On Sampling Closed Planar Curves and Surfaces

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Abstract

In this paper we consider a problem of sampling a continuous planar curve. We show that in order to sample a curve, one needs to impose a priori assumptions about the type of parameterization (coordinate system), which would provide a functional form for otherwise just a set of points on the plane. The emphasis in this paper is on curves representable in polar coordinates, but the sampling techniques can be generalized to other coordinate systems. Samples of a curve must incorporate the information about both the curve and the coordinate system that was used to produce them. The curve is then obtained from the samples by first reconstructing the coordinate system and finally the curve itself.

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1 Introduction

Sampling theory has a long history of development. The classical sampling theorem, which provides the basis for reconstructing bandlimited 1D signals from discrete samples, was first proved by Cauchy, rediscovered by Whittaker and Kotelnikov, and finally applied to problems of communications by Shannon, which ultimately resulted in the "digital revolution". The theorem is remarkable in that it allows converting an analog signal satisfying certain class constraints into

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a discrete sequence of numbers without any loss of information. Its mathematical statement is as follows.

Theorem 1 *If a function* $f : \mathbb{R} \rightarrow \mathbb{R}$ *is bandlimited to* $\omega_{\max} < \frac{\pi}{T}$*, then it can be exactly reconstructed from its values at sampling points* $t_n = nT$*:*

$$f(x) = \sum_{n=-\infty}^{+\infty} f(nT)\,\mathrm{sinc}\left(\frac{x}{T} - n\right),\ x \in \mathbb{R}. \quad (1)$$

The formula (1) is nothing but a representation of a function with respect to a particular orthogonal basis. It was noted later that spaces other than that defined by basis of sinc-functions may be considered as models for "ideal" signals, i.e., those allowing perfect reconstruction from their discrete samples.

In this paper we consider the problem of sampling a closed planar curve or a 3D surface. Just like in the case of 1D signals, the ability to convert a continuous contour to a collection of discrete points is extremely desirable and yields a number of useful applications. If the informative content of an image is limited to a single shape, then sampling the boundary of that shape as opposed to the entire image may result in significant increase of compression efficiency.

On the other hand, when a shape is analyzed in the context of computer vision, then the importance of being able to find a discrete representation cannot be overemphasized. While there exists a number of methods dealing with continuous curves, the latter must often come from a very narrow class, e.g., quadrics. At the same time representing shapes as arrays of points is fundamentally simpler, and therefore a richer theory exists for such scenarios.

The main obstacle we have to tackle when sampling a curve is lack of its *a priori* given functional form. As will be discussed in detail later, what we normally observe is a set of planar points as opposed to a curve as a function. That creates a problem of finding a parameterization that could be run through a sampling algorithm. Since a type of parameterization needs to be defined *a priori*, certain constraints have to be imposed on curves to guarantee its existence. We will show that once a parameterization of the curve is obtained, a sampling theorem based on sampling theory of 1D signals can be applied. Finally we will demonstrate that the proposed techniques, as opposed to existing algorithms, such as Fourier descriptors and splines (see [1]) , can be generalized to the case of surfaces in 3D.

Our paper is organized as follows. Our focus is mainly concentrated on 2D curves. Section 2 provides a general background about planar curves. Its purpose is to fix the notation as well as provide the insight as to the fundamental differences between 1D signals and planar curves from the point of view of sampling theory. In Section 3 we formulate the problem we address in this paper, and describe in detail the solution that we propose, which includes class-constraints on curves, sampling and reconstruction algorithms and present examples of applying those algorithms to idea curves. Technical proofs are moved to Appendix A for the sake of clarity.

In practice, most if not all curves do not exactly satisfy any *a priori* given conditions. It follows then that no technique can provide a perfect sampling and reconstruction of realistic curves. In light of this, in Section 4 we propose an optimization technique which yields an optimal approximation to an original curve. We show that it is consistent with the sampling theorem for class-constrained curves in the sense that the latter becomes a particular case of the former.

Results obtained for 2D curves can easily be generalized for 3D surfaces. A brief summary of results for 3D is presented in Section 5. Section 6 concludes the paper by providing the final remarks.

2 Planar Curves

Curves arise in practice in many ways (see Figure 1). They normally represent a silhouette or a contour of some object in an image. From mathematical point of view, what we observe on a piece of paper or a computer screen is in fact an image of a curve. In general there exists no way to uniquely extract the functional form of a curve from its image, provided no *a priori* assumptions about the nature of the curve. Note that if we are given a graph of a 1D signal, then a very specific coordinate system, namely Cartesian, is attached to it even if not explicitly mentioned. We will give the following definitions, which are commonly used in multidimensional analysis and differential geometry (see [2]).

Definition 1 *A path (parameterized curve) is a continuous function* $\gamma : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$*, i.e.,*

$$\gamma(t) = \big(x(t), y(t)\big) \in \mathbb{R}^2,\ t \in [a,b],\ \gamma \in C\big([a,b]\big). \tag{2}$$

Here t*, which varies in* $[a,b]$*, is the parameter of a curve.*

Definition 2 *The subset* $\mathscr{C} \subset \mathbb{R}^2$ *defined by*

$$\mathscr{C} = Im(\gamma) \equiv \gamma\big([a,b]\big) \equiv \big\{\gamma(t) \mid t \in [a,b]\big\} \tag{3}$$

is called the image of a path $\gamma(\cdot)$.

The set $\mathscr{C}$ is what we really call a curve in everyday life. It is therefore intuitive that the set $\mathscr{C}$ should not depend on a choice of parameterization contrary to what is suggested by Definition 2. The following construction assures that $\mathscr{C}$ is in fact independent from a parameterization so long as the latter belongs to an admissible class of paths.

Definition 3 *Two paths* $\gamma_1(t) : [a,b] \to \mathbb{R}^2$ *and* $\gamma_2(\tau) : [\alpha,\beta] \to \mathbb{R}^2$ *are said to be equivalent (write* $\gamma_1 \sim \gamma_2$*), if there exists a change of variable* $t = t(\tau)$*, satisfying the conditions*

1. $t : [\alpha,\beta] \to [a,b]$ *is a bijection between the two intervals;*

2. t *and* t^{-1} *are continuously differentiable, i.e.,*

$$t \in C^1\big([\alpha,\beta]\big),\ t^{-1} \in C^1\big([a,b]\big);$$

3. *The two parameterizations have the same direction, i.e.,*

$$\frac{d}{d\tau}t(\tau) > 0,\ \forall \tau \in [\alpha,\beta],$$

such that

$$\gamma_1\big(t(\tau)\big) = \gamma_2(\tau),\ \forall \tau \in [\alpha,\beta]. \tag{4}$$

It easily follows from Definition 3 that equivalent paths have the same image.

Theorem 2

$$\Big(\gamma_1 \sim \gamma_2\Big) \Rightarrow \Big(Im(\gamma_1) = Im(\gamma_2)\Big). \tag{5}$$

Furthermore, the relationship $\sim$ from Definition 3 is an equivalence relationship, which yields that all continuous paths are divided into disjoint equivalence classes. Those classes are called planar curves.

Definition 4 *Let* $\gamma_0 : [a,b] \to \mathbb{R}^2$ *be a path. Then the equivalence class*

$$C = \{\gamma \mid \gamma \sim \gamma_0\} \tag{6}$$

is called a curve with the representative γ_0.

Theorem 2 asserts that the image of a curve is uniquely defined.

Definition 5 *Let C be a curve, and $\gamma_0 \in C$. Then by definition*

$$Im(C) \equiv Im(\gamma_0). \tag{7}$$

Definition 6 *A curve C is called closed if it contains at least one parameterization $\gamma_0 : [a, b] \to \mathbb{R}^2$, which is closed, i.e., $\gamma_0(a) = \gamma_0(b)$.*

One easily establishes that *all* elements of a closed curve are closed parameterizations.

Definition 7 *A closed curve C is called Jordan if it does not intersect itself except at the end points, i.e., if $\gamma_0 : [a, b] \to \mathbb{R}^2$ is its parameterization, and $t_1 < t_2$ then*

$$\Big(\gamma_0(t_1) = \gamma_0(t_2)\Big) \Rightarrow \Big(t_1 = a,\ t_2 = b\Big). \tag{8}$$

In the remainder of the paper, we will focus exclusively on Jordan curves.

A curve is defined as an object independent of a particular parameterization. Other geometric objects attributed to curves may be defined in a similar manner. A concept we find particularly useful is that of a unit tangent vector to a curve.

Definition 8 *Let $\gamma : [a, b] \to \mathbb{R}^2,\ \gamma \in C^1\big([a, b]\big),\ \gamma(t) = \big(x(t), y(t)\big),\ t \in [a, b]$ be a path and $t_0 \in [a, b]$. Then the unit tangent vector to the path at the point $\gamma(t_0)$ is a vector $\boldsymbol{\tau}(t_0)$ defined as*

$$\boldsymbol{\tau}(t_0) = \frac{\gamma'(t_0)}{\|\gamma'(t_0)\|} \in \mathbb{R}^2, \tag{9}$$

where

$$\gamma'(t_0) = \big(x'(t_0), y'(t_0)\big),\ \|\gamma'(t_0)\| = \sqrt{x'(t_0)^2 + y'(t_0)^2}. \tag{10}$$

As was noted earlier, it can be easily shown that the notion of a unit tangent vector to a curve is geometric, i.e., it does not depend on a particular parameterization of the curve. More precisely, the following theorem holds.

Theorem 3 *Let $\gamma_1(t) : [a, b] \to \mathbb{R}^2$ and $\gamma_2(\tau) : [\alpha, \beta] \to \mathbb{R}^2$ be two equivalent paths, i.e. $\gamma_1 \sim \gamma_2$, such that $t = t(\tau)$ is a change of variable satisfying all the conditions listed in Definition 3. Let $\boldsymbol{\tau}_1(t), \boldsymbol{\tau}_2(\tau)$ be tangent vectors to paths γ_1 and γ_2 at points $\gamma_1\big(t(\tau)\big)$ and $\gamma_2(\tau)$ respectively. (Note that $\gamma_1\big(t(\tau)\big) = \gamma_2(\tau)$.) Then*

$$\boldsymbol{\tau}_1\big(t(\tau)\big) = \boldsymbol{\tau}_2(\tau). \tag{11}$$

The proof of this theorem can be found in [2]. We will make use of this theorem in later sections of the paper.

3 Sampling Planar Curves

3.1 General Finite Sampling Problem

Here we give a mathematical description of a finite sampling problem which is, generally speaking, suitable for any signals including 1D function, images or curves on the plane. Our description and solution of the problem will not only be rigorous and unambiguous, but it will also help us contrast and compare the relevance of traditional sampling theory of 1D and 2D signals to the problem we have at hand.

To proceed with the formulation of the finite sampling problem for an arbitrary signal, we state the following definition.

Definition 9 *Let $f : X \to Y$ be a signal (function). Completely specifying the sampling problem of this signal entails imposing conditions on f, such that it is possible to find a triple $(N, \mathscr{X}, \mathscr{I})$, where $N \in \mathbb{N}$ is a number of samples, $\mathscr{X} \equiv \{x_1, \ldots, x_N\} \in X^N$ is a collection of N distinct samples, and $\mathscr{I}$ is an interpolation function (method of reconstruction): $\mathscr{I} : X^N \times Y^N \times X \to Y$, such that*

$$\mathscr{I}\big(x_1, f(x_1), \ldots, x_N, f(x_N); x\big) = f(x),\ \forall x \in X. \tag{12}$$

The reader may find the above definition too abstract. However, it is extremely important to formally specify the problem we are going to be addressing. This definition also reveals the minimal information needed in order to formulate a sampling theorem. The following well-known theorem is a classical example of a solution to a 1D finite sampling problem.

Theorem 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a 1D signal. Suppose also that it is periodic with a fundamental period 2π and bandlimited, i.e., there exists $N \in \mathbb{N}$ (for the sake of simplicity we will assume N even) and $c_{-\frac{N}{2}}, \ldots, c_{\frac{N}{2}} \in \mathbb{C}$, such that*

$$f(t) = \sum_{n=-N/2}^{N/2} c_n e^{itn},\ \forall t \in \mathbb{R}. \tag{13}$$

Then if $t_1, \ldots, t_{N+1} \in \mathbb{R}$ is a collection of $N + 1$ arbitrary but distinct points on the real line and $f(t_1), \ldots, f(t_{N+1})$ the corresponding sample values of $f(t)$, we can reconstruct the signal

$f(t)$, and the interpolation function, $\mathscr{I}$, is defined by the right side of Equation 13, where the coefficients $\{c_n\}_{n=-\frac{N}{2},\ldots,\frac{N}{2}}$ are the unique solution of the following system of equations.

$$\begin{pmatrix} f(t_1) \\ \vdots \\ f(t_{N+1}) \end{pmatrix} = \begin{pmatrix} e^{i(-\frac{N}{2})t_1} & \ldots & e^{i\frac{N}{2}t_1} \\ \vdots & \ddots & \vdots \\ e^{i(-\frac{N}{2})t_{N+1}} & \ldots & e^{i\frac{N}{2}t_{N+1}} \end{pmatrix} \cdot \begin{pmatrix} c_{-\frac{N}{2}} \\ \vdots \\ c_{\frac{N}{2}} \end{pmatrix}. \tag{14}$$

The reader may refer to [3] among others for the proof of this theorem. It easily follows from the fact that any $N+1$ samples yield a linear system of equations with respect to $N+1$ signal Fourier coefficients, which is full-rank and thus has a unique solution. The Fourier coefficients, in turn, completely define a signal $f(t)$ (see, for example, [4]).

Definition 10 *Let $f : X \to Y$ be a given signal. We will say that a point*

$$y \in Im(f) \tag{15}$$

is an incomplete sample of f on X.

Definition 11 *Let $f : X \to Y$ be a given signal. We will call a pair*

$$\big(x, f(x)\big) \in X \times Y, \tag{16}$$

a complete sample of f on X.

Note, that when given an incomplete sample of a signal, we have its value at an unknown location. In a statement of a finite sampling problem a reconstruction of a signal is usually from its complete samples. The following simple example demonstrates that the sample locations are critical.

Consider again a 1D bandlimited signal $f(t)$, satisfying all of the conditions from Theorem 4 with $N = 3$. Then if we are given any 3 complete samples, Theorem 4 guarantees that we will be able to uniquely reconstruct $f(t)$. On the other hand, if we were given incomplete samples, then by varying the locations of those samples we would be able to construct an infinite number of signals, satisfying all the required conditions, e.g., if

$$f(t_1) = 1,\ f(t_2) = 0,\ f(t_3) = -1,$$

then the locations

$$t_1 = 0,\ t_2 = \frac{\pi}{2},\ t_3 = \pi$$

yield $f(t) = \cos t$, but for

$$t_1 = \frac{\pi}{2},\ t_2 = \pi,\ t_3 = \frac{3\pi}{2},$$

the formula for the signal would become $f(t) = \sin t$ (See Figure 2).

This very simple example underlines the difficulty we encounter when formulating a sampling problem for planar curves as we discuss next.

3.2 Sampling Closed Curves: Formulation

Recall, that a closed planar curve C is a class of equivalent (closed) parameterizations of the same image $\mathscr{C}$. We formulate the sampling problem for the curve C as follows. Find a parameterization $\gamma \in C$, whose finite sampling problem is well defined - seek a function $\gamma : [a, b] \to \mathscr{C} \subset \mathbb{R}^2$, which satisfies some conditions so that a finite number of complete samples results yielding a reconstruction of $\gamma(\cdot)$.

To proceed, it is important to make the following observation. A complete sample for a curve has the form $\big(t; \gamma(t)\big) \equiv \big(t; x(t), y(t)\big)$. In practice, however, all we have is a set of point $\big\{(x_i, y_i)\big\}_{i=1}^N$, which do not constitute *complete* samples. In fact, a closer consideration reveals that these samples are not even *incomplete* as for arbitrary points on the plane $\big\{(x_i, y_i)\big\}_{i=1}^N$ there is no information about the parameterization at hand or its domain. The following two steps are in order to mitigate such a limitation.

- Make additional assumptions about the nature of a curve, namely about the *type* of coordinate system, this curve is represented in.
- Make the information of the coordinate system selection part of the sample specification so as to avoid any potential ambiguity presenting a closed curve and subsequentles.

3.3 Curves Representable in Polar Coordinates

As was argued earlier, a choice of a coordinate system is important for extracting some meaningful information from curve samples. This may be demonstrated by a 1D function, whose samples are as such in a cartesian coordinate system, and otherwise just a set of points in the plane. We glean the information projecting points onto the two axes. While this example may look trivial, the resulting procedure is instructive and indicative of the challenges we have to tackle.

To address the representation issues of closed (Jordan) curves, one natural reference system which surfaces is that of polar coordinates. Assuming that such a curve representation exists, its seemingly restrictive scope quickly pales next to its merit in suitability as well as adaptability to a wide array of settings. In the case of interest herein, we adopt a polar parameterization of a given curve C and next discuss the technical conditions which underly its existence.

Definition 12 *Suppose we have a curve C. If for a given point $\gamma_0 \equiv (x_0, y_0)$ there exists a function $r(\theta) : [0, 2\pi] \to \mathbb{R}^2$, such that the curve C can be parameterized according to*

$$\gamma(\theta) = \gamma_0 + \big(r(\theta)\cos(\theta), r(\theta)\sin(\theta)\big),\ \theta \in [0, 2\pi],\ \gamma(0) = \gamma(2\pi), \tag{17}$$

then we call γ_0 an admissible (polar) center of C.

While such a definition does not uniquely specify an admissible center of a curve, it allows us to address the existence issue, which in turn affords our constructing of curves whose class membership is easily carried out.

Given a curve C, and using the above definition, we may proceed to determine the admissibility of any point (x_0, y_0) by ensuring that every half-line originating at this point intersects C at one and only one point. To alleviate the computational load such a procedure may entail, we state the following theorem which forms the basis for a more efficient search of admissible points.

Theorem 5 *Let C be a smooth curve, $(x_0, y_0) \in \mathbb{R}^2$, and ℓ - half-line starting from (x_0, y_0) and intersecting the image of C, set $\mathscr{C}$, in at least two points. Then there exists another half-line ℓ^t that also starts from (x_0, y_0) and is tangent to C.*

The proof of this theorem can be found in the Appendix A.

It is important to note that the existence of a tangent half-line to a curve originating at a point (x_0, y_0) does not guarantee the non-admissibility of the latter as a center for the curve. It does, however, as further discussed below, rule all of those points out of the set from further consideration and hereby identify a so-called regular admissible set.

Definition 13 *Let C be a curve and $(x_0, y_0) \in \mathbb{R}^2$ be a point. Then this point is called a regular admissible center if there exists no half-line ℓ going from (x_0, y_0) and tangent to C. A collection of all regular admissible centers will be called the admissible set of the curve C and denoted $\mathscr{A}_C$.*

To simultaneously qualify the membership of any selected point (x_0, y_0) to $\mathscr{A}_C$ and avoid a computational explosion, the following describes an easily implementable procedure.

Theorem 6 *Let C be a curve, and $\gamma : [a,b] \to \mathscr{C}$ be its parameterization. For any $t \in [a,b]$ define $\boldsymbol{\tau}(t)$ - a unit tangent vector to γ at a point $\gamma(t)$, and $\gamma_s(t)$ - a vector connecting the points $\gamma_0 \equiv (x_0, y_0)$ and $\gamma(t) \equiv \big(x(t), y(t)\big)$, i.e.,*

$$\gamma_s(t) = \big(x(t) - x_0, y(t) - y_0\big),\ t \in [a,b], \tag{18}$$

where $\gamma(t) \equiv \big(x(t), y(t)\big)$ (see Figure 3). Then $\gamma_0 \in \mathscr{A}_C$ if and only if the orientation of the pair of vectors $\big\{\boldsymbol{\tau}(t), \gamma_s(t)\big\}$ is constant (non-zero) for all $t \in [a,b]$. The latter condition means

$$sgn\left(\begin{vmatrix} \gamma_1(t) & \gamma_2(t) \\ \tau_1(t) & \tau_2(t) \end{vmatrix}\right) = const \neq 0,\ \forall t \in [a,b], \tag{19}$$

where $\gamma(t) \equiv \big(\gamma_1(t), \gamma_2(t)\big)^T,\ \tau(t) \equiv \big(\tau_1(t), \tau_2(t)\big)^T$.

See Appendix A for the proof of this theorem. Appendix B contains more details regarding the numerical implementation of the test for admissibility based on Theorem 6.

In practice, a continuous curve is, of course, a polygon with a large number of vertices and ever smaller edges, and significantly simplifies the exhaustive search as called by Equation (19). In addition, as we show next, the structure of $\mathscr{A}_C$ may be used to advantage in avoiding to have to check the totality of points individually.

Theorem 7 *Let C be a curve and $\mathscr{A}_C$ be its admissible set. Then $\mathscr{A}_C$ is convex, i.e.,*

$$\Big(\gamma_1, \gamma_2 \in \mathscr{A}_C\Big) \Rightarrow \Big([\gamma_1, \gamma_2] \subset \mathscr{A}_C\Big), \tag{20}$$

where

$$[\gamma_1, \gamma_2] = \big\{\alpha\gamma_1 + (1-\alpha)\gamma_2 \mid \alpha \in [0,1]\big\}. \tag{21}$$

The proof is presented in the Appendix A. A fast and simple approximation of $\mathscr{A}_C$ may, for instance, be carried out by a Monte Carlo method and illustrated in Figure 4.

3.4 Sampling and Reconstructing Curves: Algorithms

Provided that a non-empty admissible set $\mathscr{A}_C$ exists, as spelled out above, our goal in this section is to describe practical algorithms enabling us to sample closed curve and reconstructing them. Specifically, when given a curve C, we have to, subject to any other additional constraints, identify and acquire its appropriate samples, which will also allow us to reconstruct it.

Suppose C is a curve and $\gamma_0 \in \mathscr{A}_C$. Then by definition there exists a polar parameterization $\{r(\theta),\ \theta \in [0, 2\pi]\}$ of the curve C. Since, $r(\theta)$ is a 1D signal, we may therefore apply Theorem 4 to obtain the following result.

Theorem 8 *Suppose that we have a curve C and $\gamma_0 \in \mathscr{A}_C$. Let $r(\theta)$ be a polar representation of C centered at γ_0. Then if $r(\theta)$ is bandlimited, and there exists $N \in \mathbb{N}$, such that*

$$r(\theta) = \sum_{n=-N/2}^{N/2} c_n e^{i\theta n}, \tag{22}$$

then $r(\theta)$ can be reconstructed from any $N + 1$ samples (See Figure 5).

The reconstruction formula follows from Theorem 4. A few remarks regarding the last result are in order.

- Note that this result is not a finite sampling theorem as it heavily relies on a given point γ_0. Assuming such knowledge is quite unnatural, since in practice we are given only samples of a curve, and thus it is impossible to use this theorem.

- Providing the polar center of a curve explicitly is often undesirable because of noise that is present in virtually any real-world system. Suppose, for example, that the curve is to be transmitted over a digital channel. It then would not greatly reduce the efficiency of transmission if we simply added γ_0 to transmitted samples. The downside of this approach is in that γ_0 becomes the bottle-neck of the system, since any error added to it, would significantly affect the reconstructed curve.

Our goal is now to come up with a sampling technique that would allow us to *extract* the polar coordinate system from samples in a reliable manner. Recall, that Theorem 4 states that a bandlimited signal can be reconstructed from any (possibly non-uniform) samples so long as their number matches the bandwidth of the signal. We show that it is possible to use this freedom to choose samples in such a way that they would contain the information about the coordinate system that was used to produce them.

We formulate the following proposition which will immediately provide us with a desirable sampling technique.

Proposition 1 *Let C be a curve and $\gamma_0 \in \mathscr{A}_C$. Without loss of generality we will assume that $\gamma_0 = (0,0)$. Otherwise we can always shift the curve. Suppose that $\{r(\theta),\ \theta \in [0,2\pi]\}$ is a corresponding polar parameterization of C. Then if $r(\theta)$ is bandlimited, so that (22) holds, then we can always choose sampling points $\{\theta_i\}_{i=1}^{N+1}$ in such a way that*

$$\frac{1}{N+1}\sum_{i=1}^{N+1}\big(r(\theta_i)\cos(\theta_i), r(\theta_i)\sin(\theta_i)\big) = (0,0). \tag{23}$$

The significance of the last statement lies in the implication of an existence of a set of sample points, whose orthocenter is at $(0,0)$, representing a parameterized curve so long as the latter is bandlimited. This is tautologous, as is shown below, to saying that a unique reconstruction is possible with no additional assumptions. Thus we have the following sampling theorem.

Proposition 2 *Let C be a curve. Assume $\mathscr{A}_C \neq \emptyset$, and $\gamma_0 \equiv (x_0, y_0) \in \mathscr{A}_C$. Suppose that $r(\theta)$ is a polar parameterization of C centered at γ_0. If $r(\theta)$ is bandlimited, so that (22) holds, then there exist distinct points $\theta_1, \ldots, \theta_{N+1} \in [0,2\pi]$, such that the curve C is uniquely defined by the points*

$$\big(r(\theta_i)\cos(\theta_i), r(\theta_i)\sin(\theta_i)\big) \in \mathbb{R}^2,\ i = 1,\ldots,N+1. \tag{24}$$

The points $\{\theta_i\}$ are chosen ac cording to Proposition 1, and the reconstruction procedure easily follows from Equation (23) and Theorem 4.

The above sampling procedure is sufficiently general to be applicable to a variety of curves. The sampling and unique reconstruction of a curve C is subject to the following conditions, which are, recall

- The admissible set from Definition 13 must be non-empty, i.e., $\mathscr{A}_C \neq \emptyset$.
- There must exist a point $\gamma_0 \in \mathscr{A}_C$, such that the corresponding polar parameterization of C is bandlimited.

The conditions we impose on a curve seem restrictive until we note that, unlike the 1D case, the proposed technique affords a reconstruction in spite of lack of complete samples and a fixed and unambiguous coordinate system.

The Figure 6 shows the result of applying the described technique to a curve, where we may also note that such an encoding of γ_0 is stable against random additive noise because of the smoothing that takes effect in the course of averaging in Equation (23).

4 General Applicability: Deviation From Model

As discussed in the beginning of this paper, perfect reconstruction is impossible for arbitrary signals in general, and curves in particular. Any reconstruction algorithm requires additional information about the nature of a signal. In many cases a signal is constrained to lie in a certain functional space for the sampling theorem to be valid. The classical sampling theorem, for example, requires that a signal belong to the space of bandlimited functions.

Although class constraining, the above conditions, which a curve must satisfy to be a good candidate for sampling and exact reconstruction, are justifiable in light of the rather limited number of degrees of freedom imposed on an otherwise infinite dimensional space where our functions of interest live.

In the present context and for all practical purposes, where sampling of shapes is our main interest, a good approximation of curves will be perhaps no less important than an exact reconstruction. This in turn yields a number of interesting questions, such as:

- How close is the approximation to the original curve?
- Can we minimize the error of approximation?

An important point to note is that in the case of a non-bandlimited curve the bandlimited reconstruction may suffer from significant errors, if the proposed above procedure is directly applied. Recall, that if a curve is to be sampled and uniquely reconstructed, its sampling points are chosen so that the corresponding orthocenter coincides with the polar center. If we denote the distance between the orthocenter of samples and the polar center by $d(\theta_1, \ldots, \theta_{N+1})$, then our condition may be written as

$$d(\theta_1, \ldots, \theta_{N+1}) = 0. \tag{25}$$

We claimed earlier that it is always possible to find $\{\theta_i\}_{i=1}^{N+1}$ for a bandlimited curve, such that (25) holds and where N matches the bandwidth. It is clearly seen that finite sampling and exact reconstruction are limited to bandlimited signals/curves. In the event that the curve is non-bandlimited, we may only seek to determine the samples, which will minimize $d(\theta_1, \dots, \theta_{N+1})$ while optimizing the distance between the curve and its reconstructed approximation. Note that the previously described sampling procedure may be viewed as a particular case of this technique, which specializes to that above for the corresponding class of curves and for which the approximation error is zero, and the two criteria optimization described next reduces to one.

4.1 Sampling By Optimized Approximation

Let us assume that we have a curve C, such that $\mathscr{A}_C \neq \emptyset$, and $\gamma_0 \in \mathscr{A}_C$. Let $\{r(\theta), \theta \in [0, 2\pi]\}$ be a polar parameterization, which is not necessarily bandlimited. Let $N \in \mathbb{N}$ be a fixed integer number (for the sake of simplicity even). Towards formulating our generalized sampling define for each $(N+1)$-tuple $\theta_1, \dots, \theta_{N+1} \in [0, 2\pi]$

$$d^2(\theta_1, \dots, \theta_{N+1}) = \left(\sum_{i=1}^{N+1} r(\theta_i) \cos(\theta_i) \right)^2 + \left(\sum_{i=1}^{N+1} r(\theta_i) \sin(\theta_i) \right)^2, \tag{26}$$

and

$$l^2(\theta_1, \dots, \theta_{N+1}) = \int_0^{2\pi} \left| r(\vartheta) - \hat{r}_{\theta_1, \dots, \theta_{N+1}}(\vartheta) \right|^2 \, d\vartheta, \tag{27}$$

where $\hat{r}_{\theta_1, \dots, \theta_{N+1}}(\vartheta)$ is the reconstruction achieved by the samples defined at $\theta_1, \dots, \theta_{N+1}$. In an ideal situation we would like both d^2 and l^2 to be zero. It is clear, however, that the two functions may achieve their minima at different points in an $(N+1)$-dimensional space. We search for a so-called Pareto solution of the two-criteria optimization problem, which may be viewed as a point, from which any deviation may not decrease in one component without increasing in the other. For more details refer to [5]. Define $F : [0, 2\pi]^{N+1} \to \mathbb{R}^2$ as follows

$$F(\theta_1, \dots, \theta_{N+1}) = \left(d^2(\theta_1, \dots, \theta_{N+1}), l^2(\theta_1, \dots, \theta_{N+1}) \right). \tag{28}$$

Definition 14 *A point $(\theta_1^0, \dots, \theta_{N+1}^0) \in \mathbb{R}^{N+1}$ is called the Pareto optimal point of F, if $\forall (\theta_1, \dots, \theta_{N+1}) \in \mathbb{R}^{N+1}$ we have*

$$\begin{cases} d^2(\theta_1, \dots, \theta_{N+1}) & \leq & d^2\left(\theta_1^0, \dots, \theta_{N+1}^0\right) \\ l^2(\theta_1, \dots, \theta_{N+1}) & \leq & l^2\left(\theta_1^0, \dots, \theta_{N+1}^0\right) \end{cases} \Rightarrow (\theta_1, \dots, \theta_{N+1}) = (\theta_1^0, \dots, \theta_{N+1}^0). \tag{29}$$

Consider the case when the function $r(\theta)$ is bandlimited, in the sense of Equation (22). There then exists according to Proposition 1 the corresponding number $N+1$ and $\theta_1^0, \ldots, \theta_{N+1}^0 \in [0, 2\pi]$, such that

$$\begin{cases} d^2\left(\theta_1^0, \ldots, \theta_{N+1}^0\right) & = & 0, \\ l^2\left(\theta_1^0, \ldots, \theta_{N+1}^0\right) & = & 0. \end{cases} \tag{30}$$

Because of the non-negativity of the functions d^2 and l^2, we conclude that the point $\left(\theta_1^0, \ldots, \theta_{N+1}^0\right)$ is Pareto optimal, which also shows that our sampling technique described in the previous sections is a particular case of the multi-criteria optimization.

4.2 Implementation and Results

Multi-criteria optimization is quite well known in the mathematical community as well as engineering design, finance and other sciences. Several methods of finding a solution to a multi-criteria optimization problem (Pareto optimal point) have been proposed. In this paper we present the results obtained by using the steepest-descent method, which is while generally slow but proven to achieve the global optimal point. For the sake of space we defer the description of this method to [6] and only present the simulation results.

Figure 7 contains a silhouette of Germany. One can see that a reasonable approximation can be achieved using only 15 samples. A similar example is presented in Figure 8, where a very good reconstruction of a human kidney is made only from 7 samples. Finally, the shape of the human brain can be accurately encoded into 19 samples (Figure 9).

5 Further Extension: Sampling of Surfaces

The results of the previous sections may be straightforwardly generalized to the case of surfaces in $\mathbb{R}^3$. Much like having a curve, when choosing a polar coordinate system makes it effectively a 1D signal, a surface can be made into an image, by fixing spherical coordinates. Given that our interest focuses on 2D curves, and that most of the results for surfaces are a simple generalization of the 2D case, we forego any details and rather only state the results.

Definition 15 *Suppose we have a surface S. If a point $\gamma_0 \equiv (x_0, y_0, z_0)$ is such that there exists a function $r(\theta, \phi) : [0, 2\pi]^2 \to \mathbb{R}^3$, such that the surface S can be parameterized according to*

$$\gamma(\theta, \phi) = \gamma_0 + \big(r(\theta, \phi)\cos(\theta)\cos(\phi), r(\theta, \phi)\cos(\theta)\sin(\phi), r(\theta, \phi)\sin(\theta)\big), \tag{31}$$

where

$$\phi, \theta \in [0, 2\pi],\ \gamma(0, \phi) = \gamma(2\pi, \phi),\ \gamma(\theta, 0) = \gamma(\theta, 2\pi), \tag{32}$$

then we call γ_0 *an admissible (spherical) center of* S*.*

Theorem 9 *Let* S *be a smooth surface,* $(x_0, y_0, z_0) \in \mathbb{R}^3$*, and* ℓ *- half-line starting from* (x_0, y_0, z_0) *and intersecting the image of* S*, set* $\mathscr{S}$*, in at least two points. Then there exists another half-line* ℓ^t *that also starts from* (x_0, y_0, z_0) *and is tangent to* S*.*

Definition 16 *Let* S *be a smooth surface and* $(x_0, y_0, z_0) \in \mathbb{R}^3$ *be a point. Then this point is called a regular admissible center if there exists no half-line* ℓ *going from* (x_0, y_0, z_0) *and tangent to* S*. A collection of all regular admissible centers will be called the admissible set of a surface* S *and denoted* $\mathscr{A}_S$*.*

Theorem 10 *Let* S *be a surface and* $\mathscr{A}_S$ *be its admissible set. Then* $\mathscr{A}_S$ *is convex, i.e.,*

$$\Big(\gamma_1, \gamma_2 \in \mathscr{A}_C\Big) \Rightarrow \Big([\gamma_1, \gamma_2] \subset \mathscr{A}_C\Big), \tag{33}$$

where

$$[\gamma_1, \gamma_2] = \big\{\alpha\gamma_1 + (1-\alpha)\gamma_2 \mid \alpha \in [0, 1]\big\}. \tag{34}$$

Theorem 11 *Suppose that we have a surface* S *and* $\gamma_0 \in \mathscr{A}_S$*. Let* $r(\theta, \phi)$ *be a spherical representation of* S *centered at* γ_0*. Then if* $r(\theta, \phi)$ *is bandlimited, so that* $\exists M, N \in \mathbb{N}$*, such that*

$$r(\theta, \phi) = \sum_{m=-M/2}^{M/2} \sum_{n=-N/2}^{N/2} c_{m,n} e^{i\theta n} \cdot e^{i\phi m}, \tag{35}$$

then $r(\theta, \phi)$ *can be reconstructed from any samples so long as their number is sufficient.*

Proposition 3 *Let* S *be a surface and* $\gamma_0 \in \mathscr{A}_S$*. Without loss of generality we will assume that* $\gamma_0 = (0, 0, 0)$*. Suppose that* $r(\theta, \phi)$*,* $(\theta, \phi) \in [0, 2\pi]^2$ *is a corresponding spherical parameterization of* S*. Then if* $r(\theta, \phi)$ *is bandlimited, then we can always choose sampling grid* $\big\{(\theta_i, \phi_j)\big\}$ *in such a way that*

$$\sum_{i,j} \big(r(\theta_i, \phi_j)\cos(\theta_i)\cos(\phi_j), r(\theta_i, \phi_j)\cos(\theta_i)\sin(\phi_j), r(\theta_i, \phi_j)\sin(\theta_i)\big) = (0, 0, 0). \tag{36}$$

Proposition 4 *Let S be a surface. Assume $\mathscr{A}_S \neq \emptyset$, and $\gamma_0 \equiv (x_0, y_0, z_0) \in \mathscr{A}_S$. Suppose that $r(\theta, \phi)$ is a spherical parameterization of S centered at γ_0. If $r(\theta, \phi)$ is bandlimited, then there exist distinct points (θ_i, ϕ_j) such that the surface S is uniquely defined by the corresponding samples of the surface.*

6 Conclusions

Sampling of curves and surfaces is as important for a number of applications as sampling of 1D or 2D signals. It may help to compress certain images more efficiently, than other techniques that work with an entire image. Sampling also helps to provide us with shape landmarks, which are an essential tool for shape analysis and computer vision.

It was shown that a direct attempt to sample a curve encounters the difficulty, which is not topical for a signal or an image, namely that of finding a functional representation of the curve. Making sure that a curve will always have a parameterization of a prescribed type requires imposing constraints on the nature of the curve. It was demonstrated that for the case of polar parameterization such constraints could be found and efficient class identification numerical procedures could be implemented.

It was further discussed that unlike the 1D case, samples of a curve are meaningless unless they carry information about a coordinate system in which the curve had been parameterized. We used the flexibility in choosing positions of sampling points when dealing with a 1D bandlimited signal to incorporate the information about the polar center into the samples of a curve. The resulting procedure yielded a technique allowing us to reconstruct a curve from its samples without any additional information.

The approach we took for the case of ideal curves could be extended to curves outside that class. We proposed an optimization technique for sampling and approximating a curve for its samples, of which the previous sampling procedure arose as a particular case. That technique proved to provide us with very efficient discrete representations of a number of realistic contours.

7 Acknowledgements

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A Proofs

A.1 Proof of Theorem 5

Let us suppose that there is no $\gamma(t)$, such that the half-line that starts at γ_0 and goes through $\gamma(t)$ is tangential to the curve C. By assumption there exist $t_1, t_2 \in [a, b]$, $t_2 > t_1$, and the half-line ℓ starting from γ_0, such that it intersects $\mathscr{C}$ at the points $\gamma(t_1)$ and $\gamma(t_2)$.

- We first prove that we can choose t_1 and t_2 so that no point $\gamma(t)$ for $t \in (t_1, t_2)$ lies on ℓ. Suppose it is not true, i.e., for any chosen t_1, t_2 there exists $t_3 \in (t_1, t_2)$ such that $r(t_3) \in \ell$. Denote

$$t_1^{(1)} = t_1,\ t_2^{(1)} = t_3, \tag{37}$$

and consider the points $t_1^{(1)}$ and $t_2^{(1)}$. We have

$$\gamma\left(t_1^{(1)}\right), \gamma\left(t_2^{(1)}\right) \in \ell. \tag{38}$$

Then by assumption there must exist $t_3^{(1)} \in \left(t_1^{(1)}, t_2^{(1)}\right)$, such that $\gamma\left(t_3^{(1)}\right) \in \ell$. Denote

$$t_1^{(2)} = t_1^{(1)},\ t_2^{(2)} = t_3^{(1)}. \tag{39}$$

Proceeding in the same fashion, we see that we can construct two sequences: $\left\{t_1^{(n)}\right\}_{n\in\mathbb{N}}$, which is constant and $\left\{t_2^{(n)}\right\}_{n\in\mathbb{N}}$, which is strictly decreasing and bounded and thus convergent, i.e.,

$$\exists \lim_{n\to+\infty} t_2^{(n)} = t_1. \tag{40}$$

Because of smoothness of the curve $\gamma(t)$ we conclude, that ℓ is tangent to $\gamma(t)$ at the point t_1. This leads us to a contradiction, because we assumed that it was impossible. Thus we have indeed proved that there exist two point t_1 and t_2, such that any point between $\gamma(t_1)$ and $\gamma(t_2)$ does not lie on ℓ.

- Let us now proceed as follows. Denote by $\phi(t)$ the angle between $\gamma(t) \equiv \gamma(t) - \gamma_0$ and $\tau(t)$. Suppose without loss of generality, that $\phi(t_1) \in (0, \pi)$. (It is clear that $\phi(t) \notin \{0, \pi\}$, and the case $\phi(t_1) \in (\pi, 2\pi)$ is absolutely analogous.) We will prove that at some point on $(t_1, t_2]$ the angle eventually becomes 0 or π, which means that the half-line connecting γ_0 and that point is tangential to C.

  Suppose, that $\exists \varepsilon > 0$ - arbitrarily small but fixed, such that $\forall t \in [t_1, t_2]$ $\phi(t) \in [\varepsilon, \pi - \varepsilon]$. It is clear then that the piece of the curve $\gamma(t), t \in [t_1, t_2]$ will completely lie in the sector. $\gamma(t_2) \in \ell$, and ℓ lies outside that sector. Since $\gamma(t)$ is continuous, we have a contradiction.

  The contradiction we have just come to means that $\forall n \in \mathbb{N}$ $\exists t_n \in [t_1, t_2]$ such that either $\pi - \frac{1}{2^n} < \phi(t_n) \leq \pi$ or $0 \leq \phi(t_n) < \frac{1}{2^n}$, which in turn means that there exists a subsequence $\{t_{n_k}\}$ such that $\{\phi(t_{n_k})\}$ converges to either 0 or π. Suppose, for example, that $\exists \{t_{n_k}\} \in [t_1, t_2]$, such that $\exists \lim\limits_{k\to+\infty} \phi(t_{n_k}) = 0$. Because $\{t_{n_k}\}$ is a bounded sequence, $\exists$ yet another subsequence $\left\{t_{n_{k_l}}\right\}$, such that $\exists \lim_{l\to+\infty} t_{n_{k_l}} = s$ and $\exists \lim_{l\to+\infty} \phi\left(t_{n_{k_l}}\right) = 0$. This proves that $\phi(s) = 0$, which means that the half-line going from γ_0 ro $\gamma(s)$ is tangential to C, which leads us to a contradiction with our assumptions again.

The assumption that there is no half-line connecting γ_0 and $\gamma(t)$ tangential to C is therefore wrong, which proves the theorem. □

A.2 Proof of Theorem 6

Suppose first that there is no $\gamma(t)$ that is tangent to C, where $t \in [a, b]$. Denote the orientation of a pair of vectors $\{\gamma(t), \boldsymbol{\tau}(t)\}$ by $\mathscr{O}(t)$, i.e.,

$$\mathscr{O}(t) = sign\left(\begin{vmatrix} \gamma_1(t) & \gamma_2(t) \\ \tau_1(t) & \tau_2(t) \end{vmatrix}\right). \tag{41}$$

- *Forward part*:
  - Consider the case $\mathscr{O}(a) = 0$. Then $\gamma(0)$ and $\boldsymbol{\tau}(0)$ are collinear which is in contradiction with the assumption we just made.
  - Suppose then that $\mathscr{O}(a) \neq 0$. Without loss of generality assume $\mathscr{O}(a) = 1$. If $\exists t_0 \in [a, b]$, such that $\mathscr{O}(t_0) = -1$, then $\exists s \in [a, t_0]$, such that $\mathscr{O}(s) = 0$. Indeed, the determinant involved in the definition of the orientation is a continuous function of t, because of the smoothness of $\gamma(t)$. In addition, it is positive at a and negative at t_0. Then by Wierstrass theorem, there exists an intermediate point $s \in [a, t_0]$, where the continuous function equals 0.
- *Converse part*: Suppose that $\mathscr{O}(t) \equiv const \neq 0$. This means that that the determinant is either always positive or always negative, which indicates that $\gamma(t)$ and $\boldsymbol{\tau}(t)$ are never collinear. This proves the entire theorem. □

A.3 Proof of Theorem 7

Let C be a curve with the image $\mathscr{C}$. Let $\mathscr{A}_C$ be its admissible set and $\boldsymbol{\rho}^1, \boldsymbol{\rho}^2 \in \mathscr{A}_C$. Let $\alpha \in (0, 1)$. We will show that the point $\boldsymbol{\rho} = \alpha\boldsymbol{\rho}^1 + (1-\alpha)\boldsymbol{\rho}^2$ belongs to $\mathscr{A}_C$. To proceed, per Theorem 6, we will have to prove the corresponding orientation function never changes its sign.

We have

$$\begin{vmatrix} \tau_1(t) & \tau_2(t) \\ \gamma_1(t) - \boldsymbol{\rho}_1 & \gamma_2(t) - \boldsymbol{\rho}_2 \end{vmatrix} = \begin{vmatrix} \tau_1(t) & \tau_2(t) \\ \gamma_1(t) - \alpha\boldsymbol{\rho}_1^1 - (1-\alpha)\boldsymbol{\rho}_1^2 & \gamma_2(t) - \alpha\boldsymbol{\rho}_2^1 - (1-\alpha)\boldsymbol{\rho}_2^2 \end{vmatrix} =$$

$$= \begin{vmatrix} \alpha\tau_1(t) + (1-\alpha)\tau_1(t) & \alpha\tau_2(t) + (1-\alpha)\tau_2(t) \\ \alpha\gamma_1(t) + (1-\alpha)\gamma_1(t) - \alpha\boldsymbol{\rho}_1^1 - (1-\alpha)\boldsymbol{\rho}_1^2 & \alpha\gamma_2(t) + (1-\alpha)\gamma_2(t) - \alpha\boldsymbol{\rho}_2^1 - (1-\alpha)\boldsymbol{\rho}_2^2 \end{vmatrix} =$$

$$= \alpha \begin{vmatrix} \tau_1(t) & \tau_2(t) \\ \gamma_1(t) - \boldsymbol{\rho}_1^1 & \gamma_2(t) - \boldsymbol{\rho}_2^1 \end{vmatrix} + (1-\alpha) \begin{vmatrix} \tau_1(t) & \tau_2(t) \\ \gamma_1(t) - \boldsymbol{\rho}_1^2 & \gamma_2(t) - \boldsymbol{\rho}_2^2 \end{vmatrix}.$$

Two options are to consider.

- Suppose $\mathscr{O}_{\boldsymbol{\rho}^1}(t) = \mathscr{O}_{\boldsymbol{\rho}^2}(t) = \mathscr{O}\ \forall t$. Then it follows from the above derivation that $\mathscr{O}_{\boldsymbol{\rho}}(t) = \mathscr{O}\ \forall t$. By assumptions, $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \in \mathscr{A}_C$, so that $\mathscr{O}$ is a constant not equal to zero. Which yields that $\boldsymbol{\rho} \in \mathscr{A}_C$.

- Alternatively, assume $\mathscr{O}_{\boldsymbol{\rho}^1}(t) \not\equiv \mathscr{O}_{\boldsymbol{\rho}^2}(t)$. Let us derive a contradiction with assumptions of the theorem. Without loss of generality we can assume that

$$\mathscr{O}_{\boldsymbol{\rho}^1}(t) = 1,\ \mathscr{O}_{\boldsymbol{\rho}^2}(t) = -1. \tag{42}$$

  But there exist at least two points t_1, t_2, at which the two orientations must coincide. Indeed, the line, that contains the points $\boldsymbol{\rho}^1, \boldsymbol{\rho}^2$ intersects $\gamma(t)$ in two points. Denote one of them $\gamma(t_1)$, and the other $\gamma(t_2)$. Then the vectors $\gamma(t_i) - \boldsymbol{\rho}^1$ and $\gamma(t_i) - \boldsymbol{\rho}^2$ are collinear for $i = 1, 2$, which yields that the orientations should be the same. Thus we get a contradiction with (42), which proves the theorem. □

B Details of Implementation

In this section, we discuss in details the implementation of theoretical results presented in the earlier sections of the paper. Very much like the vast majority of techniques presented in the continuous setting, the algorithms we have derived require the process of discretization.

In practice, we are given a binary image in the form

$$I(x, y) \in \{0, 1\},\ x = 1, \dots, M,\ y = 1, \dots, N. \tag{43}$$

An image of a closed curve is then defined implicitly by points where the pixel value is 1, i.e.

$$\mathscr{C} = \{(x, y) \mid I(x, y) = 1\}. \tag{44}$$

Obviously, $\mathscr{C}$ is then a finite set and thus can be enumerated as

$$\mathscr{C} = \big\{(x_1, y_1), \ldots, (x_P, y_P)\big\}. \tag{45}$$

B.1 Admissible Set

Let $\boldsymbol{r}_0 = (x_0, y_0),\ 1 \le x_0 \le M, 1 \le y_0 \le N$ be a given pixel in the image. Our goal is to determine if this point can serve as an admissible center of polar coordinates. Assume first that $\boldsymbol{r}_0$ is indeed an admissible polar center for $\mathscr{C}$. We will proceed as follows. Define a sequence

$$\mathscr{C}_{polar} = \big\{(\phi_1, r_1), \ldots, (\phi_P, r_P)\big\},\ \phi_i \in [0, 2\pi),\ r_i \ge 0,\ i = 1, \ldots, P, \tag{46}$$

where

$$\forall i = 1, \ldots, P,\ x_i = r_i \cos\phi_i,\ y_i = r_i \sin\phi_i. \tag{47}$$

Without loss of generality, we assume that

$$\phi_1 < \phi_2 < \cdots < \phi_{P-1} < \phi_P. \tag{48}$$

Otherwise we always can rearrange points. If for some distinct i and j, $\phi_i = \phi_j$, then we, of course, get a contradiction with the assumption that $\boldsymbol{r}_0$ is an admissible point, so in fact all inequalities in (48) are strict.

Note that any polar parameterization of the digital curve, if it exists, has the form

$$\big\{(\phi_1 + \Delta, r_1), \ldots, (\phi_P + \Delta, r_P)\big\},\ \Delta \in \mathbb{R}. \tag{49}$$

On the other hand, if (49) defines a polar parameterization for some $\Delta_0 \in \mathbb{R}$, then taking any other Δ yields a valid polar parameterization. We thus have shown that our construction provides us with a unique (up to an additive constant in angle) possible polar parameterization.

Recall, that we are still reasoning under the assumption that $\boldsymbol{r}_0 = (x_0, y_0)$ is an admissible polar center. Define vectors γ_i^s and τ_i by

$$\gamma_{i,1}^s = x_i - x_0,\ \gamma_{i,2}^s = y_i - y_0,\ i = 1, \ldots, P, \tag{50}$$

and

$$\tau_i = (x_i - x_0, y_i - y_0),\ i = 1, \ldots, P. \tag{51}$$

The vectors τ_i are discrete tangent vectors, and Theorem 6 assures then that the pairs $\left\{(\tau_i, \gamma_i^s)\right\}_{i=1,\ldots,P}$ should all have the same orientation.

Summarizing, the algorithm of determining if a particular point can be a polar center for a given curve works as follows.

1. Extract points belonging to the curve.
2. Convert them to a polar coordinate system centered at the point in question.
3. Convert the points' Cartesian coordinates to polar coordinates
4. Compute pairs of discrete vectors $\left\{(\gamma_i^s, \tau_i)\right\}$.
5. Accept the hypothesis that the point is admissible if the orientations of the pairs of vectors defined above does not change. Otherwise, conclude that the point is not in an admissible set.

Other algorithms used to produce numerical implementations are either described in the referenced literature or self-evident. We thus do not elaborate on them any further.

Figure 1: Synthetic silhouette of a B-2 plane.

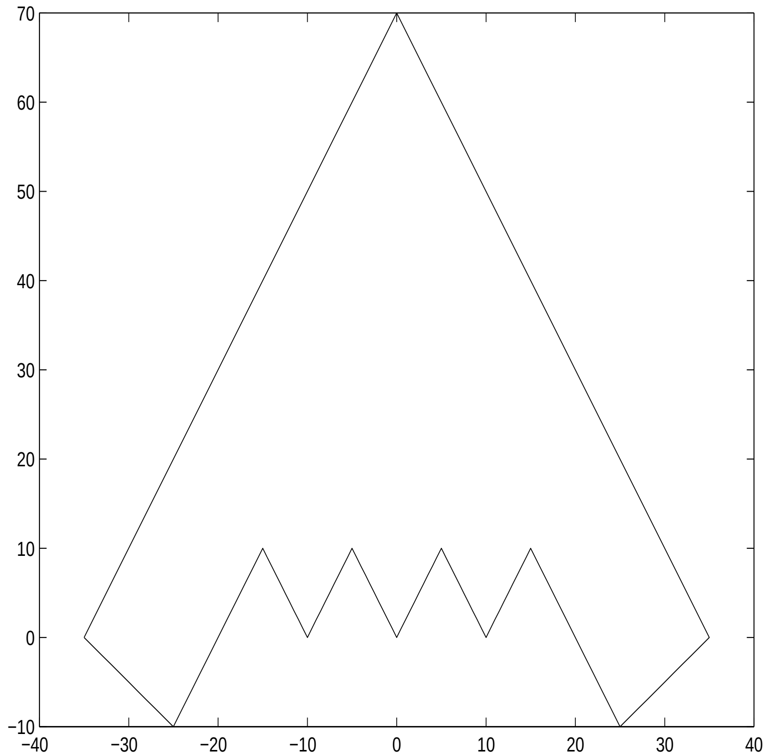

Figure 2: Samples with the same values but different locations correspond to different signals.

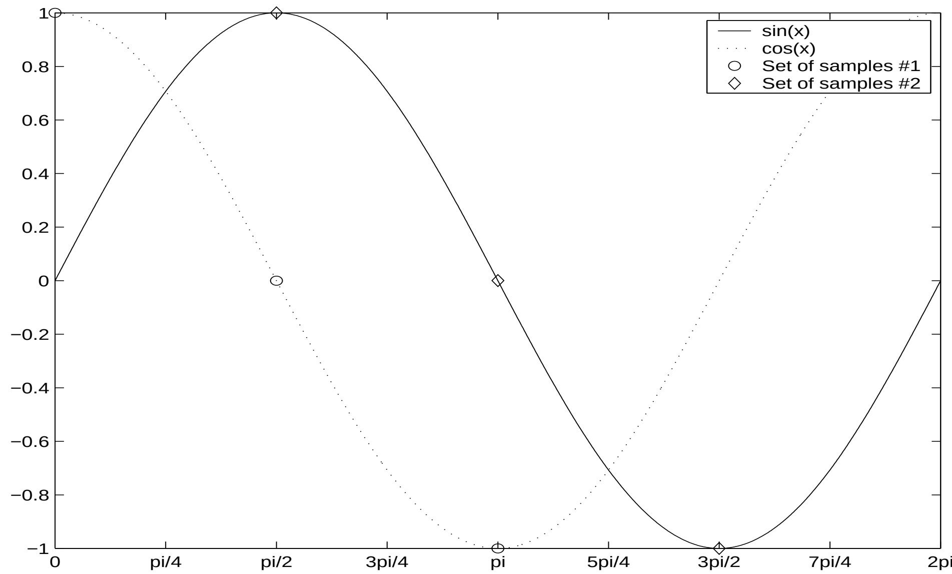

Figure 3: Orientation of a pair $\{\boldsymbol{\tau}(t), \gamma_s(t)\}$ remains the same for all $t \in [a, b]$.

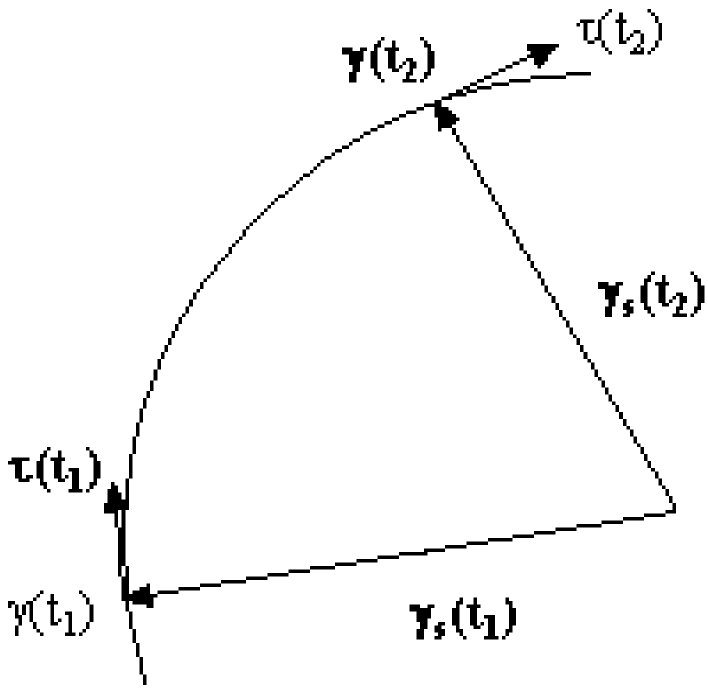

Figure 4: A curve and its admissible set.

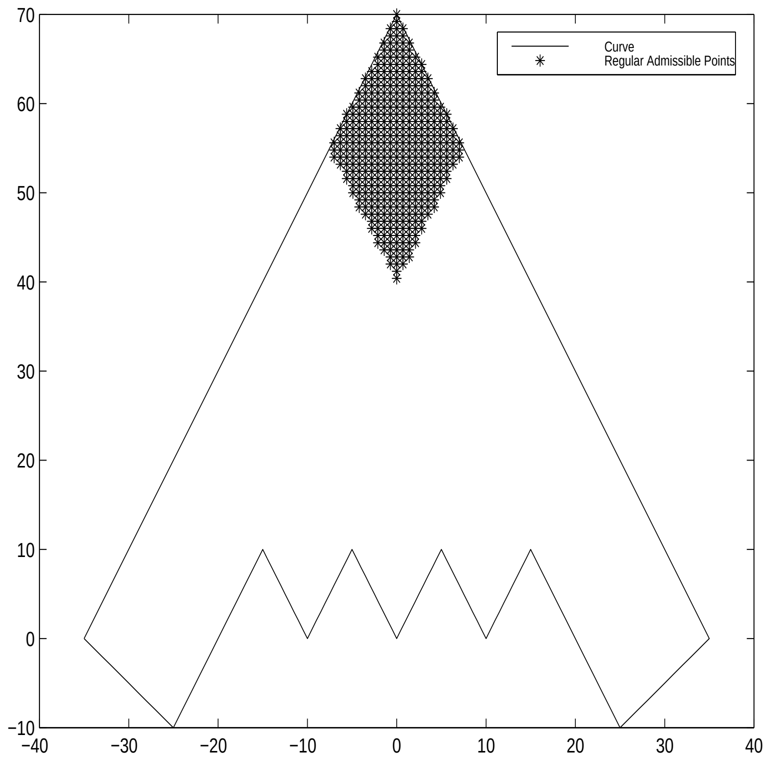

Figure 5: Given a polar center a curve can be reconstructed from any samples as long as their number is sufficient.

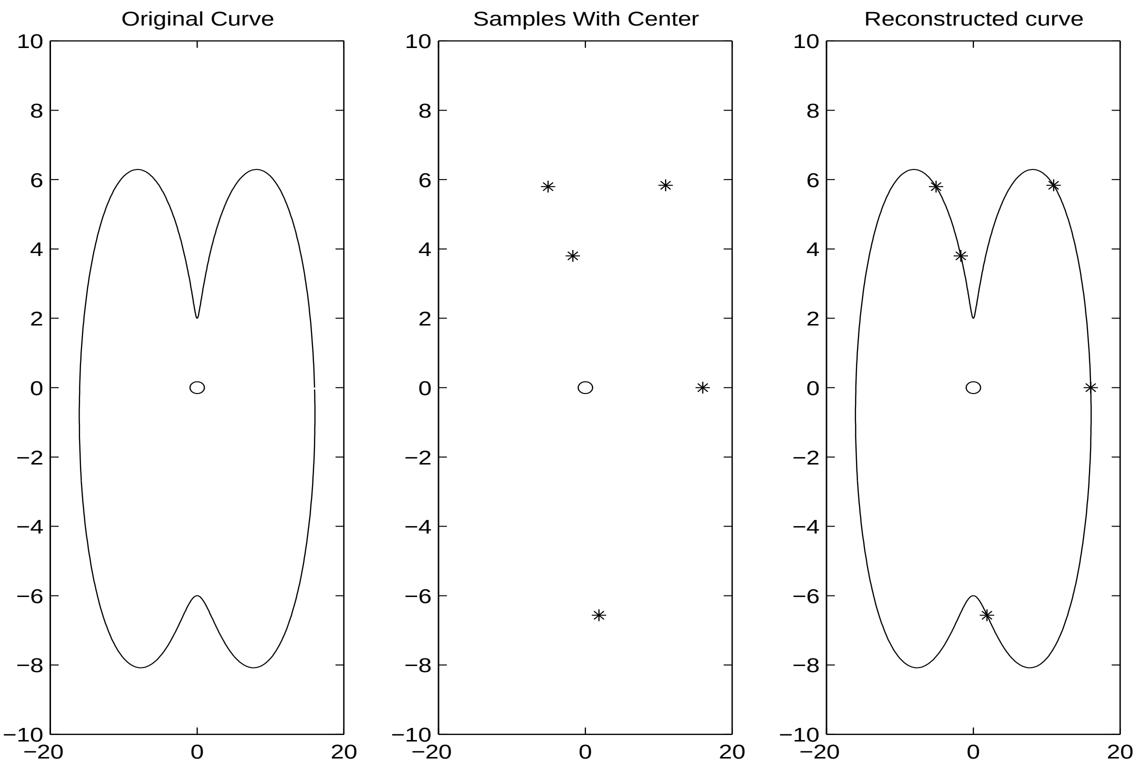

Figure 6: A curve is sampled and then perfectly reconstructed from the samples *without* using any additional information.

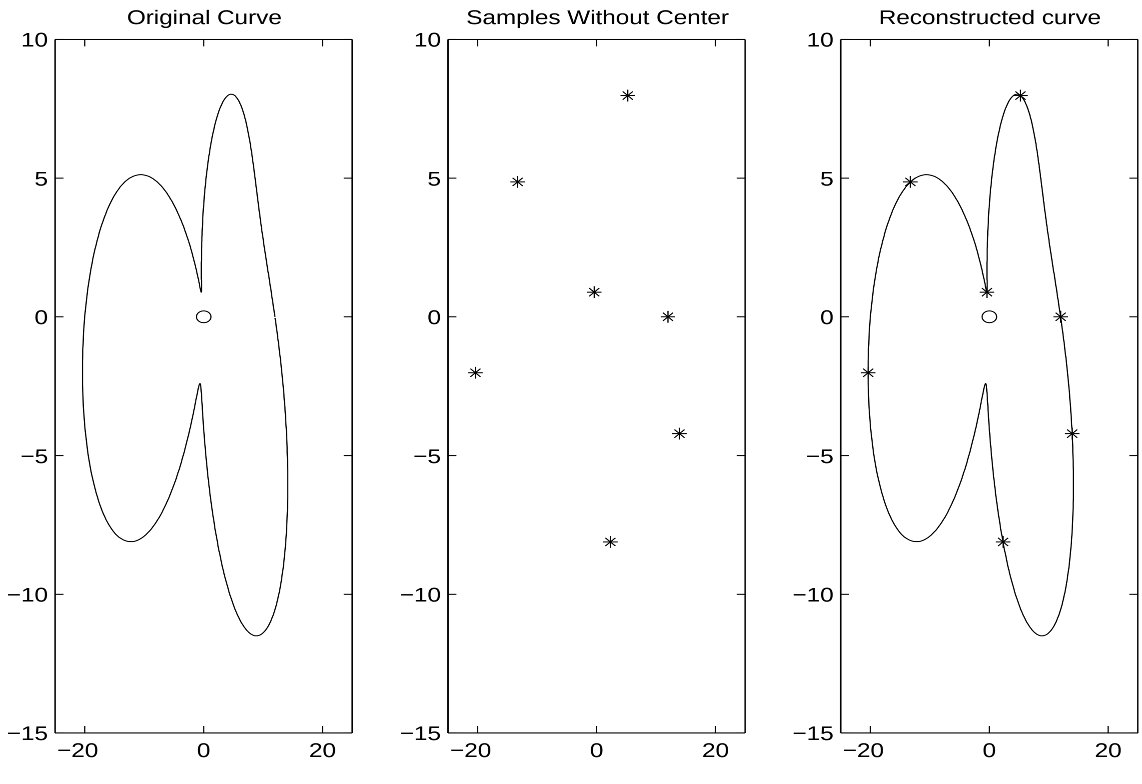

Figure 7: A good approximation of a map of Germany may be obtained from as few as 15 samples.

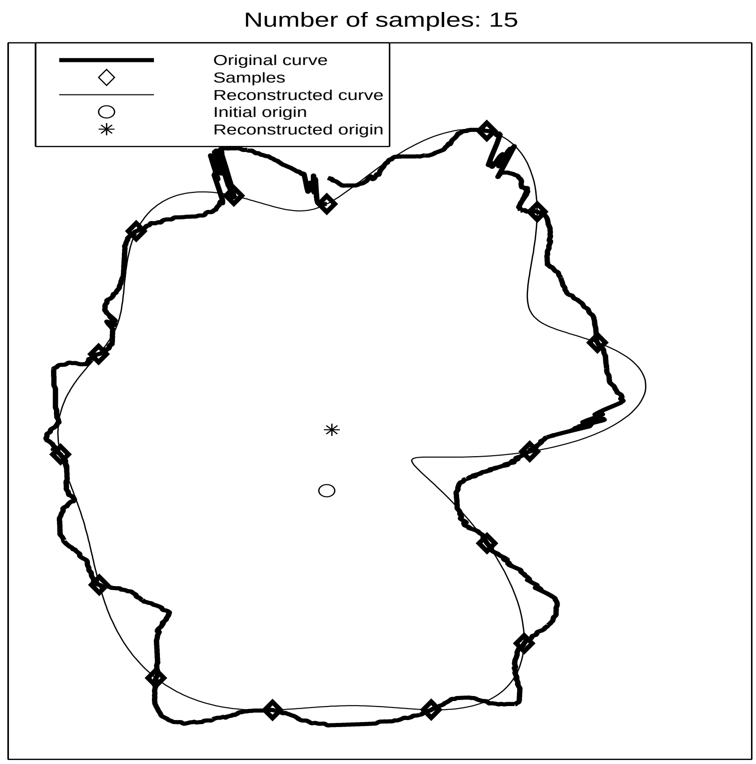

Figure 8: A contour of a human kidney can be recovered from only 7 samples.

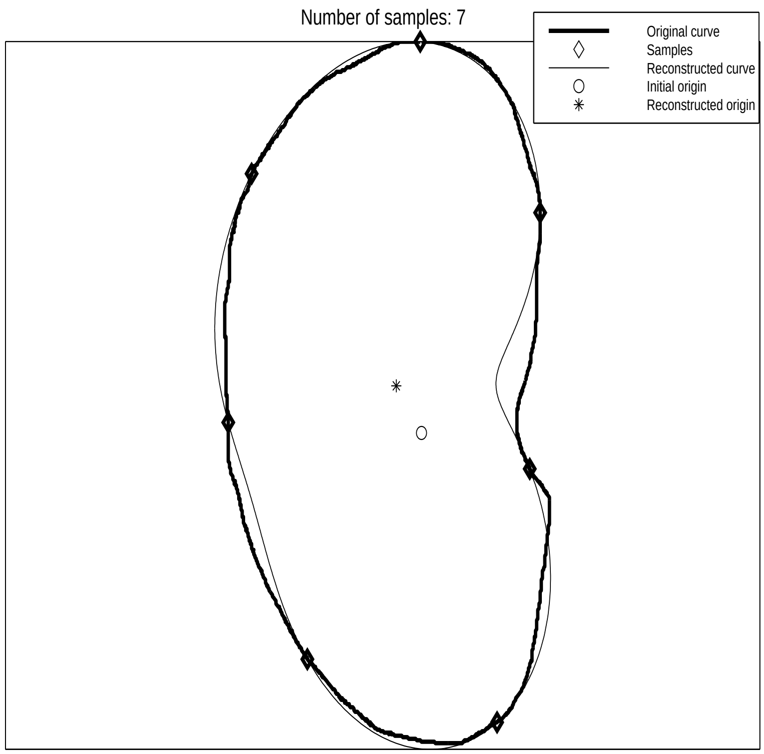

Figure 9: A contour of a human brain can be recovered from only 19 samples.