The number of limit cycles of a quintic polynomial system

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In this paper we consider the bifurcation of limit cycles of the system \( \dot{x} = y(x^5 - a^2) (y^2 - b^2) + \varepsilon P(x, y), \dot{y} = -x(x^5 - a^2)(y^2 - b^2) + \varepsilon Q(x, y) \) for \( \varepsilon \) sufficiently small, where \( a, b \in \mathbb{R} - \{0\} \), and \( P, Q \) are polynomials of degree \( n \), we obtain that up to first order in \( \varepsilon \) the upper bounds for the number of limit cycles that bifurcate from the period annulus of the quintic center given by \( \varepsilon = 0 \) are \( (3/2)(n + \sin^2(n\pi/2)) + 1 \) if \( a \neq b \) and \( n - 1 \) if \( a = b \). Moreover, there are systems with at least \( (3/2)(n + \sin^2(n\pi/2)) + 1 \) if \( a \neq b \) and, \( n - 1 \) limit cycles if \( a = b \).

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1. Introduction and statement of results

One of the main problems in the qualitative theory of real planar differential systems is the study of their limit cycles. Probably the more classical way to produce limit cycles is by perturbing a system which has a center, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits of the period annulus of the unperturbed system [see1,2]. From [3] we know that by perturbing the linear center \( \dot{x} = -y, \dot{y} = x \) using arbitrary polynomials \( P \) and \( Q \) of degree \( n \), we obtain at most \( [(n - 1)/2] \) bifurcated limit cycles up to first order in \( \varepsilon \), from the period annulus, where \([.]\) denotes the integer part function. Also the authors of [4] obtained at most \( n \) limit cycles up to first order in \( \varepsilon \) by perturbing system \( \dot{x} = -y(1 + x), \dot{y} = x(1 + x) \) inside the polynomial differential systems of degree \( n \). It is known that if we perturb the cubic center \( \dot{x} = -y(1 + x)(2 + x), \dot{y} = x(1 + x)(2 + x) \) inside the polynomial differential systems of degree \( n \), we can obtain at most \( 2n + 2 - (-1)^n \) limit cycles up to first order in \( \varepsilon \) [5]. In [6] the authors perturbing the Hamiltonian center given by \( H = y^2/2 + x^4/4(n + 1) \) inside the polynomial differential systems of degree \( n \) odd, obtained \( (n + 1)(n + 3)/8 \) limit cycles in the perturbed system. This number of limit cycles was again obtained later on, perturbing other centers [7]. Also in [8] the authors, perturbing a convenient Hamiltonian center inside the polynomial differential systems of degree \( n \) even, obtained \( n(n + 2)/8 \) limit cycles in the perturbed system. Later on, this number was improved for \( n \) even in [7] obtaining \( n(n + 6)/8 \) limit cycles. In [9] the authors studied the perturbation of the vector field \( \dot{x} = -yR(x, y), \dot{y} = xR(x, y) \), where \( R(x, y) = (a + 1)x^2 + ay^2 + b \), with \( a \) and \( b \) reals and \( ab \neq 0 \). They obtained \( (n + 1)(n + 7)/8 - 1 \), for \( n \) odd, and \( n(n + 6)/8 - 1 \), for \( n \) even, as a lower bound for the maximum number of limit cycles surrounding a unique singular point. Llibre and Buică [10] investigated the perturbed system of cubic polynomial differential system \( \dot{x} = -y(x + a)(y + b) + \varepsilon P(x, y), \dot{y} = x(x + a)(y + b) + \varepsilon Q(x, y) \), where \( P(x, y), Q(x, y) \) are polynomials with degree \( n \), and \( a, b \in \mathbb{R} - \{0\} \), and proved that upper bound for the number of isolated zeros of the Abelian integral is \( 3(n - 1)/2 + 4 \) if \( a \neq b \) and, respectively, \( 2[(n - 1)/2] + 2 \) if \( a = b \), up to first order in \( \varepsilon \), from the period annulus.

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In this paper we are interested in determining an upper bound for the number of limit cycles of
\[
\begin{align*}
\dot{x} &= y(x^2 - a^2)(y^2 - b^2) + \varepsilon \sum_{0 \leq i+j \leq n} a_{ij} x^i y^j, \\
\dot{y} &= -x(x^2 - a^2)(y^2 - b^2) + \varepsilon \sum_{0 \leq i+j \leq n} b_{ij} x^i y^j,
\end{align*}
\]
which bifurcated from the period annulus of system \((1)\) \(\varepsilon = 0\) up to first order in \(\varepsilon\). Here \(a, b \in \mathbb{R} - \{0\}\), \(|\varepsilon|\) is a sufficiently small real number, \(a_{ij}, b_{ij}\) are real and \(|a_{ij}| \leq K, |b_{ij}| \leq K\) with \(K\) a positive constant and \(n\) a positive integer. Adopting notations used in [5], we can rewrite \(c = (a_{ij}, b_{ij})_{0 \leq i+j \leq n}\) and \(B_k = \{c : |a_{ij}| \leq K, |b_{ij}| \leq K\}.\) Note that system \((1)\) \(\varepsilon = 0\) is mainly the linear center with four straight lines of singular points. It has the first integral \(H(x, y) = x^2 + y^2\) and the integrating factor \(R(x, y) = \frac{(x^2 - a^2)(y^2 - b^2)}{(x^2 - a^2)(y^2 - b^2)}\). The period annulus of \((1)\) \(\varepsilon = 0\) can be described by
\[
L_h : x^2 + y^2 = h, \quad \text{or} \quad x = \sqrt{h} \sin t, y = \sqrt{h} \cos t, 0 < h < \min(a^2, b^2) = h_0.
\]

Let
\[
\Phi(h) = \int_{I_0} \frac{P(x, y)dy - Q(x, y)dx}{(x^2 - a^2)(y^2 - b^2)}, \quad 0 \leq h < h_0
\]
be the first-order Melnikov function or Abelian integral of \((1)\) where \(P(x, y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j\) and \(Q(x, y) = \sum_{i+j=0}^{n} b_{ij} x^i y^j\).

Our main results are

**Theorem 1.1.** Suppose \(a \neq b\). For any \(K > 0\) and compact set \(V \subset \Omega\), if \(\Phi(h)\) is not identically zero for \(c = (a_{ij}, b_{ij})\) varying in a compact set \(D \subset B_k\), then there exist an \(\varepsilon_0 > 0\) such that for \(0 < |\varepsilon| < \varepsilon_0\), \(c \in D, (1)\) has at most \(\frac{1}{2}(n + \sin^2 \frac{\pi}{2}) + 1\) limit cycles in \(V\). Moreover, there are systems \((1)\) with at least \(\frac{1}{2}(n + \sin^2 \frac{\pi}{2}) + 1\) limit cycles in \(V\).

**Theorem 1.2.** Suppose \(a = b\).

(i) For any \(K > 0\) and compact set \(V \subset \Omega\), if \(\Phi(h)\) is not identically zero for \((a_{ij}, b_{ij})\) varying in a compact set \(D \subset B_k\), then \((1)\) has at most \(n - 1\) limit cycles in \(V\).

(ii) For any \(K > 0\) and compact set \(V \subset \Omega\), there exist an \(\varepsilon_0\) and \((a_{ij}^0, b_{ij}^0) \in B_k\), such that for all \(0 < |\varepsilon| < \varepsilon_0\), \(|a_{ij} - a_{ij}^0| < \varepsilon_0, |b_{ij} - b_{ij}^0| < \varepsilon_0\), \((1)\) has precisely \(n - 1\) limit cycles in \(V\).

2. Some lemmas

Let
\[
I_{ij} = \int_0^{2\pi} \frac{x^iy^jdt}{(x^2 - a^2)(y^2 - b^2)}, \quad \Phi_{ij} = a_{ij}I_{i+1,j} + b_{ij}I_{i,j+1},
\]
where \(x = \sqrt{h} \sin t, y = \sqrt{h} \cos t\). Notice that hereafter we just use \(x, y\) for convenience. To prove our main theorems we first give some lemmas.

**Lemma 2.1.** We have
\[
J_1 = \int_0^{2\pi} \frac{dt}{x - a} = \frac{-2\pi}{\sqrt{a^2 - h}}, \quad J_2 = \int_0^{2\pi} \frac{dt}{x + a} = \frac{2\pi}{\sqrt{a^2 - h}},
\]
\[
J_3 = \int_0^{2\pi} \frac{dt}{y - b} = \frac{-2\pi}{\sqrt{b^2 - h}}, \quad J_4 = \int_0^{2\pi} \frac{dt}{y + b} = \frac{2\pi}{\sqrt{b^2 - h}}.
\]

**Proof.** Results follow easily by using residue theorem. \(\square\)

**Lemma 2.2.** For \(i \geq 0, k \geq 1, I_{2k-1} = 0\). Also we have
\[
I_{1,0} = \int_0^{2\pi} \frac{xdt}{(x^2 - a^2)(y^2 - b^2)} = 0,
\]
\[
I_{0,0} = \frac{-2\pi}{h - (a^2 + b^2)} \left[ \frac{1}{a\sqrt{a^2 - h}} + \frac{1}{b\sqrt{b^2 - h}} \right].
\]
Proof. Since along \( L_h \), \( y^2 = h - x^2 \), by partial fractions, (3) becomes
\[
I_{1,0} = \frac{1}{b^2 + a^2 - h} \left[ -\frac{1}{2} (J_1 + J_2) + \int_0^{2\pi} \frac{x}{x^2 + b^2 - h} \, dt \right] = \frac{1}{b^2 + a^2 - h} \int_0^{2\pi} \frac{x}{x^2 + b^2 - h} \, dt.
\]
From Lemma 2.1, \( J_1 + J_2 = 0 \). Now let \( x = \sqrt{h} \sin t \), then we have \( \int_0^{2\pi} \frac{x}{x^2 + b^2 - h} \, dt = 0 \). Hence (3) follows. Formula (4) can be proved in the same way. Also by symmetry of \( L_h \) with respect to \( x \)-axis it is easy to see that \( I_{i,2k-1} = 0 \) for \( i \geq 0, k \geq 1 \).

Lemma 2.3.
\[
I_{2,0} = \frac{-2\pi}{h - (a^2 + b^2)} \left[ \frac{a}{\sqrt{a^2 - h}} - \frac{b}{\sqrt{b^2 - h}} + \frac{h}{b\sqrt{b^2 - h}} \right]. \tag{5}
\]
\[
I_{i,2k} = \sum_{j=0}^{k} (-1)^j C_i^j I_{i+2j,0} h^{k-j}. \tag{6}
\]
Proof. Formula (5) can be followed easily from Lemmas 2.1 and 2.2. (6) also is obtained using \( (h - x^2)^k = \sum_{j=0}^{k} (-1)^j C_i^j x^{2j} h^{k-j} \).

Lemma 2.4.
\[
I_{i,0} = \begin{cases} 
\int_0^{2\pi} \frac{1}{(X^2 - 1)(Y^2 - 1)} \, dt + P_0, & i = 2k - 1, k > 1 \\
\int_0^{2\pi} \frac{1}{(X + 1)(Y^2 - 1)} \, dt, & i = 2k, k > 1 
\end{cases} \tag{7}
\]
where \( X = x/a, Y = y/b \) and
\[
P_0 = \sum_{j=1}^{k-1} \left( \int_0^{2\pi} \frac{X^{2j-1}}{Y^2 - 1} \, dt \right) + \int_0^{2\pi} \frac{1}{(X + 1)(Y^2 - 1)} \, dt, \quad k = i + 1/2 \tag{8}
\]
\[
P_e = \sum_{j=0}^{k-1} \left( \int_0^{2\pi} \frac{X^{2j}}{Y^2 - 1} \, dt \right), \quad k = i/2. \tag{9}
\]
Proof. Let \( X = x/a, Y = y/b \). Then we have
\[
I_{i,0} = \frac{d^i}{a^2 b^2} \left[ \int_0^{2\pi} \frac{1}{(X^2 - 1)(Y^2 - 1)} \, dt + \int_0^{2\pi} \frac{1}{(X + 1)(Y^2 - 1)} \, dt \right].
\]
Let \( P = \int_0^{2\pi} \frac{1 + X + X^2 + \cdots + X^{(i-1)}}{(X + 1)(Y^2 - 1)} \, dt \). Hence for \( k > 1, P = P_0 \) if \( i = 2k - 1 \), and \( P = P_e \), if \( i = 2k \).

Lemma 2.5. The integrals \( \int_0^{2\pi} \frac{x^r}{y^2 - 1} \, dt, k > 1 \) emerged in Lemma 2.4 are
\[
\int_0^{2\pi} \frac{x^r}{y^2 - 1} \, dt = 0, \quad r = 2k - 1,
\]
\[
\int_0^{2\pi} \frac{x^r}{y^2 - 1} \, dt = \frac{-b^2}{a'(b^2 - h)} \left[ 2A^{(r)} M_1 + a_0^{(r)} + a_1^{(r)} h + \cdots + a_{k-1}^{(r)} h^{k-1} \right], \quad r = 2k.
\]
where \( a_j^{(r)} = (-1)^j (b^2 - h)^{2-j} K_j \), are real for \( j = 0, \ldots, k - 1 \). Also
\[
A^{(r)} = \frac{1}{2r} \frac{1}{b^2 - h}, \quad K_0 = 2\pi, K_{2j} = \frac{(2j - 1)!!}{(2j)!!} 2\pi, \quad M_1 = \frac{2\pi}{b} \sqrt{b^2 - h}. \tag{10}
\]
Proof. We have \( Y^2 = \frac{1}{b^2} (h - a^2 X^2) \), so
\[
\int_0^{2\pi} \frac{x^r}{y^2 - 1} \, dt = \int_0^{2\pi} \frac{-b^2}{a'(b^2 - h)} \int_0^{2\pi} \frac{x^r}{(1 - \sqrt{b^2 - h})(1 + \sqrt{b^2 - h})} \, dt.
\]
where \( i = \sqrt{-1} \). But by Lemma 2.2 in [5] if \( r = 2k \), then

\[
\int_0^{2\pi} \frac{x^r \, dt}{(1 - \alpha_1 x)(1 - \alpha_2 x)} = B_1^{(r)} M_1 + B_2^{(r)} M_2 + d_0^{(r)} + d_1^{(r)} h + \ldots + d_{k-1}^{(r)} h^{k-1},
\]

(11)

where \( \alpha_1, \alpha_2 \) are complex or real numbers and

\[
B_1^{(r)} = \frac{1}{(\alpha_1 - \alpha_2) \alpha_1^{r-1}}, \quad B_2^{(r)} = \frac{1}{(\alpha_2 - \alpha_1) \alpha_2^{r-1}},
\]

\[
d_j^{(r)} = \frac{\alpha_{j-2}^{r-1} - \alpha_2^{r-1} - \alpha_{j-2}^{r-1} K_{2j} \alpha_j}{(\alpha_1 \alpha_2)^{-1} (\alpha_1 - \alpha_2)} K_{2j}, \quad j = 0, \ldots, k - 1
\]

\[
M_j = \int_0^{2\pi} \frac{dt}{1 - \alpha_j x} = \frac{2\pi}{\sqrt{1 - \alpha_j^2 h}}, \quad j = 1, 2.
\]

Now by setting \( \alpha_1 = -\alpha_2 = \frac{i}{\sqrt{b^2 - h}} \) in (11), the conclusions of lemma are immediate. \( \square \)

Hereafter we denote any polynomial in \( \mathbb{E}[x] \) of degree \( j \) by \( p_j(x) \) or \( P_j(h) \) although its coefficients may vary from one expression to another. If it is necessary, we will also use indistinctly \( Q_j(x) \).

**Remark 2.6.** It follows from Lemma 2.5 that for \( r = 2k \), \( k > 1 \) we have

\[
\int_0^{2\pi} \frac{X^r}{Y^2 - 1} \, dt = -\frac{b^2}{a^r (b^2 - h)} [2A^{(r)} M_1 + P_2^{(r)}(h)] = -\frac{b^2}{a^r} \left[ \frac{2A^{(r)} M_1}{(b^2 - h)} + P_2^{(r)}(h) \right].
\]

(12)

**Corollary 2.7.**

\[
P_0 = \frac{2\pi a^2 b^2}{h - (a^2 + b^2)} \left[ \frac{1}{b \sqrt{b^2 - h}} + \frac{1}{a \sqrt{a^2 - h}} \right],
\]

(13)

\[
P_k = -\frac{2\pi b}{\sqrt{b^2 - h}} + \sum_{j=1}^{k-1} \frac{-b^2}{a^2 j} \left[ \frac{2A^{(2j)} M_1}{(b^2 - h)} + P_{j-1}(h) \right].
\]

(14)

**Proof.** Results can be followed by Lemmas 2.4 and 2.5 and Remark 2.6. \( \square \)

**Lemma 2.8.**

\[
l_{2k-1,0} = 0, \quad k > 1,
\]

(15)

\[
l_{2k,0} = \frac{a^{2k}}{a^2 b^2} \left[ \sum_{j=1}^{k-1} -\frac{b^2}{a^2 j} \left[ \frac{2A^{(2j)} M_1}{(b^2 - h)} + P_{j-1}(h) \right] - \frac{2\pi b}{\sqrt{b^2 - h}} \right] + \frac{-2\pi a^{2k}}{h - (a^2 + b^2)} \left[ \frac{1}{a \sqrt{a^2 - h}} + \frac{1}{b \sqrt{b^2 - h}} \right].
\]

(16)

Also \( l_{2k,0} = 0 \) at \( h = 0 \) and \( l_{2k,0} = \frac{1}{a^2 b^2} K_{2k} h^k + O(h^{k+1}) \) for \( k > 1 \), where \( K_{2k} \) is as in (10).

**Proof.** By Lemma 2.4, formulas (4) and (13) when \( i \) is an odd number greater than 1, we have \( l_{i,0} = \frac{d}{a^2 b^2} \left[ a^2 b^2 l_{i,0} + P_0 \right] = 0 \). Formula (16) can be obtained in a similar way. Also it is easy to see that \( l_{2k}(0) = 0 \). Moreover from (1) and (2) we have

\[
l_{2k,0} = h^k \int_0^{2\pi} \frac{(\sin^2 t) \, dt}{(h \sin^2 t - a^2)(h \cos^2 t - b^2)} = h^k F(h).
\]

Therefore \( l_{2k,0} = h^k \left[ F(0) + F'(0)h + \ldots \right] = \frac{1}{a^2 b^2} K_{2k} h^k + O(h^{k+1}) \). \( \square \)

**Lemma 2.9.** For \( k > 0 \) we have

\[
\sum_{i+j=2k} \Phi_{i,j} = \tilde{b}_{2k,k} l_{1,0} h^k + \ldots + \tilde{b}_{2k,1} l_{2k-1,0} h + \tilde{b}_{2k,0} l_{2k+1,0},
\]

(17)

\[
\sum_{i+j=2k-1} \Phi_{i,j} = \tilde{b}_{2k-1,k} l_{0,0} h^k + \ldots + \tilde{b}_{2k-1,1} l_{2k-2,0} h + \tilde{b}_{2k-1,0} l_{2k,0}.
\]

(18)
where $\tilde{b}_{2k,i} = \sum_{i=1}^{k} (-1)^{j-i}C_i^j \tilde{a}_{2k,j}$, $\tilde{b}_{2k-1,i} = \sum_{i=1}^{k} (-1)^{j-i}C_i^j \tilde{a}_{2k-1,j}$ for $0 \leq i \leq k$, and

$$\tilde{a}_{2k} = \begin{cases} a_{2k-2j} + b_{2k-2j+1,2j-1}, & j \neq 0, \sum_{j=0}^{k} \tilde{a}_{2k-1,j} = \begin{cases} a_{2k-2j+1,2j-1}, & j \neq 0, k \\ a_{2k-1,0}, & j = 0 \\ b_{0,2k-1}, & j = k. \end{cases}$$

**Proof.** By (2) and Lemma 2.2, we have

$$\sum_{i+j=2k} \Phi_{i,j} = \sum_{i=1}^{k-1} (\Phi_{2k-1-2i+2j-1} + \Phi_{2k-1-2i+1,2j-1}) + \Phi_{2k-1,0} + \Phi_{0,2k-1} = \sum_{i=0}^{k} \tilde{a}_{2k-1,i} \tilde{b}_{2k-2i,2i}.$$

By formula (6) the above relation becomes

$$\sum_{i+j=2k} \Phi_{i,j} = \tilde{a}_{2k-1,i} \tilde{c}_i^j I_{0,0} h^k + (\tilde{a}_{2k-1,k-1} \tilde{c}_k^0 - \tilde{a}_{2k-1,k} \tilde{c}_k^i) I_{2,0} h^{k-1}$$

$$+ \cdots + (\tilde{a}_{2k-1,0} \tilde{c}_0^0 - \tilde{a}_{2k-1,1} \tilde{c}_1^i) I_{2,0} h^{k-1}$$

But $\tilde{b}_{2k-1,i} = \sum_{j=1}^{k} (-1)^{j-i}C_i^j \tilde{a}_{2k-1,j}$. Hence (18) follows. (17) can be proved in a similar way. \( \square \)

**Remark 2.10.** By Lemmas 2.8 and 2.9 we have

$$\sum_{i+j=2k} \Phi_{i,j} = 0,$$

$$\sum_{i+j=2k} \Phi_{i,j} = \frac{h^k}{a^2 b^2} \left[ \tilde{b}_{2k-1,4} K_0 + \cdots + \tilde{b}_{2k-1,1} K_{2k-2} + \tilde{b}_{2k-1,0} K_{2k} \right] + O(h^{k+1}). \tag{19}$$

3. Proof of main Theorems

In this section using the results of Section 2, we prove Theorems 1.1 and 1.2.

3.1. Proof of Theorem 1.1

**Proof.** Without loss of generality assume that $a^2 \leq b^2$. Let $n = 2s$, in this case we have (neglecting the minus sign)

$$\Phi(h) = \int_0^{2\pi} \frac{1}{(x^2 - a^2)(y^2 - b^2)} \sum_{0 \leq i \leq j \leq 2s} (a_i x^{i+1} y^j + b_j x^i y^{j+1}) dt$$

$$= \sum_{i+j=2k} \Phi_{i,j} = \sum_{k=1}^{s} \left( \sum_{i+j=2k} \Phi_{i,j} + \sum_{i+j=2k} \Phi_{i,j} \right) + \Phi_{0,0}. \tag{20}$$

By (19) and the fact that $\Phi_{0,0} = 0$ we conclude that $\Phi(h) = \sum_{k=1}^{s} \sum_{i+j=2k} \Phi_{i,j}$. Now using formulas (18), (4), (5) and (16) we get

$$\sum_{i+j=2k} \Phi_{i,j} = \frac{1}{h - (a^2 + b^2)} \left[ \frac{1}{\sqrt{a^2 - h}} P_1(h) + \frac{1}{\sqrt{b^2 - h}} Q_k(h) \right]$$

$$+ \frac{1}{\sqrt{b^2 - h}} P_{k-2}(h) + \frac{1}{h - (a^2 + b^2)} Q_{k-2}(h). \tag{21}$$

Here we used \( \frac{2A(2k)}{(b^2-h)} = \frac{2\pi}{b^3} (b^2-h)^{-1} \sqrt{b^2-h} \), where

$$P_k(h) = -2\pi \left[ \frac{\tilde{a}_{2k-1,1} h^k}{a} + a \left( \sum_{j=0}^{k} (-1)^{j-k+1} C_{j}^{k-1} \tilde{a}_{2k-1,j} \right) h^{k-1} + a^3 \left( \sum_{j=0}^{k} (-1)^{j-k+2} C_{j}^{k-2} \tilde{a}_{2k-1,j} \right) h^{k-2} + \cdots \right]$$

$$+ a^{2k-3} \left( \sum_{j=1}^{k} (-1)^{j-1} C_{j}^{1} \tilde{a}_{2k-1,j} \right) h + a^{2k-1} \sum_{j=0}^{k} (-1)^{j} b_c^2 \tilde{a}_{2k-1,j}. \right]$$
\[ Q_k(h) = \frac{-2\pi}{b} \left[ (1 - k)\tilde{a}_{2k-1,k} + \tilde{a}_{2k-1,k-1} \right] h^k - b^2 \left( \sum_{j=k-1}^{k} (1)^{j-k+1} c_{j}^{k-1} \tilde{a}_{2k-1,j} \right) h^{k-1} \\
+ a^4 \left( \sum_{j=k-2}^{k} (-1)^{j-k+2} c_{j}^{k-2} \tilde{a}_{2k-1,j} \right) h^{k-2} + \ldots \\
+ a^{2^{k-2}} \left( \sum_{j=1}^{k} (-1)^{j-1} c_{j}^{1} \tilde{a}_{2k-1,j} \right) h + a^{2^{k}} \sum_{j=0}^{k} (-1)^{j} c_{j}^{0} \tilde{a}_{2k-1,j} \right] . \]

\[ P_{k-2}(h) = \frac{-2\pi}{a^2 b} \left[ a^4 \left( \sum_{j=k-2}^{k} (-1)^{j-k+2} c_{j}^{k-2} \tilde{a}_{2k-1,j} \right) h^{k-2} + \ldots \\
+ a^{2^{k-2}} \left( \sum_{j=1}^{k} (-1)^{j-1} c_{j}^{1} \tilde{a}_{2k-1,j} \right) h + a^{2^{k}} \sum_{j=0}^{k} (-1)^{j} c_{j}^{0} \tilde{a}_{2k-1,j} \right] . \]

\[ Q_{k-2}(h) = \frac{-2\pi}{a^2 b} \left[ - a^2 \left( \sum_{j=k-2}^{k} (-1)^{j-k+2} c_{j}^{k-2} \tilde{a}_{2k-1,j} \right) h^{k-2} + \ldots + a^{2^{k-2}} \left( \sum_{j=1}^{k} (-1)^{j-1} c_{j}^{1} \tilde{a}_{2k-1,j} \right) \right] \times \left( \sum_{j=1}^{k} \left( \frac{b^2 - h}{(1a)^j} \right) h + a^{2^{k}} \sum_{j=0}^{k} (-1)^{j} c_{j}^{0} \tilde{a}_{2k-1,j} \right) \sum_{j=1}^{k} \left( \frac{b^2 - h}{(1a)^j} \right) \]

\[ Q'_{k-2}(h) = \frac{-1}{a^2} \left[ a^2 p_0(h) \left( \sum_{j=k-2}^{k} (-1)^{j-k+2} c_{j}^{k-2} \tilde{a}_{2k-1,j} \right) h^{k-2} + \ldots \right. \\
+ a^{2^{k-2}} \left. \left( \sum_{j=1}^{k} (-1)^{j-1} c_{j}^{1} \tilde{a}_{2k-1,j} \right) \left( \sum_{j=1}^{k-1} p_{j-1}(h) a^{2j} \right) h + a^{2^{k}} \sum_{j=0}^{k} (-1)^{j} c_{j}^{0} \tilde{a}_{2k-1,j} \right) \sum_{j=1}^{k-1} \left( \frac{q_{j-1}(h)}{(1a)^{2j}} \right) \right] . \]

where \( \tilde{a}_{2k-1,j} \) is given in Lemma 2.9 and \( P_k(h), Q_k(h), P_{k-2}(h), Q_{k-2}(h), Q'_{k-2}(h) \) are real polynomials of \( h \). Notice that each of them has a different set of coefficients. So the coefficients of the functions \( \frac{h^k}{(h-(a^2+b^2))\sqrt{a^2-h}}, \frac{h^k}{(h-(a^2+b^2))\sqrt{b^2-h}}, \frac{h^k}{\sqrt{a^2-h}}, \frac{h^k}{\sqrt{b^2-h}} \)

\( h^k \sqrt{b^2-h} \) and \( h^k \) are independent. From this we deduce that \( P_k(h), Q_k(h), P_{k-2}(h), Q_{k-2}(h), Q'_{k-2}(h) \) can be taken as polynomials with arbitrary coefficients. Hence we get the following formula for the Abelian integral \( \tilde{\Phi}(h) \)

\[ \Phi(h) = \sum_{i=1}^{\infty} \sum_{j=2k-1}^{\infty} \Phi_{i,j} = \frac{1}{h - (a^2 + b^2)} \left[ \frac{1}{\sqrt{a^2-h}} P_k(h) + \frac{1}{\sqrt{b^2-h}} Q_k(h) \right] \\
+ \frac{1}{\sqrt{b^2-h}} P_{k-2}(h) + \sqrt{b^2 - h} Q_{k-2}(h) + Q'_{k-2}(h) . \] (22)

Where \( P_k(h), Q_k(h), P_{k-2}(h), Q_{k-2}(h), Q'_{k-2}(h) \) are new polynomials with arbitrary coefficients, as above. Let \( z = h \), from (22) we have \( ((z - (a^2 + b^2))\sqrt{a^2-z}\sqrt{b^2-z}) \tilde{\Phi}(z) = \psi(z) \), where

\[ \psi(z) = \sqrt{b^2 - z} P_k(z) + \sqrt{a^2 - z} Q_k(z) + \sqrt{a^2 - z}\sqrt{b^2 - z} P_{k-2}(z) , \] (23)

in which \( P_k, Q_k, P_{k-2} \) are new, but with the same qualities as before. Now we can say:

**Lemma 3.1.*** The following set of \( 3s + 2 \) linearly independent functions

\[ \{ z^i \sqrt{a^2 - z}, z^i \sqrt{b^2 - z}, 0 \leq i \leq s \} \cup \{ z^i \sqrt{a^2 - z}\sqrt{b^2 - z}, 0 \leq i \leq s - 1 \} \] (24)

is a basis of the linear space \( \{ \psi : \psi \text{ given by (23)} \} \). Also there exists a function \( \psi \) of the form (23) such that \( \psi \) has at least \( 3s + 1 \) simple zeros for \( z \in (0, a^2) \).

**Proof.*** In order to prove that (24) is a basis, since they are linearly independent, it is sufficient to show that \( \psi \) is a linear combination with arbitrary coefficient of (24) whereby (23) and the properties of \( P_k, Q_k, P_{k-2} \), mentioned above, we get this. Now applying Lemma 4 in [10] with \( U = (0, a^2) \), the second part of lemma can be proved. \( \square \)

Let \( G = G_{\rho, \varepsilon} \) be a simple connected region with \( \partial G = C \), where \( C = C_{\rho, \varepsilon} := C_r \cup L_\rho(\rho, \varepsilon) \cup C_{\rho} \; ; \; C_r := \{ |z - a^2| = \varepsilon \ll 1 \} \; ; \; C_\rho := \{ |z| = \rho \gg 1 \} \; ; \; \text{and} \; L_\pm(\rho, \varepsilon) := L_{\pm} \cap |\varepsilon \leq |z| \leq \rho \} \), where \( L_{\pm} \) are the upper and lower banks of the cut \( \{ z \in \mathbb{R}, z \geq a^2 \} \), respectively; see Fig. 1.

We use the notation \( \sharp \{ z \in D | f(z) = 0 \} \) to indicate the number of zeros of the function \( f \) in the set \( D \) taking into account their multiplicities, and \( \sharp \{ z \in \partial D | f(z) \} \) to indicate the number of the complete turns of the vector \( f(z) \) around the path \( \partial D \) in the counterclockwise direction.
One of the standard tools to give an upper bound for the number of zeros of \( \psi(z) \) is to extend this function to a suitable subset of the complex plane, and afterwards applying the Argument Principle to the extended function. To best of our knowledge Petrov [11–13] was the first to use this method which was then adopted by others (e.g. [10, 14]). In this case, from (23) it is easy to see that the zeros of the function \( \psi \) are among the zeros of some polynomial of degree \( 4s + 2 \). Therefore there will be a disk large enough so that it includes all zeros of \( \psi \). Therefore we may apply the Argument Principle to \( G = G_{\rho, \varepsilon} \) for \( \rho \) and \( 1/\varepsilon \) positive and large enough.

First we have \( \Im[z \in C_{\varepsilon} | \psi(z) | \leq v(\varepsilon) \), where \( v(\varepsilon) \) tends to zero as \( \varepsilon \to 0 \). Noting (23), we obtain \( \Im[z \in C_{\rho} | \psi(z) | \leq s + \frac{1}{2} + \mu(\rho) \), where \( \mu(\rho) \) tends to zero as \( \rho \to +\infty \), since \( \psi(z) = O(|z|^{s+\frac{1}{2}}) \) as \( z \to +\infty \). In \( L_{\pm} \), we have for \( z \in (a^2, b^2) \) and \( z \in (b^2, \infty) \), respectively,

\[
\begin{align*}
\psi_1 &= \sqrt{b^2 - z p_0(z)} + i\sqrt{z - a^2 q_0(z)} + i\sqrt{z - a^2 b^2 - z p_{-1}(z)}, \\
\psi_2 &= i\sqrt{z - b^2 p_0(z)} + i\sqrt{z - a^2 q_0(z)} - \sqrt{z - a^2 b^2 - z b^2 p_{-1}(z)}.
\end{align*}
\]

Noting (25) and the number of zeros of \( \Re \psi_1 \) and \( \Re \psi_2 \) in \( (a^2, b^2) \) and \( (b^2, \rho) \), respectively, we have

\[
\Im[z \in L_{\pm} | \psi(z)] \leq \Im[z \in (a^2, b^2)] \Re \psi_1 + \Im[z \in (b^2, \rho)] \Re \psi_2 + 1 \leq 2s + 1 + \mu(\rho).
\]

Putting all the results together we obtain that \( \Im[z \in C_{\rho} | \psi(z) | = 0 \) \leq 3s + 2. Hence the number of zeros of \( \Phi(h) \) are not larger than \( 3s + 2 \). For the case of \( n = 2s - 1 \), similarly we can prove that the number of zeros of \( \Phi(h) \) are not larger than \( 3s + 2 \). On the other hand from Lemma 2.8 we have that \( \Phi(0) = 0 \). From Remark 2.10 and (20), we know that \( \Phi(h) = \sum_{k=1}^{\infty} \sum_{j=2k-1}^{\infty} \Phi_{ij} = h[p_{-1}(h) + O(h^3)] \). Now from Lemma 1.3 in [5] we know that there exists an \( \varepsilon_0 > 0 \) such that for \( 0 < |\varepsilon| < \varepsilon_0, c = (a_j, b_j) \) with \( |a_j| \leq k, |b_j| \leq k \), the system (1) has at most \( \frac{1}{2}(n + \sin^2 \frac{\pi}{2}) + 1 \) limit cycles. \( \square \)

### 3.2. Proof of Theorem 1.2

**Proof.** Suppose \( a = b \). First let \( n = 2s \).

(i) Set \( a = b \) in Eqs. (15)–(18) and denote the resulting \( I_{ij} \) and \( \Phi_{ij} \) by \( I_{ij}^{(2)}, \Phi_{ij}^{(2)} \) respectively. So for \( k > 1 \) we have

\[
I_{2k-1, 0}^{(2)} = 0, \quad I_{2k, 0}^{(2)} = \frac{d^2}{h^2} \sum_{j=1}^{k-1} \frac{-a^2}{a^2 j!} \left[ \frac{2B_{1}^{(2)} M_{1}^{(2)}}{(a^2 - h)} + p_{j-1}(h) \right] - \frac{2\pi a^2 h}{a^3 \sqrt{a^2 - h(h - 2a^2)}}.
\]

Now let \( \sqrt{a^2 - h} = l \), then \( h = a^2 - l^2 \). By this change of variable, (26) becomes

\[
I_{2k, 0}^{(2)} = \frac{a^2 l}{l(a^2 + l^2)} \left( \frac{d^2}{a^4} \sum_{j=1}^{k-1} \frac{-a^2}{a^2 j!} \left[ (-1)^j l^{2j-2} 2\pi a l + (-1)^{j-1} l^{j-1} 2\pi + \ast l^{j-2} (a^2 - l^2) \right] + \cdots + \ast l^{2j-1} (a^2 - l^2) \right) + \frac{2\pi a^2}{a^3 (l^2 + a^2)} (a^2 - l^2)
\]

\[
= \frac{a - l}{l(a^2 + l^2)} \left( \frac{a^2 l}{l(a^2 + l^2)} \sum_{j=1}^{k-1} \frac{-a^2}{a^2 j!} \left[ \frac{2\pi (-1)^{j+1}}{a} l^{j-1} + \ast l^{j-3} (a + l) \right] + \cdots + \ast l^{2j-1} (a + l)(a^2 - l^2) \right) + \frac{2\pi a^2}{a^3} (a + l) = \frac{a - l}{l(a^2 + l^2)} p_{j-1}(l).
\]
where $\ast$ denotes some constant, $[.]$ is the integer part function and $p_{l-1}(l)$ is a real coefficient polynomial of $l$ of degree $(i−1)$. (Notice that the coefficients of $p_{l-1}(l)$ are independent from $a_{ij}$ and $b_{ij}$.)

Similar to the case of $a \neq b$ we have $\sum_{i+j=2k} \phi_{ij}^{(2)} = 0$ and

$$
\sum_{i+j=2k-1} \phi_{ij}^{(2)} = \tilde{b}_{2k-1,k} l^{(2)}_{0,k} + \tilde{b}_{2k-1,k-1} l^{(2)}_{1,k-1} + \cdots + \tilde{b}_{2k-1,1} l^{(2)}_{2k-2} + \tilde{b}_{2k-1,0} l^{(2)}_{2k,0}
$$

$$
= \frac{a - l}{l(a^2 + l^2)} \left( c_{2k-1}^{(2k-1)} l^{2k-1} + \cdots + c_{k}^{(2k-1)} l + c_{k-1}^{(2k-1)} \right),
$$

where $c_{j}^{(2k-1)}$ are linear combinations of $(a_{ij}, b_{ij})$ with $i+j = 2k-1$. Therefore

$$
\phi(h) = \frac{a - l}{l(a^2 + l^2)} \left( \frac{1}{2} \sum_{j=1}^{2k-1} c_{j}^{(2k-1)} l^j \right) = \frac{a - l}{l(a^2 + l^2)} \sum_{j=0}^{2s-1} c_{j} l^j.
$$

(27)

Hence, from Lemma 1.3 in [5] we know that there exists an $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| < \varepsilon_0$, $c = (a_{ij}, b_{ij})$ with $|a_{ij}| \leq k$, $|b_{ij}| \leq k$, the system (1) has at most $n-1$ limit cycles.

(ii) From the proof of part (i) we know that the coefficients $c_{2s-1}$, ..., $c_1$, $c_0$ of (27) satisfy

$$
c_{2s-1} = L \left( c_{2s-1}^{(2s-1)} \right),
$$

$$
c_{2s-2} = L \left( c_{2s-2}^{(2s-1)} \right),
$$

$$
c_{2s-3} = L \left( c_{2s-3}^{(2s-3)} \right),
$$

$$
\vdots
$$

$$
C_1 = L \left( c_1^{(2s-3)}, \cdots, c_1^{(1)} \right),
$$

$$
C_0 = L \left( c_0^{(2s-3)}, \cdots, c_0^{(0)} \right),
$$

where $L(\ldots)$ denotes the linear function and $c^{(k)}$ is linear combinations of $a_{ij}, b_{ij}$ with $i+j = k$, $k = 1, 3, 5, \ldots, 2s-1$. From (28) we have that the linear map $K : (a_{ij}, b_{ij}) \rightarrow (c_{2s-1}, c_{2s-2}, \ldots, c_1, c_0)$ is surjective. That is, the coefficients $c_{2s-1}, \ldots, c_1, c_0$ of $p_{2s-1}(l)$ are independently varied. So there exist $(a_{ij}, b_{ij})$, such that $p_{2s-1}(l)$ has exact $2s-1$ simple zeros for $\varepsilon_0 > 0$ and $|a_{ij} - a_{ij}^0|, |b_{ij} - b_{ij}^0|$ small. Then, from Lemma 1.3 and Remark 1.1 of [5], we have proved that system (1) has precisely $n-1$ limit cycles. For the case $n = 2s-1$, the proof is similar. □

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References


