2010) considers the more realistic networked agents with nonidentical masses, and the lower bound of the damping is derived in the case of exact knowledge of the mass properties of the agents.

Yet, the above results are insufficient or inapplicable for many real applications, such as formation of multiple spacecraft or mobile autonomous robots, planetary exploration by multiple rover agents, and cooperation of multiple robot manipulators. These plants are usually known as mechanical systems in the literature, and their dynamic behaviour is generally described by the standard Euler–Lagrange equations of motion. The challenge in the flocking controller design for multiple mechanical systems comes from both

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the nonlinearity and uncertainty of the Euler–Lagrange dynamics (see, e.g., Slotine & Li, 1991; Spong, Hutchinson, & Vidyasagar, 2006), inhibiting the direct application of the flocking algorithms for double-integrator agents. The work that attempts to resolve the flocking problem of multiple mechanical systems appears in Chopra, Stipanović, and Spong (2008), Sarlette, Sepulchre, and Leonard (2009) and Wang (2013a). In Chopra et al. (2008), the proportional-derivative (PD)-like velocity/position coupling control is employed to realise the flocking objective on undirected topologies (the velocity coupling topology is allowed to be balanced), and to avoid inter-agent collision, a special position coupling control is adopted. However, it should be emphasised that the flocking objective achieved in Chopra et al. (2008) only involves the velocity consensus, and the position consensus cannot be ensured. The rotational flocking of multiple exactly identical rigid bodies (i.e., attitude and angular velocity consensus) on undirected topologies is achieved locally by a PD controller (see, e.g., Sarlette et al., 2009), which, yet, is generally not achievable in the case of nonidentical rigid bodies. It becomes even more challenging when the interaction topology among the mechanical systems is directed. One solution to the flocking problem on directed graphs is given in Wang (2013a), which achieves both the position and velocity consensus by employing a new reference velocity for each system, and the stability of the network is derived by the similarity decomposition approach.

In this paper, we study the flocking problem for multiple nonidentical mechanical systems on directed topologies from a perspective different from Wang (2013a). We propose an integral-sliding adaptive controller to realise the flocking objective, which results in a cascade closed-loop system. Using a new input–output analysis tool—iBIBO (integral-bounded-input bounded-output) stability and Lyapunov-like approach, we illustrate the convergence of the position/velocity consensus errors between the mechanical systems, i.e., the flocking objective is achieved. We also show that the proposed control ensures the weighted average velocity consensus of the networked mechanical systems (in contrast to the consensus controllers in Chopra and Spong (2006), Nuño, Ortega, Basañez, and Hill (2011), Min, Wang, Sun, Gao, and Zhang (2012), Mei, Ren, and Ma (2012) and Wang (2013b), which result in position consensus with the velocities of all systems converging to zero), and additionally ensures that the positions of the mechanical systems converge to the ramp trajectory determined by the initial states of the mechanical systems, analogous to the results for the double-integrator agents (Lee & Spong, 2007; Ren & Atkins, 2007; Yu et al., 2010). Compared with the flocking controller for networked mechanical systems and the analysis in Wang (2013a), the major contribution lies in (1) the new analysis/perspective based on the proposed new notion—iBIBO stability and (2) the incorporation of the integral-sliding action. The introduction of the new stability notion originates from the observation that stability of the network dynamics without a leader is not so strong, e.g., exponential stability of the whole system states is usually not achievable, and we can, at best, achieve marginal stability (since there is a simple zero pole in the closed-loop system). In Wang (2013a), the similarity decomposition approach and input–output analysis for exponentially stable and strictly proper linear systems are used for the position/velocity consensus analysis. Here, we use a new input–output analysis that exploits the input–output property of a marginally stable second-order linear interconnection system with all the poles excluding a simple zero pole being in the open left half plane (LHP) (i.e., iBIBO stability), which does not rely on any coordinate transformation. The incorporation of the integral-sliding action into the controller here ensures the convergence of the systems’ positions to the initial-states-dependent ramp trajectory, in contrast to Wang (2013a). We should also emphasise that the proposed notion of iBIBO stability characterises a new input–output property of a class of dynamical systems [in contrast to the standard BIBO (bounded-input bounded-output) stability (van der Schaft, 2000) and ISS (input-to-state stability) (Dashkovskiy, Efimov, & Sonntag, 2011; Khalil, 2002)], which may possibly be used for the analysis and design of other complex networked systems, and in this sense, the proposed iBIBO stability bears its value in its own right.

There is some other work in the literature focusing on the flocking problem of nonlinear systems or uncertain mechanical systems (Dashkovskiy, Rüffer, & Wirth, 2008; Liu & Jiang, 2013; Meng, Lin, & Ren, 2012; Su, Chen, Wang, & Lin, 2011; Yu, Chen, Cao, & Kurths, 2010). The flocking controllers given in Yu et al. (2010) and Su et al. (2011) take the Lipschitz nonlinearity into consideration. But the Lipschitz nonlinearity is far from enough to describe the nonlinearity of mechanical systems, e.g., the nonlinear Coriolis and centrifugal forces generally depend on the squares and mutual multiplications of the derivatives of the generalised coordinates. The work in Dashkovskiy et al. (2008) and Liu and Jiang (2013) considers the formation of multiple nonholonomic mobile robots with a leader, where the stability of the closed-loop system is shown by the ISS small-gain analysis approach. The work in Meng et al. (2012) considers the swarm tracking of networked Lagrange systems with multiple leaders, where the followers are assumed to interact on undirected topologies and adaptive control is designed to account for the model uncertainties. Nevertheless, the fundamental problem of flocking without a leader on the general directed topologies is not resolved in Dashkovskiy et al. (2008), Liu and Jiang (2013) and Meng et al. (2012). A preliminary version of the paper appears in Wang and Xie (2013).

The rest of the paper is organised as follows. Section 2 gives some preliminaries. Section 3 presents the detailed description of the concept of iBIBO stability. The adaptive
flocking controller for mechanical systems and the ensuing stability analysis are given in Section 4. The simulation results are provided in Section 5. Section 6 concludes the paper.

2. Preliminaries

2.1 Graph theory

Let us begin with the introduction of the theory of directed graphs (Godsil & Royle, 2001; Lin, Francis, & Maggiore, 2005; Olafati-Saber & Murray, 2004; Ren & Beard, 2005; Ren, Beard, & McClain, 2005). In this work, we consider a network of \( n \) mechanical systems. As is now standard, we employ a directed graph \( G = (\mathcal{V}, \mathcal{E}) \) to describe the interaction topology among the \( n \) systems, where \( \mathcal{V} = \{1, 2, \ldots, n\} \) is the vertex set that denotes the collection of the \( n \) systems, and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the edge set that denotes the information flow among the \( n \) systems. The neighbours of system \( i \) constitute a set denoted by \( \mathcal{N}_i = \{j | (i, j) \in \mathcal{E}\} \). A graph is said to have a directed spanning tree if there exists a vertex \( k_0 \in \mathcal{V} \) such that any other vertex of the graph has a directed path to vertex \( k_0 \). Let \( \mathcal{W} = [w_{ij}] \) be the weighted adjacency matrix associated with the graph, which is defined according to the rule that \( w_{ij} > 0 \) if \( j \in \mathcal{N}_i \), and \( w_{ij} = 0 \) otherwise. Furthermore, we make the standard assumption that \( w_{ii} = 0 \), \( \forall i = 1, 2, \ldots, n \). The Laplacian matrix \( \mathcal{L}_w = \{\ell_{w,ij}\} \) is defined as

\[
\ell_{w,ij} = \begin{cases} 
\sum_{k=1}^{n} w_{ik} & \text{if } i = j, \\
-w_{ij} & \text{otherwise.}
\end{cases}
\]

Some important properties of the Laplacian matrix \( \mathcal{L}_w \) are described by the following lemma.

**Lemma 1** (Lin et al., 2005; Ren & Beard, 2005, 2008): Let the Laplacian matrix \( \mathcal{L}_w \) be associated with a directed graph containing a spanning tree. Then, \( \lambda_1 = 0 \) is a simple eigenvalue of \( \mathcal{L}_w \) and all other eigenvalues of \( \mathcal{L}_w \) (denoted by \( \lambda_i \), \( i = 2, 3, \ldots, n \)) are in the open right half plane (RHP), i.e., \( \text{Re} \lambda_i > 0 \), \( i = 2, 3, \ldots, n \), and rank \( \mathcal{L}_w \) = \( n - 1 \). Moreover, there exists a nonnegative vector \( \gamma = [\gamma_1, \gamma_2, \ldots, \gamma_n]^T \) (i.e., \( \gamma_i \geq 0 \), \( i = 1, 2, \ldots, n \)) such that \( \gamma^T \mathcal{L}_w = 0 \) and \( \sum_{k=1}^{n} \gamma_k = 1 \) where \( \gamma_i > 0 \) if and only if vertex \( i \) is a root of the graph, and \( \mathcal{L}_w 1_n = 1 \) where \( 1_n = [1, 1, \ldots, 1]^T \).

2.2 Equations of motion of mechanical systems

The dynamic behaviour of the \( i \)-th mechanical system can be described as (Slotine & Li, 1991; Spong et al., 2006)

\[
M_i(q_i) \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + g_i(q_i) = \tau_i,
\]

where \( q_i \in \mathbb{R}^m \) is the generalised position (or configuration), \( M_i(q_i) \in \mathbb{R}^{m \times m} \) is the inertia matrix, \( C_i(q_i, \dot{q}_i) \in \mathbb{R}^{m \times m} \) is the Coriolis and centrifugal matrix, \( g_i(q_i) \in \mathbb{R}^m \) is the gravitational torque, and \( \tau_i \in \mathbb{R}^m \) is the control torque exerted on the system. Three well-known properties associated with the nonlinear dynamics (2) that shall be useful for the subsequent controller design and stability analysis are listed as follows (see, e.g., Slotine & Li, 1991; Spong et al., 2006).

**Property 1:** The inertia matrix \( M_i(q_i) \) is symmetric and uniformly positive definite.

**Property 2:** The Coriolis and centrifugal matrix \( C_i(q_i, \dot{q}_i) \) can be appropriately selected such that \( M_i(q_i) - 2C_i(q_i, \dot{q}_i) \) is skew-symmetric.

**Property 3:** The nonlinear dynamics (2) depends linearly on a physical parameter vector \( a_i \), which gives rise to

\[
M_i(q_i) \dot{\zeta} + C_i(q_i, \dot{q}_i) \zeta + g_i(q_i) = Y_i(q_i, \dot{q}_i, \zeta, \dot{\zeta}) a_i,
\]

where \( Y_i(q_i, \dot{q}_i, \zeta, \dot{\zeta}) \) is the dynamic regressor matrix, \( \zeta \in \mathbb{R}^m \) is a differentiable vector, and \( \dot{\zeta} \) is the time derivative of \( \zeta \).

3. iBIBO stability

In this paper, we propose a new notion—iBIBO stability to facilitate the stability analysis for networked multi-agent systems.

**Definition 1:** Consider a dynamical system \( y = Gu \), where \( G \) represents a general input–output mapping. Then, the system \( y = Gu \) is said to be iBIBO stable if \( y(t) \) is bounded for any integral-bounded input \( u(t) \), where integral boundedness means that there exists a positive constant \( c_0 \) such that \( \int_0^t |u(r)|dr \leq c_0 \) for \( \forall t \geq 0 \), where \( | \cdot | \) denotes the norm of a vector.

This notion originates from the study of consensus of multi-agent systems (e.g., Mei et al., 2012; Min et al., 2012; Nuño et al., 2011; Ren & Beard, 2008; Wang, 2013b).

Specifically, in the consensus problem for networked mechanical systems as is studied in Nuño et al. (2011), Min et al. (2012) and Mei et al. (2012), the consensus value cannot be ensured to be bounded since the integral of an external (‘external’ is said with respect to the network dynamics) quantity cannot be guaranteed to be bounded. The work in Wang (2013b) gives a control scheme to ensure the boundedness and further convergence of the integral of this external quantity, so that the consensus value is bounded.

Other related results appear in the context of stabilisation of a single linear system using saturated feedback under external disturbances (see, e.g., Wang, Saberi, Grip, & Stoorvogel, 2013; Wen, Roy, & Saberi, 2008), and it is shown that disturbances having a uniformly bounded integral will result in bounded states for marginally stable chain-integrator systems (Wen et al., 2008) and that a special class of ‘quasi-integral-bounded’ disturbances gives rise to bounded states for multi-frequency linear systems with decomposed...
structure (Wang et al., 2013). The notion given by Definition 1 can therefore be considered as a generalisation of these early results to the more general dynamical systems. While the single iBIBO stable system is rarely seen in practical applications, iBIBO stability is, in certain sense, ubiquitous in networked multi-agent systems without a leader.

As a preliminary step, let us take three simple linear systems to illustrate the concept of the iBIBO stability and also acquire some insight into the feature of the systems that are iBIBO stable.

**Example 1:** The most simple iBIBO stable system may be the single integrator \( \dot{y} = u \). The application of Laplace transformation yields \( Y(p) = \frac{1}{p} U(p) \), where \( p \) denotes the Laplace variable, and it can be seen that the system transfer function \( G(p) = \frac{1}{p} \) has a simple zero pole. An apparent fact is that for integral-bounded input, \( y(t) = y(0) + \int_0^t u(r) dr \) must be bounded.

**Example 2:** Let us consider a relatively complex system \( \dot{y} + \dot{y} = u + 2u \), which has a simple pole at the origin and a pole in the open LHP. Suppose that \( \int_0^t u(r) dr \) is bounded, and is the output \( y \) bounded? To see this clearly, let us take the Laplace transform of the system and then obtain

\[
Y(p) = \frac{p + 2}{p(p + 1)} U(p),
\]

where \( G(p) = \frac{p+2}{p(p+1)} \) denotes the transfer function of the plant. We may rewrite the system as \( Y(p) = \frac{p+2}{p+1} U(p) \), where \( \frac{U(p)}{p} \) is the Laplace transform of the bounded signal \( \int_0^t u(r) dr \), and since \( p G(p) = \frac{p+2}{p+1} \) is exponentially stable (Desoer & Vidyasagar, 1975, p. 58), we obtain the boundedness of the output \( y \) by exploiting the standard fact that an exponentially stable linear system is BIBO stable.

**Example 3** (Ren & Beard, 2008, p. 35, 36): Consider the following well-studied multi-agent system under the external disturbances

\[
\dot{x} = -L_w x + \tilde{d},
\]

where \( \tilde{d} = [\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n]^T \), \( x = [x_1, x_2, \ldots, x_n]^T \), \( \tilde{d}_i \) denotes the external disturbance acting on the \( i \)-th agent, \( x_i \) denotes the state of the \( i \)-th agent, \( i = 1, 2, \ldots, n \), and the agents are assumed to interact on a directed graph with a spanning tree. In the reduced case that \( \tilde{d}_i = d^i \), \( i = 1, 2, \ldots, n \), it is shown in Ren and Beard (2008) that \( x_i(t) \to \Sigma_{k=1}^n y_i(0) + \int_0^t d^i(r) dr \) as \( t \to \infty \), \( i = 1, 2, \ldots, n \). This leads us to straightforwardly obtain the result that if the disturbance \( d^i \) is integral bounded, i.e., \( \int_0^t d^i(r) dr \in L_{\infty} \) for \( \forall t \geq 0 \), then, \( x_i(t) \in L_{\infty} \) for \( \forall t \geq 0 \), \( i = 1, 2, \ldots, n \).

The common characteristics of the above three simple examples is that all of them belong to the category of marginally stable and strictly proper linear systems with a simple pole at the origin and the rest of the poles in the open LHP. Interestingly, the above results can be generalised to the general marginally stable linear systems.

**Proposition 1:** Let \( Y(p) = G(p)U(p) \) be a marginally stable and strictly proper linear system with a simple pole at the origin and the rest of the poles in the open LHP, and if the integration of the input \( u(t) \) is bounded regardless of \( t \) (i.e., \( \int_0^t u(r) dr \) is bounded for \( \forall t \geq 0 \)), then, the output \( y(t) \) must be bounded. That is, the linear system \( G(p) \) is iBIBO stable.

**Proof:** Rewrite the system \( Y(p) = G(p)U(p) \) as

\[
Y(p) = pG(p) \frac{U(p)}{p},
\]

where the new input \( \frac{U(p)}{p} \) is the Laplace transform of the signal \( \int_0^t u(r) dr \).

Let \( \tilde{G}(p) = pG(p) \). Since \( G(p) \) is strictly proper, \( \tilde{G}(p) \) is obviously properly according to Desoer and Vidyasagar (1975, p. 58). In addition, since the simple zero pole of \( G(p) \) is cancelled by the factor \( p \), we have that \( G(p) \) is exponentially stable (Desoer & Vidyasagar, 1975, p. 58). Recall the standard result that the bounded input for an exponentially stable linear system yields the bounded output. Therefore, the bounded input \( \int_0^t u(r) dr \) for \( \tilde{G}(p) \) gives rise to the bounded output \( y(t) \). That is, the system \( G(p) \) is iBIBO stable. \( \square \)

**Remark 1:**

1. Different from the standard BIBO stability (van der Schaft, 2000) or ISS notion (Khalil, 2002), the iBIBO stability refers to an integral-bounded input and a bounded output. For marginally stable and strictly proper linear systems with a simple zero pole and the rest of the poles in the open LHP, BIBO stability or ISS does not hold due to the existence of a pole at the origin. The notion of iBIBO stability presented here characterises this special linear system.

2. Another notion in the literature that may be related to iBIBO stability is iISS (integral-input-to-state stability) (Angeli, Sontag, & Wang, 2000; Dashkovskiy et al., 2011; Sontag, 1998), which refers to the integral of certain norm of the input (i.e., in the form \( \int_0^t \beta(u(r)) dr \)) and the output, and the difference is that the proposed iBIBO stability refers to an input whose integral (i.e., \( \int_0^t u(r) dr \)) is bounded. Furthermore, as demonstrated in Sontag (1998), a linear system with external input is iISS if and only if all the poles of the system are in the open LHP. The marginally stable and strictly proper system with a simple pole at the origin and the rest of the poles in the open LHP studied in this work, yet, does not satisfy the condition required by iISS.
Remark 2: Consider again Example 3, and it is demonstrated in Ren and Beard (2008) that in the case that $\bar{d}^i$, $i = 1, 2, \ldots, n$, are nonidentical, the mapping from the input $\bar{d}$ to the consensus error vector $[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]'$ is ISS (Khalil, 2002). Let $G^*(p)$ denote the transfer function of the system (4) with $\bar{d}$ as the input and $x$ as the output. From Proposition 1, we can obtain a result stronger than that in Ren and Beard (2008), i.e., if $\bar{d}$ is integral bounded, then, the states of the agents will be bounded since according to Min et al. (2012) and Nuñ0 et al. (2011), $G^*(p)$ has a simple zero pole and the rest poles of $G^*(p)$ are all in the open LHP. Another interesting result is given in Min et al. (2012) which also examines the same system as the one given in Example 3, and it derives the BIBO stability from $\bar{d}$ to $\hat{x}$ by exploiting the fact that the zero pole of $G^*(p)$ is cancelled by a zero which appears due to the fact that the output is chosen as $\hat{x}$. This result can actually be considered as a special application of the proposed iBIBO notion if we rewrite the system (4) as its differentiated form

$$\dot{x} = -L_w\dot{x} + \bar{d},$$

and note the fact that the system (6) has essentially the same poles as $G^*(p)$ if $\bar{d}$ is taken as the input and $\dot{x}$ as the output.

Example 4: Let us now consider the following two-agent system with nonlinear coupling

$$\begin{align*}
\dot{x}_1 &= -(x_1 - x_2)^3 + \bar{d}_1, \\
\dot{x}_2 &= -(x_2 - x_1)^3 + \bar{d}_2,
\end{align*}$$

(7)

where $x_1$ and $x_2$ denote the states of the first and second agents, respectively, and suppose that external disturbances $\bar{d}_1$ and $\bar{d}_2$ are both integral bounded.

1. Adding both sides of the two subsystems of (7) gives

$$\dot{x}_1 + \dot{x}_2 = \bar{d}_1 + \bar{d}_2,$$

(8)

which implies that $x_1 + x_2 \in \mathcal{L}_\infty$ since $\bar{d}_1 + \bar{d}_2$ is integral bounded (i.e., $\int_0^t [\bar{d}_1(r) + \bar{d}_2(r)]dr \in \mathcal{L}_\infty$).

2. From (7), we also have that

$$\dot{x}_1 - \dot{x}_2 = -2(x_1 - x_2)^3 + \bar{d}_1 - \bar{d}_2.$$  

(9)

Note that here $\Delta = \bar{d}_1 - \bar{d}_2$ is not bounded but only integral bounded. Consider the following nonnegative function

$$V = \frac{1}{2} \left( x_1 - x_2 - \int_0^t \Delta(r)dr \right)^2.$$  

(10)

The derivative of $V$ can be written as

$$\dot{V} = (x_1 - x_2 - \int_0^t \Delta(r)dr)(\dot{x}_1 - \dot{x}_2 - \Delta),$$

$$= -2(x_1 - x_2 - \int_0^t \Delta(r)dr) \times \left[ (x_1 - x_2 - \int_0^t \Delta(r)dr) + \int_0^t \Delta(r)dr \right].$$

Let $z = x_1 - x_2 - \int_0^t \Delta(r)dr$, and we have

$$\dot{V} = -2z^2.$$  

Obviously, if $|z| > |\int_0^t \Delta(r)dr|$ (i.e., $V = (1/2)z^2 > (1/2)|\int_0^t \Delta(r)dr|^2$), then $\dot{V} < 0$. According to the standard result (see, e.g., Khalil, 2002), this implies that $V \in \mathcal{L}_\infty$ and consequently $z \in \mathcal{L}_\infty$ since $\int_0^t \Delta(r)dr \in \mathcal{L}_\infty$. From the fact that $z \in \mathcal{L}_\infty$, we have that $x_1 - x_2 \in \mathcal{L}_\infty$, which, combined with the result that $x_1 + x_2 \in \mathcal{L}_\infty$, yields the result that $x_1 \in \mathcal{L}_\infty$ and $x_2 \in \mathcal{L}_\infty$. Therefore, we obtain that the nonlinear system (7) with $\bar{d}_1$, $\bar{d}_2$ as the input and $x_1$, $x_2$ as the output is iBIBO stable.

Remark 3: The input–output property for marginally stable and strictly proper linear systems given in Proposition 1 is potentially/possibly useful in the analysis and controller design for networked complex systems (e.g., via a cascade-like analysis). In later sections of this paper, we will show its use in the analysis of the flocking controller for networked mechanical systems, which indeed provides a new perspective on analysis of complex networked systems. Also note that the proposed new concept is applicable to certain nonlinear systems such as the one given in Example 4.

4. Adaptive flocking controller

In this section, we consider the flocking problem for the $n$ mechanical systems with parametric uncertainties, and our control objective is to realise flocking of the $n$ mechanical systems, i.e., $q_i - q_j \to 0$ and $\dot{q}_i - \dot{q}_j \to 0$ as $t \to \infty$, $\forall i, j = 1, 2, \ldots, n$.

The position/velocity consensus ensures the holding of the first and second Reynolds rules. Although the above statement of flocking does not explicitly state the collision avoidance objective (i.e., the third Reynolds rule), it can be easily extended to ensure this, e.g., by guaranteeing that $q_i - q_j$ converges to a given constant offset vector $d_i$, $\forall i, j = 1, 2, \ldots, n$ (Lee & Spong, 2007).
Following Wang (2013a), we first define a sliding vector as

\[ s_i = \dot{q}_i - \dot{q}_i(0) + b \sum_{j \in \mathcal{N}_i} w_{ij} (q_i - q_j) + k \sum_{j \in \mathcal{N}_i} w_{ij} \int_0^t [q_i(r) - q_j(r)] dr, \]  

where \( b, k > 0 \) are the damping and stiffness gains, respectively. We may notice that the sliding vector \( s_i \) depends on the initial velocity \( \dot{q}_i(0) \), which, yet, does not encounter any problem since it is convenient for the \( i \)th system to store the initial velocity of its own.

Differentiating (11) with respect to time gives

\[ \dot{s}_i = \ddot{q}_i + b \sum_{j \in \mathcal{N}_i} w_{ij} (\dot{q}_i - \dot{q}_j) + k \sum_{j \in \mathcal{N}_i} w_{ij} (q_i - q_j). \]  

Based on the sliding vector \( s_i \) given in (11), we define a reference velocity

\[ \dot{q}_{r,i} = \dot{q}_i(0) - b \sum_{j \in \mathcal{N}_i} w_{ij} (q_i - q_j) - k \sum_{j \in \mathcal{N}_i} w_{ij} \int_0^t [q_i(r) - q_j(r)] dr - \alpha_i \int_0^t s_i(r) dr, \]  

where \( \alpha_i \) is a positive design constant. Differentiating (13) with respect to time gives the reference acceleration

\[ \ddot{q}_{r,i} = -b \sum_{j \in \mathcal{N}_i} w_{ij} (\dot{q}_i - \dot{q}_j) - k \sum_{j \in \mathcal{N}_i} w_{ij} (q_i - q_j) - \alpha_i \dot{s}_i. \]  

Next, define another sliding vector

\[ \xi_i = \dot{q}_i - \dot{q}_{r,i} = s_i + \alpha_i \int_0^t s_i(r) dr. \]  

The sliding vector \( \xi_i \) is the composition of the sliding vector \( s_i \) and its integral \( \int_0^t s_i(r) dr \) (similar to Wang, 2013b), which will be demonstrated to be useful for guaranteeing the convergence of positions of the systems to the initial-states-dependent ramp trajectory. This convergence is not achievable without this integral, as is the case in Wang (2013a).

We propose the following control law for the \( i \)th system

\[ \tau_i = Y_i (q_i, \dot{q}_i, \ddot{q}_i, \dot{q}_{r,i}) \dot{a}_i - K_i \xi_i, \]  

where \( K_i \) is a symmetric positive definite matrix and \( \dot{a}_i \) is the estimate of the parameter \( a_i \), which is updated by the adaptation law

\[ \dot{\dot{a}}_i = -\Gamma_i Y_i^T (q_i, \dot{q}_i, \ddot{q}_i, \dot{q}_{r,i}) \xi_i, \]  

where \( \Gamma_i \) is the symmetric positive definite adaptation gain matrix.

**Remark 4:** The adaptive controller given by (16) and (17) extends the Slotine and Li scheme (Slotine & Li, 1987) to take into consideration the inter-coupling among the systems by using new reference velocity and reference acceleration, where the reference velocity includes the integral-sliding vector \( \int_0^t \dot{s}_i(r) dr \) [which is inspired by Wang (2013b)].

Substituting the control law (16) into the dynamics (2) yields the following equation

\[ M_i(q_i) \dot{\xi}_i + C_i(q_i, \dot{q}_i) \dot{\xi}_i = -K_i \xi_i + Y_i (q_i, \dot{q}_i, \ddot{q}_i, \dot{q}_{r,i}) \Delta a_i, \]  

where \( \Delta a_i = \dot{a}_i - a_i \) is the parameter estimation error.

The dynamic behaviour of the \( i \)th system can then be described by

\[ \begin{aligned}
\dot{q}_i &= -b \sum_{j \in \mathcal{N}_i} w_{ij} (\dot{q}_i - \dot{q}_j) - k \sum_{j \in \mathcal{N}_i} w_{ij} (q_i - q_j) + \dot{s}_i, \\
\dot{s}_i &= -\alpha_i \int_0^t s_i(r) dr + \xi_i, \\
M_i(q_i) \dot{\xi}_i + C_i(q_i, \dot{q}_i) \dot{\xi}_i &= -K_i \xi_i + Y_i (q_i, \dot{q}_i, \ddot{q}_i, \dot{q}_{r,i}) \Delta a_i, \\
\dot{\dot{a}}_i &= -\Gamma_i Y_i^T (q_i, \dot{q}_i, \ddot{q}_i, \dot{q}_{r,i}) \xi_i,
\end{aligned} \]  

where \( s_i \) and \( \xi_i \) act as the cascade variables.

Stacking up all the equations expressed as the first one in (19) yields

\[ \ddot{q} = -b (L_w \otimes I_m) \dot{q} - k (L_w \otimes I_m) q + \dot{s}, \]  

where \( \otimes \) denotes the Kronecker product (Brewer, 1978), \( I_m \) is the \( m \times m \) identity matrix, \( q = [q_{1T}, q_{2T}, \ldots, q_{nT}]^T \), and \( s = [s_{1T}, s_{2T}, \ldots, s_{nT}]^T \). The vector \( \dot{s} \) is taken as the input in (20).

Under the action of the input signal \( \dot{s} \), it is challenging to identify the properties of the outputs \( q \) and \( \dot{q} \) of (20), mainly due to the asymmetry of \( L_w \) (since it is associated with a general directed graph) and the fact that \( L_w \) has a simple zero eigenvalue (see Lemma 1) (which means that (20) is not possible to be an exponentially stable and strictly proper linear system, and thus (20) is obviously not BIBO stable). Here, we show how the proposed iBIBO concept renders this problem to be conveniently handled, and in fact, we will show that if \( \dot{s} \) is integral bounded, the output \( \dot{q} \) will be bounded under certain conditions. For this, let \( Q_s(p) \) denote the Laplace transform of \( \dot{q} \), and rewrite Equation (20) as

\[ \begin{aligned}
p Q_s(p) - \dot{q}(0) &= -b (L_w \otimes I_m) Q_s(p) \\
& \quad - k (L_w \otimes I_m) \frac{Q_s(p) + q(0)}{p} \\
& \quad + p S(p) - s(0),
\end{aligned} \]  

where \( S(p) \) is an auxiliary bounded signal.
where $S(p)$ is the Laplace transform of $s$. Equation (21) can be further written as

$$Q_x(p) = \left[ \left( pI_n + bL_w + \frac{k}{p}L_w \right)^{-1} \otimes I_n \right]$$

$$\left[ \dot{q}(0) - \frac{k}{p} (L_w \otimes I_n) q(0) - s(0) + pS(p) \right].$$

(22)

Regarding the transfer function $G(p)$ in (22), the following two propositions hold.

**Proposition 2:** Suppose that $L_w$ is associated with a directed graph containing a spanning tree. Then, the poles of $G(p)$ excluding the simple pole at the origin are all in the open LHP if and only if the damping and stiffness gains satisfy

$$\frac{b^2}{k} > \max_{i \geq 2} \left( \frac{(Im\lambda_i)^2}{(Re\lambda_i)|\lambda_i|^2} \right).$$

(23)

where $\lambda_i$, $i = 2, 3, \ldots, n$, are the nonzero eigenvalues (possibly repeated) of the Laplacian matrix $L_w$, $|\lambda_i|$ is the modulus of $\lambda_i$, and $Re\lambda_i$ and $Im\lambda_i$ are the real and imaginary parts of $\lambda_i$, respectively.

**Proposition 3:** If the gains $b, k$ satisfy (23), the system $y = G(p)u$ is iBIBO stable. That is, if the integration of the input, i.e., $\int_0^\infty u(r)dr$ is bounded, the output $y(t)$ is bounded for $\forall t \geq 0$.

**Proof of Proposition 2:** Let $e_i$ denote the eigenvector of $L_w$ associated with its eigenvalue $\lambda_i$, $i = 1, 2, \ldots, n$. Obviously, from the standard matrix theory, the poles of $G(p)$ satisfy

$$G^{-1}(p)e_i = \left( p + b\lambda_i + \frac{k}{p}\lambda_i \right)e_i = 0.$$  

(24)

Since $e_i \neq 0$, we have the characteristic equation $h_i(p) = (p + b\lambda_i + \frac{k}{p}\lambda_i) = 0$. For the case $i = 1$, due to $\lambda_1 = 0$, the characteristic root is $p = 0$, which must be a simple pole of $G(p)$ since $\lim_{p \to 0} \text{rank} \left[ G^{-1}(p) \right] = \text{rank}(L_w) = n - 1$ (by Lemma 1). For the case $i \geq 2$, we are only certain that $p \neq 0$, and the judge of $p$ in the LHP or RHP, yet, becomes complicated.

1) A direct result is that if the stiffness $k = 0$, the roots of the characteristic equation are $p = -b\lambda_i$, which are all in the open LHP.

2) The increasing of $k$ yields another boundary condition, which allows $G(p)$ to have a pole on the imaginary axis. Let $p = j^*\omega \neq 0$ be the pole on the imaginary axis where $j^* = \sqrt{-1}$ denotes the imaginary unit, and substituting $p = j^*\omega$ into $h_i(p) = 0$ yields

$$h_i(j^*\omega) = j^*\omega + b\lambda_i - \frac{k}{\omega}j^* = bRe\lambda_i + \frac{k}{\omega} Im\lambda_i$$

$$+ j^* \left( \omega + bIm\lambda_i - \frac{k}{\omega} Re\lambda_i \right) = 0.$$

(25)

for $i \geq 2$. Equation (25) suggests that the real and imaginary parts of $h_i(j^*\omega)$ are both equal to zero, i.e.,

$$\left\{ \begin{array}{l}
 bRe\lambda_i + \frac{k}{\omega} Im\lambda_i = 0,
 \omega + bIm\lambda_i - \frac{k}{\omega} Re\lambda_i = 0,
 \end{array} \right.$$

(26)

which yields $\frac{b^2}{k} = \frac{Im\lambda_i}{Re\lambda_i}$. Since the poles of $G(p)$ depend continuously on the stiffness $k$ and the poles of $G(p)$ excluding the simple zero pole are in the open LHP for $k = 0$, therefore, $G(p)$ has only one simple pole at the origin and its all other poles are in the open LHP if and only if the gains $b, k$ satisfy

$$\frac{b^2}{k} > \max_{i \geq 2} \left( \frac{(Im\lambda_i)^2}{(Re\lambda_i)|\lambda_i|^2} \right).$$

**Remark 5:** The derivation of the lower bound for $\frac{b^2}{k}$ given in (23) follows an analysis similar to Wang (2013a), which generalises the result in Olfati-Saber and Murray (2004) (which is concerned with the consensus problem of single-integrator agents on undirected graphs in the case of existence of time delays and seeks the maximum delay allowed by the closed-loop system) to the flocking problem on the more general directed graphs. It seems interesting that the condition (23) coincides with the result in Yu et al. (2010), though from different perspectives. The demonstration of the property of the interconnection system (20) at the velocity level, yet, is crucial for solving the flocking problem of networked mechanical systems, as is illustrated below.

**Proof of Proposition 3:** From Proposition 2, we know that $G(p)$ has a simple pole at the origin and the other poles of $G(p)$ are all in the open LHP. Furthermore, $\lim_{p \to \infty} G(p) \to 0$, implying that $G(p)$ is strictly proper (Desoer & Vidyasagar, 1975, p. 58). Thus, $G(p)$ satisfies all the conditions stated in Proposition 1, and then, the system $y = G(p)u$ must be iBIBO stable.

We are presently ready to give the following theorem.

**Theorem 1:** The control law (16) and the parameter adaptation law (17) result in flocking of networked mechanical systems on a directed graph containing a spanning tree if and only if the damping and stiffness gains $b, k$ satisfy (23), i.e., $\dot{q}_i - \dot{q}_j \to 0$ and $\dot{q}_i \to \Sigma_{i=1}^n \gamma_i \dot{q}_i(0)$ as $t \to \infty$, $\forall i,$
where \( J \) is the Jordan form of \( G^{-1}(p) \) and \( D \in \mathbb{R}^{n \times n} \) is an invertible matrix, and without loss of generality, let the left-top entry of \( J \) be associated with the simple zero eigenvalue of \( L_{ee} \), i.e., \( j_{11} = p + b \lambda_1 + k_1 \eta_1 = p \). Furthermore, the first column of \( D \) is set as \( 1_n \), and in this case, the first row of \( D^{-1} \) must be \( \gamma^T \), which is due to the structure of \( J \) and the fact that \( l_n \) and \( \gamma \) are the right and left eigenvectors of \( G^{-1}(p) \) associated with its simple eigenvalue \( j_{11} = p \), respectively. From (29), we obtain

\[
G(p) = DJ^{-1}D^{-1}, \tag{30}
\]

where the diagonal entries of \( J^{-1} \) are composed of

\[
\frac{p}{p^2 + b \lambda_i + k_i}, \quad i = 1, 2, \ldots, n.
\]

Since the left-top entry of \( J^{-1} \) is \( \frac{1}{p} \), we obtain that \( l_n \) and \( \gamma \) are the right and left eigenvectors of \( G(p) \) associated with its simple eigenvalue \( \frac{1}{p} \), respectively. Therefore, \( G(p) \) can be decomposed into two parts, i.e.,

\[
G(p) = \frac{1}{p} l_n \gamma^T + \Delta G(p), \tag{31}
\]

whose time-domain counterpart is bounded since all the poles of \( \Delta G(p) \) are in the open LHP since \( \Delta G(p) \) excludes the zero pole of \( G(p) \). According to the final value theorem (Dorf & Bishop, 2008), we have that \( \lim_{p \to 0} \frac{p}{p^2 + b \lambda_i + k_i} = 0 \) for \( i \geq 2 \), since all the poles of the transfer function \( \frac{p}{p^2 + b \lambda_i + k_i} \), \( i \geq 2 \), are in the open LHP (from Proposition 2 and its proof), and further \( \lim_{p \to 0} \Delta G(p) = 0 \). Therefore, with the result \( \gamma^T L_{ee} = 0 \) (by Lemma 1), \( \Pi_2(p) \) can be rewritten as

\[
\Pi_2(p) = -\frac{k}{p} [\Delta G(p)L_{ee} \otimes I_m] q(0), \tag{32}
\]

whose time-domain counterpart is bounded since all the poles of \( \frac{\Delta G(p)}{p} \) are in the open LHP. Furthermore, the application of the final value theorem (Dorf & Bishop, 2008) gives \( \lim_{p \to 0} q \Pi_2(p) = 0 \).

The boundedness of the time-domain counterparts of \( \Pi_1(p), \Pi_2(p), \) and \( \Pi_3(p) \) implies that \( \bar{u} \in \mathcal{L}_\infty \) according to the standard principle of superposition for linear systems. From (11) and the result that \( s_i \in \mathcal{L}_\infty, \forall i \), we have that \( z^* = bL_uq + kL_u \int_0^t q(r)dr \in \mathcal{L}_\infty \). Obviously, \( L_u \int_0^t q(r)dr \) can be considered as the output of a stable system \( 1/(bp + k) \) with \( z^* \) as the input, and therefore, \( \int_0^t q(r)dr \in \mathcal{L}_\infty \) and \( L_uq \in \mathcal{L}_\infty \) according to the input-output properties of linear systems (Desoer & Vidyasagar, 1975, p. 59). From (13) and (14), we obtain that \( \tilde{q}_{r,i} \in \mathcal{L}_\infty \) and \( \tilde{q}_{r,i} \in \mathcal{L}_\infty, \forall i \). Due to the uniform positive definiteness of \( M(q) \) (i.e., Property 1), we have that \( \tilde{y}_i \in \mathcal{L}_\infty \) from (18), and consequently, \( \tilde{q}_i \in \mathcal{L}_\infty, \forall i \). Therefore, \( \tilde{V}_i \in \mathcal{L}_\infty \) giving rise to the uniform continuity of \( V_i, \forall i \). The application of Barbalat’s lemma (Slootine & Li, 1991) yields \( \tilde{V}_i \to 0 \) as \( t \to \infty \), which leads to the result \( \tilde{x}_i \to 0 \) as \( t \to \infty \), \( \forall i \). From (18), we know that \( \tilde{x}_i \) must be uniformly continuous since all other signals in (18) are uniformly
continuous, \( \forall i \). From Barbalat’s lemma (Slotine & Li, 1991), we obtain that \( \dot{\xi}_i \to 0 \) as \( t \to \infty \), \( \forall i \). The boundedness of \( s_i \) and \( \dot{\xi}_i \) gives that \( \dot{s}_i = \dot{\xi}_i - \alpha_i s_i \in L_\infty \), which means that \( s_i \) is uniformly continuous, \( \forall i \). Therefore, from the properties of square-integrable and uniformly continuous functions (Lozano, Brogliato, Egelund, & Maschke, 2000, p. 117), we have that \( s_i \to 0 \), which further implies that \( \dot{s}_i = \dot{\xi}_i - \alpha_i s_i \to 0 \) as \( t \to \infty \), \( \forall i \).

Based on (30) and exploiting the fact that \( \lambda_1 = 0 \), we obtain

\[
\lim_{p \to 0} pG(p) = \lim_{p \to 0} 1_n \gamma^T \frac{p}{p + b\lambda_1 + \frac{k}{p} \lambda_1} = 1_n \gamma^T. \tag{33}
\]

From the final value theorem (Dorf & Bishop, 2008), we have \( \lim_{p \to 0} qS(p) = \lim_{t \to \infty} s(t) = 0 \), and additionally obtain the limit of the velocity [using (33)]

\[
\lim_{t \to \infty} \dot{q}(t) = \lim_{p \to 0} \{pG(p) \otimes I_m\} \left[ \dot{q}(0) - k \frac{1}{p} (L_w \otimes I_m) q(0) + pS(p) - s(0) \right]
= \lim_{p \to 0} \{pG(p) \otimes I_m\} \left[ \dot{q}(0) - s(0) \right]
= \lim_{p \to 0} p \Pi_2(p) + \left[ (1_n \gamma^T) \otimes I_m \right] \left[ \dot{q}(0) - s(0) \right]. \tag{34}
\]

Due to the result that \( \lim_{p \to 0} qS(p) = 0 \) given previously and the equality \( \{1_n \gamma^T \otimes I_m\} s(0) = b[\{1_n \gamma^T \otimes I_m\} q(0)] = 0 \) (from Lemma 1), we can write Equation (34) as

\[
\lim_{t \to \infty} \dot{q}(t) = \left[ (1_n \gamma^T) \otimes I_m \right] \dot{q}(0)
= (1_n \otimes I_m) \sum_{k=1}^{n} \gamma_k \dot{q}_k(0). \tag{35}
\]

Equation (35) implies that \( (L_w \otimes I_m) \dot{q} \to 0 \) as \( t \to \infty \) according to Lemma 1. From (14), we can directly obtain that \( \ddot{q}_{i,j} \in L_\infty \), implying that \( \ddot{q}_{i,j} \) is uniformly continuous, \( \forall i \). The uniform continuity of \( \dot{\xi}_i \) and \( \dot{q}_{i,j} \) results in that of \( \dot{q}_i \) since \( \dot{q}_i = \dot{\xi}_i + \dot{q}_{i,j}, \forall i \). From (35), we know that \( \lim_{t \to \infty} \int_0^t \dot{q}_i(t)dr \) exists, \( \forall i \). Using Barbalat’s lemma (Slotine & Li, 1991), we obtain that \( \ddot{q}_i \to 0 \) as \( t \to \infty \), \( \forall i \).

From (20), we have that \( (L_w \otimes I_m) q \to 0 \) as \( t \to \infty \), and therefore, according to Lemma 1, we obtain that \( q_i - q_j \to 0 \) as \( t \to \infty \), \( \forall i, j \) since the graph contains a spanning tree.

Let us now seek the limit value of the positions of the systems. For this, we use \( (1_n \gamma^T \otimes I_m) \) to pre-multiply both sides of (20) and obtain (using the result \( \gamma^T L_w = 0 \) in Lemma 1)

\[
\sum_{k=1}^{n} \gamma_k \dot{q}_k(t) = \sum_{k=1}^{n} \gamma_k \dot{s}_k(t). \tag{36}
\]

Integrating (36) with respect to time gives (since \( \sum_{k=1}^{n} \gamma_k \dot{s}_k(0) = 0 \), as previously shown)

\[
\sum_{k=1}^{n} \gamma_k \dot{q}_k(t) - \sum_{k=1}^{n} \gamma_k \dot{q}_k(0) = \sum_{k=1}^{n} \gamma_k \dot{s}_k(t). \tag{37}
\]

Integrating (37) with respect to time yields

\[
\sum_{k=1}^{n} \gamma_k q_k(t) - \sum_{k=1}^{n} \gamma_k q_k(0) - \sum_{k=1}^{n} \gamma_k \dot{q}_k(0) \cdot t
= \sum_{k=1}^{n} \gamma_k \int_0^t s_k(r)dr. \tag{38}
\]

Since \( q_1(\infty) = q_2(\infty) = \ldots = q_n(\infty) = \sum_{k=1}^{n} \gamma_k q_k(\infty) \) and \( \int_0^t s_i(r)dr \to 0 \) as \( t \to \infty \), \( \forall i \), from (38), we have that \( q_i \to \sum_{k=1}^{n} \gamma_k q_k(0) + \left[ \sum_{k=1}^{n} \gamma_k \dot{q}_k(0) \right] t \) as \( t \to \infty \), \( \forall i \).

If the gains \( b, k \) do not satisfy (23), some poles of \( G(p) \) will be on the imaginary axis or even in the open RHP. Based on (28), we know from the standard linear system theory that the velocities of the systems will become oscillatory or even divergent. Therefore, in this case, the flocking objective can no longer be achieved.

\[\square\]

Remark 6: From (38), we know that the explicit derivation of the final positions of the mechanical systems relies on the integration of the sliding vector \( s_i, i = 1, 2, \ldots, n \). The incorporation of the integral-sliding vector \( \int_0^t s_i(r)dr \) into the controller ensures that \( \int_0^t s_i(r)dr \to 0 \) as \( t \to \infty \), \( i = 1, 2, \ldots, n \), and further allows the final positions to be explicitly derived. In Wang (2013a), however, it only guarantees that the sliding vector \( s_i \in L_2, i = 1, 2, \ldots, n \). In that case, the final positions of the mechanical systems are hard/impossible to be given in the explicit form since it is well known that \( s_i \in L_2 \) cannot ensure the convergence of \( \int_0^t s_i(r)dr \) (see, e.g., Lozano et al., 2000, p. 113, 114).

5. Simulation study

Let us demonstrate the performance of the proposed adaptive flocking controller via a numerical simulation using seven standard two-DOF (degree-of-freedom) planar robots interacting on a directed graph shown in Figure 1, from which, we observe that vertices 2, 5, and 7 act as the three

![Figure 1. Interaction graph among the robots.](image-url)
roots of the graph. Each two-DOF robot contains two serially connected links. Denote by $m_k$, $I_{C,k}$, $l_k$, and $l_{C,k}$ respectively, the mass, the moment of inertia about the centre of mass, the length, and the distance from the centre of mass to the prior joint of the $k$th link of the robot, $k = 1, 2$. The inertia matrices and the Coriolis and centrifugal matrices of the two-DOF robots can be, respectively, written as (see, e.g., Spong et al., 2006)

$$M_i(q_i) = \begin{bmatrix} d_1 + 2d_2 \cos \theta_1 & d_3 + d_2 \cos \theta_2 \\ d_3 + d_2 \cos \theta_2 & d_3 \end{bmatrix}$$ (39)

and

$$C_i(q_i, \dot{q}_i) = \begin{bmatrix} -d_2 \sin(\theta_2) \dot{\theta}_2 & -d_3 \sin(\theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ d_2 \sin(\theta_2) \dot{\theta}_1 & 0 \end{bmatrix},$$ (40)

where the joint position $q_i = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$, and the unknown parameter $a_i$ is given as

$$a_i = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} I_{C,1} + m_1l_{C,1}^2 + m_3l_{C,2}^2 + I_{C,2} + m_2l_{C,2}^2 \\ m_2l_{C,2} \\ I_{C,2} + m_2l_{C,2}^2 \end{bmatrix},$$ (41)

For simplicity, the gravitational forces are assumed to be absent in the simulation. The physical parameters of the seven robots are listed in Table 1. The sampling period is set to be 5 ms.

The entries of the weighted adjacency matrix $W$ associated with the directed graph in Figure 1 are selected as $w_{ij} = 1.0$ if $j \in \mathcal{N}_i$, and $w_{ij} = 0$ otherwise. The controller parameters $K_i$, $\alpha_i$, and $\Gamma_i$ are chosen as $K_i = 20.0l_2$, $\alpha_i = 5.0$, and $\Gamma_i = 5.0l_3$, respectively, $i = 1, 2, \ldots, 7$. The initial parameter estimates are chosen as $\hat{a}_i(0) = [0, 0, 0]^T$, $i = 1, 2, \ldots, 7$. Based on Equation (23), we obtain the condition that the damping and stiffness gains $b$ and $k$ must satisfy, i.e., $\frac{k}{b} > \max_{i \geq 2} \left[ \frac{\sigma_{\min}I_i}{\|R_{ij}\|_{ij}} \right] = 1/6$. We choose the stiffness $k = 1.0$ and the damping $b = 0.8 > \sqrt{1/6}$. Simulation results are shown in Figures 2–7. Figures 2 and 3 give the position consensus errors between the robots. Figures 4 and 5 plot the velocities of the robots.

### Table 1. Physical parameters of the seven robots.

<table>
<thead>
<tr>
<th>Robot</th>
<th>$m_1$, $m_2$(kg)</th>
<th>$l_{C,1}$, $l_{C,2}$($m$)</th>
<th>$l_1$, $l_2$(m)</th>
<th>$l_{C,1}$, $l_{C,2}$(m)</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1.2, 1.0</td>
<td>0.2250, 0.1875</td>
<td>1.5, 1.5</td>
<td>0.75, 0.75</td>
</tr>
<tr>
<td>2</td>
<td>1.3, 1.1</td>
<td>0.3131, 0.2347</td>
<td>1.7, 1.6</td>
<td>0.85, 0.80</td>
</tr>
<tr>
<td>3</td>
<td>0.9, 1.2</td>
<td>0.1687, 0.3240</td>
<td>1.5, 1.8</td>
<td>0.75, 0.90</td>
</tr>
<tr>
<td>4</td>
<td>1.3, 1.2</td>
<td>0.2773, 0.2560</td>
<td>1.6, 1.6</td>
<td>0.80, 0.80</td>
</tr>
<tr>
<td>5</td>
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<td>0.3911, 0.1875</td>
<td>1.9, 1.5</td>
<td>0.95, 0.75</td>
</tr>
<tr>
<td>6</td>
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<td>0.2890, 0.3131</td>
<td>1.7, 1.7</td>
<td>0.85, 0.85</td>
</tr>
<tr>
<td>7</td>
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<td>0.4050, 0.2250</td>
<td>1.8, 1.5</td>
<td>0.90, 0.75</td>
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</tbody>
</table>
As can be seen from Figures 2–5, the proposed adaptive approach indeed achieves the objective of flocking. The left eigenvector of the Laplacian $L_w$ associated with its zero eigenvalue is $\gamma = [0, 1/3, 0, 1/3, 0, 1/3]^T$. Therefore, the weighted average of the initial velocities of the three robots that correspond to the rooted vertices of the graph is $\gamma_2 \dot{q}_2(0) + \gamma_5 \dot{q}_5(0) + \gamma_7 \dot{q}_7(0) = [−0.3667, 0.4333]^T$, and the weighted average of the initial positions of the three robots is $\gamma_2 q_2(0) + \gamma_5 q_5(0) + \gamma_7 q_7(0) = [1.0996, −2.0001]^T$. Then, from Theorem 1, we obtain the convergent ramp trajectory of the seven robots, i.e., $[1.0996 − 0.3667t, −2.0001 + 0.4333t]^T$. The simulation results given in Figures 4 and 5 demonstrate that the velocities of the robots converge to their initial weighted average value. From Figures 6 and 7, we see that the positions of the seven robots indeed converge to the ramp trajectory (i.e., the solid red lines in Figures 6 and 7), mimicking the case of networked double-integrator agents (Ren & Atkins, 2007; Yu et al., 2010).

For comparison, we also perform the simulation under the controller in Wang (2013a), which does not incorporate the integral action $\int_0^t s_i(r)dr$, i.e., set $\alpha_i = 0$, $i = 1, 2, \ldots, 7$ and the other controller parameters are chosen to be the same as those above. The simulation results are given in Figures 8 and 9, where Figure 8 shows the velocities of the robots and Figure 9 shows the positions of the robots. Since, in this case, the convergence of $\int_0^t s_i(r)dr$, $i = 1, 2, \ldots, 7$, is not ensured, the positions of the robots do not converge to the ramp trajectory $[1.0996 − 0.3667t, −2.0001 + 0.4333t]^T$, as is reflected in Figure 9 (the first coordinate). It may also be interesting to note that the convergent speeds under the integral-sliding adaptive controller in this paper ($\alpha_i = 5.0, i = 1, 2, \ldots, 7$) and the controller in Wang (2013a)
\( \alpha_i = 0, i = 1, 2, \ldots, 7 \) are different, as shown in Figures 4, 6, 8 and 9. The comparison between Figures 4 and 8 and that between Figures 6 and 9 demonstrate that the incorporation of the integral-sliding control action may increase the speed of the position/velocity consensus process.

6. Conclusion

In this research, we have studied the flocking problem for multiple nonlinear mechanical systems on directed topologies. We propose an integral-sliding adaptive controller to achieve the flocking objective of networked mechanical systems, irrespective of the parametric uncertainties. The proposed control gives rise to a cascade closed-loop system. We formulate a new notion—iBIBO stability to characterise the input–output property of a special class of dynamical systems (if we confine our attention to linear time-invariant systems, the systems with a simple pole at the origin and the rest of the poles in the open LHP are iBIBO stable). Using the proposed iBIBO analysis tool plus the frequency-domain input–output analysis and the Lyapunov-like approach, we illustrate that the position and velocity consensus errors among the networked mechanical systems converge to zero if and only if the damping is greater than certain lower bound for a given stiffness. We also demonstrate that the proposed controller attains the weighted average velocity consensus of the mechanical systems and ensures that their positions converge to the ramp trajectory determined by the initial weighted average of their velocities and positions. The performance of the proposed integral-sliding adaptive approach is shown by numerical simulation results with a network of seven uncertain two-DOF robots.

In our future work, we hope to extend the present result to the case that there exist communication delays between the mechanical systems. In addition, the proposed iBIBO stability concept is believed to be general in the sense that it may possibly be utilised in the consensus controller design and analysis for other complex networks. Future research will be devoted to exploiting its potential applications.

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