Tighter representations for set partitioning problems

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Abstract

In this paper, we consider the set partitioning polytope and we begin by applying the reformulation-linearization technique of Sherali and Adams (1990, 1994) to generate a specialized hierarchy of relaxations by exploiting the structure of this polytope. We then show that several known classes of valid inequalities for this polytope, as well as related tightening and composition rules, are automatically captured within the first- and second-level relaxations of this hierarchy. Hence, these relaxations provide a unifying framework for a broad class of such inequalities. Furthermore, it is possible to implement only partial forms of these relaxations from the viewpoint of generating tighter relaxations that delete the underlying linear programming solution to the set partitioning problem, based on variables that are fractional at an optimum to this problem.

Keywords: Reformulation-linearization technique; Set partitioning polytope; Valid inequalities; Cutting planes

1. Introduction

The set partitioning problem can be stated as follows:

\[ \text{SP: Minimize } \{cx: Ax = e, x_j = 0 \text{ or } 1 \ \forall j \in N\}, \]

where \( A = (a_{ij}) \) is an \( m \times n \) matrix of 0's and 1's, \( e \) is an \( m \) vector of 1's and \( N = \{1, \ldots, n\} \). Also, let us denote \( M = \{1, \ldots, m\} \), and let \( a_j \) represent the \( j \)th column of \( A \). We will assume that \( A \) has no zero rows or columns, that \( \text{rank}(A) = m < n \), and that \( \text{SP} \) is feasible.

Problem \( \text{SP} \) has been extensively investigated by several researchers for the last 30 years because of its special structure and its numerous practical applications. Among

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the applications described in the literature are crew scheduling, truck scheduling, information retrieval, circuit design, capacity balancing, capital investment, facility location, political districting, and radio communication planning. Several such applications along with solution procedures are described in [3, 5, 8–13]. As discussed in the survey by Balas and Padberg [6], two well-known approaches for problem SP are implicit enumeration and simplex based cutting plane methods. In particular, as observed by Chan and Yano [8], and Marsten et al. [12], a linear programming based branch-and-bound/cut code, is still the most popular tool for solving problem SP among practitioners. An essential component of any such tool is a tight linear programming relaxation afforded by the generation of strong valid inequalities. Several innovative schemes for generating such inequalities have been proposed in a seminal paper by Balas [3]. Also, in their recent paper, Hoffman and Padberg utilize the related structures of the set covering, set packing, and knapsack polytopes, inherent within relaxations of SP, in order to tighten its linear programming representation. However, in this paper, we will be mainly concerned with obtaining improved polyhedral representations while working directly on problem SP itself, as well as with the unification of existing valid inequalities, along with the derivation of new classes of cutting planes for this problem.

The following is an outline of this paper. Recently, Sherali and Adams [15, 16] have proposed a new reformulation-linearization technique (RLT) for generating a hierarchy of relaxations for linear and polynomial zero–one programming problems, spanning the spectrum from the continuous relaxation to the convex hull representation. By specializing the application of this technique to the set partitioning polytope, we are able to derive various polyhedral representations or relaxations for this problem. Similar to the pure zero–one programming case, for some fixed \( \delta \in \{0, \ldots, n\} \), we multiply the problem constraints using all possible factors composed of \( \delta \) binary variables and their complements, where the zero-degree factors are taken as unity. We then linearize the resulting polynomial program through a suitable redefinition of variables, and hence derive the \( \delta \)-th level relaxation. By exploiting the set partitioning structure, namely, zero–one coefficients of the constraint matrix \( A \) and unit right-hand sides, we obtain in Section 3 a hierarchy of explicit, specialized, polyhedral representations for this problem. Using these representations, we show in Section 4 that many of Balas’ [3] valid inequalities and strengthening procedures for the set partitioning polytope are automatically subsumed within the foregoing first- and second-level relaxations. Hence, this provides a unifying framework for viewing such inequalities and, moreover, it indicates that even partial constructions of these relaxations can yield tight representations. Section 5 concludes the paper and suggests avenues for further research.

2. Notation pertaining to the structure of the set partitioning problem

To facilitate the reading of this paper in conjunction with the existing literature, we will find it convenient to adopt the notation of Balas [3]. For ease in reading and as
a quick reference guide, we summarize this notation below, providing only a verbal
description to enhance understanding whenever the meaning is clear. Let us begin by
rewriting problem SP as follows, and then provide a list of related notation.

\[
\begin{align*}
&\text{SP: Minimize } \sum_{i \in N} c_i x_i : \sum_{j \in N} x_j = 1 \forall i \in M, x_j \text{ binary } \forall j \in N.
\end{align*}
\]

SP: linear programming relaxation of SP obtained by replacing \( x_j \) binary by \( x_j \geq 0 \)
\( \forall j = 1, \ldots, n \).

\( N = \{1, \ldots, n\} \) \( \equiv \) index set of variables, \( M = \{1, \ldots, m\} \) \( \equiv \) index set of constraints.

\( N_i = \{\text{set of variables appearing in constraint } i\}, \forall i \in M.\)

\( M_k = \{\text{set of constraints that contain the variable } x_k\}, \forall k \in N.\)

\( \bar{N}_i = N - N_i \forall i \in M, \) and \( \bar{M}_k = M - M_k \forall k \in N \) (respective complements).

\( M_J = \bigcup_{k \notin J} M_k = \{\text{set of constraints containing any variable from the set } J \subseteq N.\)

\( M_J = \text{complement of } M_J = \{\text{set of constraints not containing any variable from the set } J \subseteq N.\)

\( N_{i,k} = \{\text{set of variables in constraint } i \text{ that do not appear with } x_k \text{ in any row of } SP \text{ for } i \in \bar{M}_k, k \in N.\)

\( N_{i,J} = \{\text{set of variables in constraint } i \text{ that do not appear with any variable from } J \subseteq N \text{ in any row of } SP \equiv \bigcap_{k \notin J} N_{i,k}, \forall i \in \bar{M}_J, J \subseteq N.\)

\( L(k) = \{\text{set of variables in problem } SP \text{ that do not appear in any constraint along with } x_k, \forall k \in N \equiv \bigcup_{i \in \bar{M}_k} N_{i,k}, \forall k \in N.\)

\( L(k) = N - L(k) - \{k\}, \) the complement of \( L(k)\), other than \( k \) itself.

3. A specialized hierarchy of relaxations for the set partitioning polytope

As mentioned in the foregoing section, for any 0–1 programming problem, Sherali
and Adams [15] have proposed a heirarchy of relaxations spanning the spectrum
from the continuous linear programming relaxation to the complete convex hull
representation. When the 0–1 problem includes only equality constraints, as does
problem SP, this hierarchy would be constructed as follows. First, for any
\( d \in \{0, \ldots, n\}\), define certain (nonnegative) polynomial \textit{factors of degree} \( d \) as

\[
F_d(J_1, J_2) = \prod_{j \in J_1} x_j \prod_{j \in J_2} (1 - x_j) \text{ for each } J_1, J_2 \subseteq N \text{ such that } J_1 \bigcap J_2 = \emptyset.
\]

\[|J_1 \cup J_2| = d.\] (2)

Any \( (J_1, J_2) \) satisfying the condition in (3) is said to be of \textit{order} \( d \). For example, if
\( d = 3, \) and if \( J_1 = \{1, 4\} \) and \( J_2 = \{5\}, \) then \( F_3(J_1, J_2) = x_1 x_4 (1 - x_5) \). For convenience, we will take \( F_0(0, 0) = 1, \) and, accordingly, assume products over null sets to be unity. For a given \( d \in \{0, \ldots, n\}, \) let us also define \( F_d(J) = F_d(J, \emptyset) \). Hence, for example, \( F_3(\{1, 4, 5\}) = x_1 x_4 x_5 \). Using these factors, Sherali and Adams [15] con-
struct a relaxation at level \( \delta \) in the hierarchy, for any given \( \delta \in \{0, \ldots, n\}, \) using the
following two steps:

Step 1: For all \( d \in \{0, \ldots, \delta\} \), multiply each of the equalities (1) by each of the factors \( F_d(J) \) of degree \( d \). Include the constraints representing the nonnegativity of all possible factors \( F_d(J_1, J_2) \) of degree \( d, d = 1, \ldots, \min \{\delta + 1, n\} \). Use the identity \( x_i^2 = x_i \), i.e., \( x_j(1 - x_j) = 0 \), for each binary variable \( x_j, j = 1, \ldots, n \), in the resulting polynomial constraints. (Actually, Sherali and Adams show that it is sufficient to include \( F_d(J_1, J_2) \geq 0 \) only for \( (J_1, J_2) \) of order \( d = \min \{\delta + 1, n\} \), since the other nonnegativity constraints are implied by these constraints. However, we retain these implied constraints for convenience, as motivated by Proposition 1 below.)

Step 2: Linearize the resulting polynomial constraints by substituting the variable \( w_j \) in place of the product term \( \prod_{j \in J} x_j \) for each \( J \subseteq N \). Here, we adopt the notation that \( w_j = x_j \forall j = 1, \ldots, n \), and we take \( w_0 = 1 \). Also, for any polynomial expression \( [\cdot] \), we will denote by \( [\cdot]_L \) the corresponding linearized expression obtained via the foregoing variable substitution. In particular, for convenience, we will denote \( [F_d(J_1, J_2)]_L \equiv f_d(J_1, J_2) \) as in Sherali and Adams. This produces the required polyhedral relaxation at level \( \delta \), in the higher dimensional space of \( x \) and \( w \) variables.

Directly applying the above two steps to the set partitioning problem \( SP \) given by (1), we obtain the following polyhedral relaxation \( SPP_\delta \) at level \( \delta \):

\[
SPP_\delta = \{(x, w) : \sum_{j \in \{N, -J\}} w_j + \sum_{i \in M} w_j = w_j \forall J \subseteq N, |J| = d, d = 0, \ldots, \delta \}
\]

(3)

\( f_d(J_1, J_2) \geq 0, \forall (J_1, J_2) \) of order \( d, d = 1, \ldots, \min \{\delta + 1, n\} \). (4)

Note that for the case \( \delta = 0 \), using the fact that \( f_0(\emptyset, \emptyset) = 1 \), and that \( f_1(j, \emptyset) \equiv x_j \) and \( f_1(\emptyset, J) \equiv (1 - x_j) \) for \( j = 1, \ldots, n \), it follows that \( SPP_0 \) given by (3) and (4) is simply the feasible region of \( \overline{SP} \). Moreover, if we denote the projection of the set \( SPP_\delta \) onto the space of the original variables \( x \) by \( SPP_{\delta x} \), Sherali and Adams [15] show that for \( \delta = 0, \ldots, n \), the sets \( SPP_{\delta x} \) represent a sequence of nested relaxations leading up to the convex hull representation, that is,

\[
SPP \equiv \text{conv}\{x \in \mathbb{R}^n : Ax = e, x \text{ binary}\}
\]

\( = SPP_{\Delta x} \subseteq SPP_{\Delta (n-1)} \subseteq \cdots \subseteq SPP_1 \subseteq SPP_0 \). (5)

Before proceeding further, let us provide a simplification for \( SPP_\delta \) in two steps. First, as the following result shows, we can equivalently replace the constraints (4) with the following set of simple nonnegativity constraints:

\[ w_j \geq 0, \forall J \subseteq N, |J| = d, d = 1, \ldots, \min \{\delta + 1, n\}. \]

(6)

**Proposition 1.** For any \( \delta \in N \), the constraints (3) and (6) imply the constraints (4) in \( SPP_\delta \).
Proof. Consider the set SPP_δ for any δ ∈ N. We will use induction on |J_2| to prove the theorem.

(a) Consider any d ∈ {1, ..., min{δ + 1, n}}. If |J_2| = 0, then f_δ(J_1, J_2) ≡ w_{J_1} ≥ 0 is implied by (6). Next, suppose that |J_2| = 1, say, J_2 = {k} ∈ N. Then,

\[ f_δ(J_1, J_2) \equiv \left(1 - x_k\right) \prod_{j \in J_1} x_j \equiv w_{J_1} - w_{J_1 + k}. \] (7)

Now, for some i ∈ M_k, we have from (3) for J = J_1 that

\[ w_{J_1} - w_{J_1 + k} = \sum_{j \in \{N_i - J_1 - k\}} w_{J_1 + j} + |N_i \cap J_1| w_{J_1}. \] (8)

From (6)–(8), it follows that f_δ(J_1, J_2) ≥ 0.

(b) Assume that f_δ(J_1, J_2) ≥ 0, d = 1, ..., min{δ + 1, n}, is implied by the constraints (3) and (6) whenever |J_2| = 1, ..., (p - 1), and consider the case of |J_2| = p, where p ≥ 2. Suppose that k ∈ J_2. Hence, for any appropriate d, we can write

\[ f_δ(J_1, J_2) \equiv \left(1 - x_k\right) \prod_{j \in J_1} x_j \prod_{j \in |J_2 - k|} (1 - x_j). \]

Now, for some i ∈ M_k, we have the set partitioning constraint, x_k + \sum_{j \in \{N_i - k\}} x_j = 1. Note that (3) includes constraints obtained by multiplying the foregoing constraint with all factors F_δ(J), |J| = 0, 1, ..., δ, and that \( \prod_{j \in J_1} x_j \prod_{j \in |J_2 - k|} (1 - x_j) \) is a linear combination of such factors. Hence, by surrogating the constraints obtained in (3) by multiplying the (signed) factors in this combination with the foregoing constraint, we get

\[ \left( \sum_{j \in \{N_i - k\}} x_j \right) \prod_{j \in J_1} x_j \prod_{j \in |J_2 - k|} (1 - x_j) \equiv \left(1 - x_k\right) \prod_{j \in J_1} x_j \prod_{j \in |J_2 - k|} (1 - x_j) \equiv f_δ(J_1, J_2). \]

Letting J_2' = \{J_2 - k\}, the left-hand side of the above equation is comprised of terms of the type f_δ(J_1 + j, J_2') for j ∈ \{N_i - k\} \#\{J_1 \cup J_2', and of the type f_δ-1(J_1, J_2') for j ∈ \{N_i - k\} \# J_1, and zeros in case j ∈ \{N_i - k\} \# J_2'. Since |J_2'| = (p - 1), the induction hypothesis implies that all these terms are nonnegative. Hence, f_δ(J_1, J_2) ≥ 0 is also implied, and this completes the proof. □

The second simplification in SPP_δ results upon deleting certain null variables and the resulting trivial constraints, as follows. Examine constraint (3) for any J ⊆ N, |J| = d ∈ \{1, ..., δ\}, and for any i ∈ M_J, where M_J = \bigcup_{k \in J} M_k as defined in Section 2. Since |N_i \cap J| ≥ 1, the nonnegativity constraints (6) imply that w_{J_1 + j} = 0 \( \forall j \in \{N_i - J\} \). Furthermore, if |N_i \cap J| > 1, then we also have w_{J_1} ≡ 0. Hence, for any such J, we need to write (3) only for i ∈ M_J. Moreover, noting that for J = 0, (3) represents the original set partitioning constraints, we obtain upon eliminating the
identified null variables, a revised equivalent representation of SPP as specified below. (See Section 2 for the relevant notation.)

\[
\text{SPP}_\delta = \{ (x, w) : \sum_{j \in N_i} x_j = 1 \; \forall i \in M \} \quad \text{(9)}
\]

\[
\left[ \sum_{j \in N_i} w_{j+i} = w_j \; \forall i \in M_i \right] \; \forall J \subseteq N \exists |J| = d, \; d = 1, \ldots, \delta \quad \text{(10)}
\]

\[
[w_j \geq 0, \text{ and } w_{j+i} \geq 0 \; \forall j \in N_{i,J}, \forall i \in M_i] \; \forall J \subseteq N \exists |J| = d,
\]

\[
d = 1, \ldots, \delta \}.
\]

In particular, we can make the following observation with respect to the convex hull representation SPP in the hierarchy (5). (A referee indicated that Ceria [7] makes a similar observation in relation to the stable set polyhedron.) Let G be the intersection graph associated with SP, and let \( \alpha(G) \) be its independence number (see [14] for these standard definitions). Assume that G is connected (otherwise, SP is separable) and that G is not a complete graph (or else, SP is trivial). Since \( \sum_{i \in N} x_j \leq \alpha(G) \forall x \text{ feasible to SP} \), we have that \( \prod_{i \in N} x_j = 0 \forall J \subseteq N \exists |J| > \alpha(G) \), i.e., \( w_j = 0 \) for all \( |J| = \alpha(G) + 1, \ldots, n \). Hence, we have that \( \text{SPP} = \text{SPP}_{P\delta(G)} \) in (5) and that no higher-level relaxations are necessary.

4. A family of valid inequalities for the set partitioning problem

In this section, we examine the specialized forms of the first- and second-level relaxations \( \text{SPP}_1 \) and \( \text{SPP}_2 \), and demonstrate that these relaxations automatically subsume (in a continuous sense) known classes of valid inequalities, along with various strengthened and composed versions of these inequalities, as proposed by Balas [3]. Hence, these relaxations afford a unifying framework for viewing such inequalities, and admit tight representations that subsume them.

The first-level RLT relaxation \( \text{SPP}_1 \) of SPP, given by (9)–(11), can be written as follows. Note that in this relaxation, \( w_{jk} \) is the linearized term for the product \( x_j x_k \), \( j < k \). We will denote \( w_{(jk)} \) to be \( w_{jk} \) if \( j < k \) and \( w_{kj} \) if \( k < j \).

\[
\text{SPP}_1 = \{ (x, w) : \sum_{j \in N_i} x_j = 1 \; \forall i \in M \} \quad \text{(12)}
\]

\[
\sum_{j \in N_{i,k}} w_{(jk)} = x_k \; \forall i \in M_k, \forall k \in N \quad \text{(13)}
\]

\[
[x_k \geq 0, \; w_{(jk)} \geq 0 \; \forall j \in N_{i,k}, \forall i \in M_k] \; \forall k \in N \}. \quad \text{(14)}
\]

Similarly, by (9)–(11), we can write the second-level (\( \delta = 2 \)) RLT relaxation \( \text{SPP}_2 \) of SPP as follows. Note here that \( w_{jkl} \) is the linearized term for the product \( x_j x_k x_l \), for
Proposition 2 below reveals that the polyhedral representation SPP₁ implies the
class of valid elementary inequalities, given by (19) below, as introduced by Balas [3].

\textbf{Proposition 2.} For every \( k \in N \) and \( i \in \bar{M}_k \), the inequalities

\[ x_k - \sum_{j \in N_{i,k}} x_j \leq 0 \quad (19) \]

are satisfied by all \( x \in \text{SPP}_1 \).

\textbf{Proof.} For any \( k \in N \) and \( i \in \bar{M}_k \), consider the constraint (13). By the nonnegativity constraints (14), this implies that \( w_{(jk)} \leq x_k \ \forall j \in N_{i,k} \). Similarly, for each \( j \in N_{i,k} \), we have that \( k \in N_{i,j} \) for some \( t \in M_j \). Examining (13) written for this combination of \( t \) and \( j \), we get \( w_{(jk)} \leq x_j \). Hence, the constraint (13) implies that \( x_k = \sum_{j \in N_{i,k}} w_{(jk)} \leq \sum_{j \in N_{i,k}} x_j \), and this completes the proof. \( \square \)

Balas [3] develops several strengthening procedures and composition rules to
generate additional valid inequalities from such elementary inequalities. We will show
that these strengthening procedures are imbedded within the structure of SPP₁ and
SPP₂, and so, by directly employing the reformulations SPP₁ or SPP₂ in solving the
set partitioning problem, we \textit{automatically} incorporate many of these strong valid
inequalities. In fact, as shown by Adams and Sherali [1, 2] such first-level relaxations
can themselves provide very tight relaxations, leading to computationally attractive
procedures. Recently, Balas et al. [4] have computationally demonstrated that even
partial first-level representations provide tight relaxations.

To discuss the relationship between the aforementioned strengthening and com-
position rules with the relaxations SPP₁ and SPP₂, let \( L(k) \) and its complement \( \bar{L}(k) \)
be as defined in Section 2. For a given \( k \in N \), let \( N_k^0 = \{ j \in L(k) : x_j = 0 \ \forall x \ \text{feasible to SP and having} \ x_k = 1 \} \). While finding the entire set \( N_k^0 \) is impractical, we can easily
construct a subset of \( N_k^0 \) for some \( k \in N \) as follows. Note that \( w_{(kl)} = 0 \) for all feasible
solutions to any RLT formulation implies that \( x_k + x_l \leq 1 \) for any vertex \( x \) of SPP,
i.e., \( l \in N^0_k \). Now, consider the following constraints (17) of SPP_2 for a given \( k \in N \), and some \( l \in L(k) \):

\[
W(kl) = \sum_{j \in M_{(k,l)}} w_{ijkl} \quad \forall i \in M_{(k,l)}.
\]

Let us define \( z_{ijkl} \) as

\[
z_{ijkl} = \max \{ w_{ijkl} : w_{ijkl} = \sum_{j \in M_{(k,l)}} w_{ijkl}, \forall i \in M_{(k,l)}, \ w_{ijkl} \geq 0 \ \forall (jkl), \ w_{ijkl} \leq 1 \}.
\]

By its structure, \( z_{ijkl} \) equals zero or one. Hence, if \( z_{ijkl} = 0 \), then \( W(kl) = 0 \), i.e., \( l \in N^0_k \). Therefore, we can delete \( W(kl) \) from the first-level RLT formulation SPP_1.

The above procedure for detecting a set of zero \( W(kl) \)'s generalizes Balas' two procedures for strengthening elementary inequalities. We show below that these two procedures yield simple sufficient conditions for the optimal solution of (20) to be zero. The first of these procedures is considered in the following proposition.

**Proposition 3** (Special case of Proposition 3.1, Balas [3]). For some \( k \in N \), consider the valid elementary inequalities \( x_k - \sum_{j \in M_k \cap N_{i,k}} x_j \leq 0 \), for \( i \in M_k \). For each \( j \in L(k) \), define \( N(j) = \cup_{h \in N_{i,k} \cap N_{i,k}} N_{i,k} \setminus \{j\} \), and for each \( i \in M_k \), let \( T_i = \{j \in N_{i,k} : N_{i,k} \subseteq N(j) \text{ for some } h \in N_{i,k}\} \). Then, the inequalities \( x_k - \sum_{j \in M_k \cap T_i} x_j \leq 0, \forall i \in M_k \), are valid for SPP.

Since for any \( l \in T_i, i \in M_k \), we have \( N_{i,k} \subseteq N(l) \) for some \( h \in M_k \), we then have that \( N_{i,k} \cap N_{i,k} = \emptyset \). Moreover, \( h \in M_l \) or else we would have \( l \in N_{i,k} \), while \( l \in N^0(l) \). From (17) written for this \( h \in M_{(k,l)} \), we get \( w_{ijkl} = \sum_{j \in M_{(k,l)}} w_{ijkl} = 0 \). Hence, Proposition 3 is a trivial sufficient condition to guarantee that \( z_{ijkl} = 0 \) in (20). In particular, using \( w_{ijkl} = 0 \ \forall j \in T_i \) in (13), and applying the argument in Proposition 2, we see that the strengthened valid inequality of Proposition 3 is implied by SPP_2. Hence, SPP_2 automatically incorporates such strengthened versions of (19) within itself.

To further generalize this discussion related to Proposition 3, consider the following result.

**Proposition 4.** Consider any \( k \in N \) and \( i \in M_k \). Then, given a \( Q \subseteq N_{i,k} \), the inequality

\[
x_k - \sum_{j \in Q} x_j \leq 0
\]

is valid for SPP if and only if \( w_{ijk} = 0 \ \forall j \in (N_{i,k} - Q) \) for any feasible solution \((x, w)\) to SPP_1 having \( x \) binary.

**Proof.** Note from Sherali and Adams [15] that SPP_1 with the added restriction that \( x \) is binary valued (call this problem SPP_1 (\( x \) binary)) is equivalent to SP. Hence, if (21)
is valid for SPP, then it is also valid for SPP$_1$ (x binary), and so by multiplying this with $x_k$ and linearizing, the constraint $x_k - \sum_{j \in Q} w_{jk} x_j \leq 0$ is valid for SPP$_1$ (x binary). From (13), it then follows that $w_{jk} = 0$ $\forall j \in (N_{i,k} - Q)$, because $w_{jk} > 0$ $\forall j,k$. Conversely, if $w_{jk} = 0$ $\forall j \in (N_{i,k} - Q)$ in SPP$_1$ (x binary), and this is directly imposed in SPP$_1$, then (13) becomes $\sum_{j \in Q} w_{jk} x_k = x_k$. As in Proposition 2, this implies that $x_k - \sum_{j \in Q} x_j \leq 0$ is valid for SPP$_1$ (x binary), and hence for SPP. This completes the proof. □

In the light of Proposition 4, for some $k \in N$, if we were given instead in Proposition 3 that the inequalities $x_k - \sum_{j \in Q} x_j \leq 0$ for $i \in M_k$, for some index sets $Q_i \subseteq N_{i,k}$, are valid for SPP, then a similar tightening of these inequalities is possible, with the tightened versions being automatically subsumed within SPP$_2$, by simply replacing $N_{i,k}$ by $Q_i$ in SPP$_1$ and SPP$_2$, after fixing $w_{jk} = 0$ $\forall j \in (N_{i,k} - Q_i)$, $i \in M_k$. This reconstructs Balas’ Proposition 3.1.

The following examples illustrate that not only does SPP$_2$ subsume the tightened inequalities of Proposition 3, but because this proposition is only a sufficient condition for $z_{kl} = 0$ in (20), it inherently accommodates other strengthened versions of (19) as well.

**Example 1** (Example 3.1 in Balas [3]). Consider the following coefficient matrix $A$ (where the blank spaces are zeros) for a set partitioning polytope having $m = 5$ and $n = 15$.

$$
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & & & & & & & & & & & & & \\
2 & 1 & 1 & 1 & 1 & & & & & & & & & \\
3 & 1 & 1 & 1 & 1 & & & & & & & & & \\
4 & 1 & 1 & 1 & & & & & & & & & & \\
5 & & & & & & & & & & & & & \\
\end{array}
$$

Consider $k = 1$, for which $M_1 = \{3, 4, 5\}$, and $N_{3,1} = \{3, 12\}$, $N_{4,1} = \{3, 4\}$, and $N_{5,1} = \{3, 4, 5, 12\}$. Let us examine the procedure for Proposition 3 to strengthen the inequality $x_1 - x_3 - x_{12} \leq 0$ associated with $N_{3,1}$. We have that $N(3) = \{4, 5, 12\}$, $N(12) = \{3, 4, 5\}$, and we find that $N_{4,1} \subseteq N(12)$. Hence, $T_3 = \{12\}$, and the above inequality can be replaced by $x_1 - x_3 \leq 0$.

We now demonstrate how this strengthened inequality is automatically implied by SPP$_2$. Consider the equalities of type (16) for $k = 1$: $x_1 = w_{1,3} + w_{1,12}$ for $i = 3$, $x_1 = w_{1,3} + w_{1,4}$ for $i = 4$, and $x_1 = w_{1,3} + w_{1,4} + w_{1,5} + w_{1,12}$ for $i = 5$. Let us examine the form of equality (17) for $l = 12$ in this second-level RLT formulation SPP$_2$. Since $M_{12} = \{1, 2, 4\}$, we have that $M_{1,12} = M_{1} \cap M_{12} = \{4\}$, and for $h = 4$, we get $N_{4,1} = \{3, 4\}$ and $N_{4,12} = \{9, 11\}$. Therefore, $N_{4,1,12} = N_{4,1} \cap N_{4,12} = \emptyset$. 

Hence, the inequality of type (17) for \( i = 4, k = 1 \) and \( l = 12 \) is 
\[ w_{(1,1,12)} = 0. \]
In particular, this yields in the above constraint (16) written for \( i = 3 \) that 
\[ x_1 = w_{1,3} \leq x_3. \]

**Example 2** (Example 3.2 in Balas [3]). This example illustrates that SPP\(_2\) captures
strengthened version of (19) beyond that of Proposition 3. Consider the following
coefficient matrix \( A \) for a set partitioning polytope having \( m = 7 \) and \( n = 10 \).
\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} \\
2 & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} \\
3 & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} \\
4 & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} \\
5 & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} \\
6 & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} \\
7 & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} \\
\end{array}
\]

For \( k = 1 \), we have that \( \bar{M}_1 = \{3,4,5,6,7\} \), \( N_{3,1} = \{2,5,7\} \), \( N_{4,1} = \{2,6,8\} \),
\( N_{5,1} = \{3,5,8\} \), \( N_{6,1} = \{3,4,6\} \), and \( N_{7,1} = \{4,5,7\} \). The reader can verify that
Balas' first strengthening procedure does not apply to any of the elementary inequalities associated with \( k = 1 \). On the other hand, consider SPP\(_2\). The constraints (16) for
\( k = 1 \) can be stated as follows:
\[ x_1 = w_{1,2} + w_{1,5} + w_{1,7}, x_1 = w_{1,2} + w_{1,6} + w_{1,8}, \]
\[ x_1 = w_{1,3} + w_{1,5} + w_{1,8}, x_1 = w_{1,3} + w_{1,4} + w_{1,6}, \text{ and } x_1 = w_{1,4} + w_{1,5} + w_{1,7}. \]
Furthermore, consider the second-level constraints of type (17) for \( i = 2 \) and \( k = 1 \),
where \( \bar{M}_2 = \{1,2,5,6,7\} \), so that \( \bar{M}_{(1,2)} = \bar{M}_1 \cap \bar{M}_2 = \{5,6,7\} \). Also, we have
\( N_{5,2} = \{3,10\} \), \( N_{6,2} = \{3,4\} \), and \( N_{7,2} = \{4\} \). This gives \( N_{5,(1,2)} = \{3\} \),
\( N_{6,(1,2)} = \{3,4\} \), and \( N_{7,(1,2)} = \{4\} \). Consequently, the constraints of type (17) for \( k = 1 \)
and \( l = 2 \) are of the form
\[ w_{1,2} = w_{1,2,3}, w_{1,2} = w_{1,2,3} + w_{1,2,4}, \text{ and } w_{1,2} = w_{1,2,4}. \]

This system implies that \( w_{1,2} = w_{1,2,3} = w_{1,2,4} = 0. \) Hence, the elementary inequality associated with \( N_{3,1} \), namely, \( x_1 - x_2 - x_5 - x_7 \leq 0 \), can be strengthened to
\( x_1 - x_5 - x_7 \leq 0 \) using (16) for \( k = 1 \) and \( i = 3 \). Note again that this strengthened inequality is automatically implied within SPP\(_2\).

We now consider Balas' second strengthening procedure.

**Proposition 5** (Proposition 3.2, Balas [3]). For some \( k \in \mathbb{N} \), let the index sets
\( Q_{ik} \subseteq N_{i,k}, i \in \bar{M}_k \), be such that the inequalities 
\( x_k - \sum_{j \in Q_{ik}} x_j \leq 0, i \in \bar{M}_k \), are satisfied by all \( x \in \text{SPP} \).
For each \( i \in \bar{M}_k \), define
\[ U_{ik} = \{ j \in Q_{ik} : Q_{hk} \cap Q_{kj} = \emptyset \text{ for some } h \in \bar{M}_{(k,j)} \}. \]
Then, the inequalities 
\( x_k - \sum_{j \in Q_{ik} \setminus U_{ik}} x_j \leq 0, i \in \bar{M}_k \), are satisfied by all \( x \in \text{SPP} \).
The foregoing strengthening procedure of Proposition 5 can be easily verified to be inherent within SPP2 as follows. As before, given the validity of \( x_k - \sum_{j \in Q_i} x_j \leq 0 \), we can set \( w_{(jk)} = 0 \) \( \forall j \in (N_{i,k} - Q_{ik}) \) for each \( i \in M_k \), so that the revised \( N_{i,k} \equiv Q_{ik} \). Now, condition (22) of Proposition 5 is a trivial sufficient condition for ensuring that the corresponding problem (20) written for indices \( j \) and \( k \), where \( j \in U_{ik} \), has an objective value of zero. This follows because (20) directly includes the simple constraints \( w_{(jk)} = 0 \) for any \( j \in U_{ik} \), noting that \( N_{h,k} \cap N_{h,j} \equiv Q_{hk} \cap Q_{hj} = \emptyset \) for some \( h \in M_{|k,l'|} \). Consequently, with \( w_{(jk)} = 0 \) \( \forall j \in Q_{ik} \setminus U_{ik} \) that appear in (16), by using the argument of Proposition 2, we see that the strengthened valid inequality of Proposition 5 is also implied by SPP2.

Note that in a similar spirit, we can employ RLT relaxations higher than the second level to further strengthen the valid inequalities obtained from the first-level formulation. For example, in the second-level RLT formulation, suppose that \( w_{(jkl)} = w_{(jk)} \) for some \( l \), and consider the third-level formulation constraint \( w_{(jkl)} = \sum_{j \in N_{i,k,j}} w_{(jk)} \), for \( i \in M_{|j,l'|} \). If \( N_{i,k,j} = \emptyset \), then \( w_{(jkl)} = 0 \), and, consequently, \( w_{(jk)} = 0 \). This information can be transferred to SPP1 to further tighten its formulation. Also, note that Balas’ strengthening procedures use only partial information regarding the logical implications of SPP1 for tightening SPP1. On the other hand, if one has the facility to handle SPP1 itself directly, then stronger relaxations can be enforced via such an explicit representation of SPP2.

Balas (1977) has also developed a particular composition rule that considers known valid inequalities of the type

\[
x_k - \sum_{j \in S} x_j \leq 0,
\]

where \( S \subseteq L(k) \), \( k \in N \), and composes specific pairs of such inequalities, deriving for each pair another valid inequality of the type (23) that is tighter than the sum of the two inequalities that generated it. We show below that when the two parent inequalities are of the type (19) or (21), then the resulting inequality obtained by applying this composition rule is implied by SPP2. Hence, SPP2 automatically accommodates such additional valid inequalities as well. However, as one might guess, if such a composition is repeatedly applied sequentially to a pair of inequalities selected from the combined set (19), (21) and (23) thus generated, then suitable higher-level RLT representations need to be considered to automatically imply the new composed valid inequalities. Otherwise, if only SPP1 is considered, then a coefficient reduction step needs to be interspersed as discussed below in order to derive such inequalities. Below, we state Balas’ composition rule and then present our analysis that views this process as a consequence of the RLT procedure.

**Proposition 6** (Proposition 5.1, Balas [3]). For \( k, h \in N \), let \( S_k \subseteq L(k) \) and \( S_h \subseteq L(h) \) be such that \( h \in S_k \), but \( k \not\in S_h \), and that the inequalities

\[
x_k - \sum_{j \in S_k} x_j \leq 0 \quad \text{and} \quad x_h - \sum_{j \in S_h} x_j \leq 0
\]

(24)
are satisfied by all \( x \in SPP \). Then all \( x \in SPP \) satisfy the inequality

\[
x_k - \sum_{j \in S} x_j \leq 0,
\]

where \( S = (S_k - h) \cup [S_h \cap L(k)] \). Furthermore, \((25)\) is stronger than the sum of the two inequalities \((24)\) if and only if \( S_h \cap [S_k \cup L(k)] \neq \emptyset \).

We want to first show that the composite inequality of type \((25)\) is implied by the constraints of \( SPP_2 \), assuming in addition that \( S_k \subseteq N_{i,k} \) for some \( i \in \overline{M}_k \), and \( S_h \subseteq N_{i,h} \) for some \( i \in \overline{M}_h \). (The statement regarding the relative strength of \((25)\) versus the sum of \((24)\) is readily evident.) Since the inequalities of type \((24)\) are given as being valid for \( SPP \), by Proposition 4, we can set as before \( w_{(jk)} = 0 \) \( \forall j \in N_{i,k} - S_k \), and \( w_{(jh)} = 0 \) \( \forall j \in N_{i,h} - S_h \). We then have the following first-level RLT constraints \((13)\) that imply the corresponding constraints \((24)\):

\[
x_k = \sum_{j \in S_k} w_{(jk)} = w_{(kh)} + \sum_{j \in (S_k - h)} w_{(jk)} \quad \text{(since \( h \in S_k \))}
\]

and

\[
x_h = \sum_{j \in S_h} w_{(jh)}.
\]

Hence, noting the consequence of multiplying \((26)\) by \( x_k \) and linearizing, we have inherent in \( SPP_2 \) that

\[
w_{(jkh)} = 0 \quad \forall j \in (S_k - h).
\]

Similarly, examining the constraint obtained by multiplying \((27)\) with \( x_h \) and linearizing, we have that the second-level RLT constraints of type \((17)\) imply that \( w_{(kh)} = \sum_{j \in S_k} w_{(jkh)} \). Since for all \( h \), \( w_{(jkh)} = 0 \) for \( j \notin L(k) \) because \( w_{(jk)} = 0 \), it follows from \((28)\) that

\[
w_{(kh)} = \sum_{j \in (S_h \cap L(k) - [S_k - h])} w_{(jkh)}.
\]

By substituting this identity for \( w_{(kh)} \) in \((26)\), we have that

\[
x_k = \sum_{j \in (S_h \cap L(k) - [S_k - h])} w_{(jkh)} + \sum_{j \in (S_k - h)} w_{(jk)}.
\]

Since \( w_{(jkh)} \leq x_j \) and \( w_{(jkh)} \leq x_j \) by \( SPP_2 \), \((29)\) implies that

\[
x_k \leq \sum_{j \in (S_h \cap L(k) - [S_k - h])} x_j + \sum_{j \in (S_k - h)} x_j + \sum_{j \in S} x_j,
\]

where \( S = (S_k - h) \cup (S_h \cap L(k)) \). Hence, the valid inequality \((25)\) is implied if we implement the second-level RLT formulation \( SPP_2 \) in this case.
For the general case when $S_k$ and $S_h$ are as defined in Proposition 6 and are not necessarily a subset of some $N_{i,k}$ and $N_{i,h}$, respectively, as assumed above, suppose that the valid inequalities (24) have been derived and added to the problem, and are then also subjected to the reformulation-linearization technique. Multiplying these inequalities by $x_k$ and linearizing yields

$$x_k - w_{ikh} - \sum_{j \in S_k - h} w_{ijk} \leq 0 \quad \text{(since } h \in S_k)$$

and

$$w_{ikh} - \sum_{j \in S_h \cap L(k)} w_{ijk} \leq 0 \quad \text{(since } w_{ijk} \equiv 0 \text{ for } j \in L(k)).$$

Hence, the revised formulation of SPP1 would contain the foregoing two inequalities. Summing them, we get the following implied inequality, noting that $S_k \subseteq L(k)$:

$$x_k \leq \sum_{j \in S_k - h - (S_h \cap S_k)} w_{ijk} + \sum_{j \in (S_h \cap L(k) - S_k)} w_{ijk} + 2 \sum_{j \in S_h \cap S_k} w_{ijk}. \quad (30)$$

Now, recognizing the binariness of the $x$ and $w$ variables at a set partitioning solution, we can perform a valid coefficient reduction in (30) by replacing the coefficient 2 in the last term in (30) by 1. Doing this and combining the resultant term with the first one on the right-hand side in (30), we get $x_k \leq \sum_{j \in S_k} w_{ijk} \leq \sum_{j \in S} x_j$ as asserted in (25).

Note that the step of tightening (30) via the coefficient reduction strategy in order to derive (25) is not necessarily an implication of SPP2 as for the case analyzed in (26)-(29). However, this inequality (25) can indeed be directly derived via a higher-level RLT process. Hence, the inequality (25) derived via the above coefficient reduction can be viewed as a projection from a higher-level RLT constraint. To see this, define the product factor $P = \prod_{j \in S_h \cap S_k} (1 - x_j)$ and multiply the inequalities in (24) by the (nonnegative) term $x_k P$. Linearizing by substituting a variable for each product term after using $x_f^2 = x_f$, i.e., $x_f (1 - x_f) = 0$ for each binary variable $x_f$, we obtain the following two inequalities, where $\lfloor \cdot \rfloor_L$ denotes the linearized substitute for $\lfloor \cdot \rfloor$ as before, and where $S_{kh}$ abbreviates $S_k \cap S_h$:

$$\lfloor x_k P \rfloor_L - \lfloor x_h x_k P \rfloor_L - \sum_{j \in S_h - h} \lfloor x_j x_k P \rfloor_L \leq 0,$$

$$\lfloor x_h x_k P \rfloor_L - \sum_{j \in (S_h \cap L(k) - S_k)} \lfloor x_j x_k P \rfloor_L \leq 0.$$

Summing these two inequalities yields

$$\lfloor x_k P \rfloor_L - \sum_{j \in S - S_{kh}} \lfloor x_j x_k P \rfloor_L \leq 0. \quad (31)$$
Now, \( P = 1 - \sum_{j \in S_{kh}} x_j + G \), where \( G = \prod_{j \in S_{kh}} (1 - x_j) + \sum_{j \in S_{kh}} x_j - 1 \geq 0 \) since either \( x_j = 0 \) \( \forall j \in S_{kh} \) whence \( G = 0 \), or else \( x_j = 1 \) for some \( j \in S_{kh} \) whence again \( G > 0 \). Using this expansion for \( P \) in (31) yields

\[
x_k - \sum_{j \in S_{kh}} w_{(jk)} + [x_kG]_L - \sum_{j \in S - S_{kh}} \{ w_{(jk)} - [x_jx_k(1 - P)]_L \} \leq 0,
\]

i.e.,

\[
x_k - \sum_{j \in S} w_{(jk)} \leq - [x_kG]_L - \sum_{j \in S - S_{kh}} [x_jx_k(1 - P)]_L.
\]

Note that \( x_kG \geq 0 \) and \( x_jx_k(1 - P) \geq 0 \). Moreover, it can be verified that by including nonnegativities on all linearized factors of type (2) for \( d = |S_{kh}| + 2 \), we also have that \( [x_kG]_L \geq 0 \) and \( [x_jx_k(1 - P)]_L \geq 0 \) are implied. Hence, (32) implies that \( x_k \leq \sum_{j \in S} w_{(jk)} \leq \sum_{j \in S} x_j \), which again yields (25).

5. Summary and conclusions

This paper has focused on the specialization of a reformulation-linearization technique (RLT) to the set partitioning problem. We have shown that the first- and second-level RLT formulations, \( \text{SP}_1 \) and \( \text{SP}_2 \), contain some rich structural properties with respect to generating a tight representation for Problem \( \text{SP} \). In particular, several known classes of valid inequalities, as well as related tightening and composition rules, are subsumed within these relaxations. However, in the case of large problem instances, we may not afford the luxury of being able to cope with the size of these resulting reformulations if they are generated in their entirety. In such cases, we might wish to construct only a partial first- or second-level reformulation, viewing only the fractional variables at an optimum basic feasible solution to \( \overline{\text{SP}} \) as being binary valued, and treating the remaining variables as being continuous, in light of Sherali and Adams [16] and Balas et al. [4]. By generating RLT constraints using a subset of such fractionating variables along with the constraints in which they appear, partial relaxations that delete the obtained fractional linear programming solution can be derived in a manner similar to that presented herein. Such strategies are applicable to other (mixed-integer) zero–one programming problems as well, and will be computationally explored in future work.

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