Parameterization Models for Sampling Planar Curves

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ABSTRACT

Discretizing planar curves yields many useful applications. We show that in order to sample a curve, one needs to impose a priori assumptions about the type of parameterization (coordinate system), which would provide a functional form for otherwise just a set of points on the plane. The choice of parameterization is dependent on the nature of typical objects one intends to work with, and thus depends on a particular application. We show the general approach to the problem of choosing a suitable parameterization based on the prevailing shape of objects of interest. Samples of a curve must incorporate the information about both the curve and the coordinate system (type of parameterization) that was used to produce them. The curve is then obtained from the samples by first reconstructing the coordinate system and finally the curve itself. Provided examples illustrate sufficiently general applicability of the proposed techniques.

Keywords: Sampling, planar curve, coordinate system, polar coordinates

1. INTRODUCTION

Sampling of 1D and 2D signals is an area of active scientific research by both engineering community and applied mathematicians. By far the most celebrated result in this area is the classical sampling theorem for 1D functions, which states that any bandlimited signal can be represented with a finite number of discrete samples without any loss of information. This theorem was first discovered by Cauchy, then rediscovered by Whittaker and Kotel'nikov and finally applied to communications problems by Shannon.

Despite the undisputed importance of the classical sampling theorem, the complexity of realistic signals dictated that the class of functions of interest was often wider than that of bandlimited signals. It, therefore, became an area of non-stopping efforts to generalize the result of Cauchy so as to accommodate for more general settings. The summary of the state-of-the-art results can be found, for example, in [1] or [2].

The classical sampling theorem was easily generalized to the 2D case, which resulted in the simple techniques for discretizing images provided analogous technical constraints. Much like the 1D case, those constraints were later significantly lifted in the effort to build mathematical methods of dealing with realistic images.

In this paper we consider the problem of sampling a planar curve. In practice curves arise as some contours in the image. Often they represent the shape of an object we are interested to study. Being able to convert a real-world object into a collection of discrete points is important for a number of reasons not the least of which is that points as opposed to curves are much easier to analyze numerically.

The fundamental property that makes a curve so much different from a 1D signal or an image is that the latter is a function, whereas the former is not. For a given curve, there is no *unique* way to represent it as a function, which automatically makes all function sampling algorithms meaningless and inapplicable long before we even fully stated the problem. On the other hand, samples of a curve are as such only if additional information about their nature and the nature of an underlying curve are provided, and otherwise they are just points on the plane that cannot be traced back to any uniquely identifiable object.

To assign a functional form to a given curve is tautologous to parameterizing it. Once such a parameterization has been specified, an attempt to sample the resulting function can be made. It is clear that no existing sampling

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algorithm can provide us with the desired sampling technique because for our problem the samples of a curve must carry the full description of the contour *as well as* the characterization of the parameterization that was used to produce them. In the latter sections of this paper will propose the general approach how to do that as well as illustrate its use for a special class of parameterizations.

Our paper is organized as follows. Section 2 is devoted to the development of the theory of 1D parameterizations of curves. Section 3 contains the description of the approach to sampling of a curve in a generalized 1D coordinate system. Section 4 applies this general technique to a particular class of curves that are representable in polar coordinates. It is shown that sampling in polar coordinates can be regarded as an optimization problem with multiple objectives. This optimization problem can be efficiently solved using numerical methods, and examples of applying those methods to real contour are presented. Section 5 concludes the paper with final remarks.

2. SAMPLING PLANAR CURVES

Let us start with the following definitions. Suppose that $\mathscr{C}$ is a given collection of points on the plane, i.e. $\mathscr{C} \subset \mathbb{R}^2$. In practice $\mathscr{C}$ is what we think of as a contour in the image.

DEFINITION 2.1. *A function $\gamma : [a,b] \to \mathscr{C} \subset \mathbb{R}^2$ is called a parameterization of $\mathscr{C}$. If a parameterization γ is a continuous function, then $\mathscr{C}$ is a contour on the plane.*

As was argued above, for a given $\mathscr{C}$ there may exist more than one parameterizations.

DEFINITION 2.2. *A curve, C, is an equivalence class of parameterizations of the same image, that is continuous functions of the form*

$$C = \{\gamma \mid \gamma : [a,b] \subset \mathbb{R} \to \mathscr{C} \subset \mathbb{R}^2,\ \gamma(t) = \big(x(t), y(t)\big) \in \mathbb{R}^2,\ t \in [a,b],\ \gamma \in C\big([a,b]\big)\} \tag{1}$$

Choosing a parameterization of the curve is the crucial first step of any sampling technique. The sampling procedure should then be constructed in such a way that the samples encompass the information about the chosen parameterization as well as the curve itself.

2.1. Parameterization and Coordinates

We will start by looking at two particular examples. First, consider a planar contour $\mathscr{C}$, which represents a graph of a 1D signal, whose functional form is unknown. The following procedure then constructs a parameterization of this curve which is the most natural given the assumptions that we deal with a 1D function. Consider the line $\mathscr{L} = \{y = 0\}$. Each point $(x,y) \in \mathscr{C}$ can be projected onto $\mathscr{L}$, obtaining the point $(x,0)$, where x is the arclength of $\mathscr{L}$ between the origin, $O = (0,0)$, and the projection point. (see Figure 2.1). The uniqueness of such a projection is guaranteed by the assumption that $\mathscr{C}$ is a graph of a 1D function. The (signed) distance between the point (x,y) and its projection onto $\mathscr{L}$ is clearly y. Despite its triviality this example is a perfect illustration of the general approach that could be utilized to convert a curve satisfying certain assumptions into a 1D signal.

We will now consider a slightly more complicated case. Let $\mathscr{C}$ be a curve and assume that we have fixed polar coordinates in the plane. Define a unit circle $\mathscr{L}$ and the "origin", $O = (1,0)$, which lies on $\mathscr{L}$. Consider an arbitrary point $(x,y) \in \mathscr{C}$ and its orthogonal projection on $\mathscr{L}$, which has coordinates $\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$. If we denote the polar angle by θ and the polar radius by r, so that

$$r\cos\theta = x, \tag{2}$$

$$r\sin\theta = y. \tag{3}$$

then θ is the arclength of $\mathscr{C}$ between O and the projection of the point on the curve, and $r-1$ is the (signed) distance between the point and its projection (see Figure 2.1).

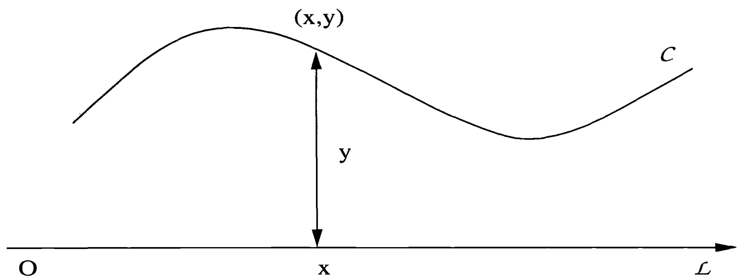

Figure 1: Cartesian coordinates

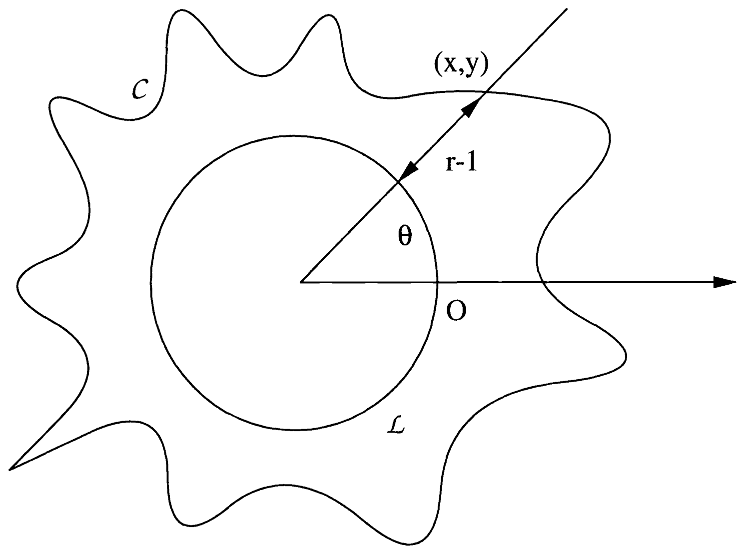

Figure 2: Polar coordinates

The above two examples suggest the general approach to converting certain classes of contours to 1D signals. This approach can be summarized as follows. Fix a curve $\mathscr{L}$ and pick $O \in \mathscr{L}$, which we will call an "origin".

DEFINITION 2.3. *We will call the curve $\mathscr{L}$ together with the origin O a 1D coordinate system.*

If a contour $\mathscr{C}$ is such that for each point $(x, y) \in \mathscr{C}$ the orthogonal projection on $\mathscr{L}$ is well-defined and unique, then it can be represented as a 1D signal with respect to the chosen coordinate system. Alternatively, each coordinate system defines a class of contours that are representable in those coordinates, and furthermore have a unique representation as a 1D signal. Upon ensuring that a contour is representable in a chosen coordinate system, the sampling and reconstruction of contours becomes a straightforward application of the sampling theory of 1D signals.

2.2. Parameter-dependent Coordinate Systems

The approach to sampling contours described in earlier sub-sections of the paper proves working for certain classes of shapes, however it also reveals serious shortcomings. To see them, let us come back to the case when a contour is representable in polar coordinates on the plane. This coordinate system can be used to parameterize a circle centered at the point $(0, 0)$ and the radius 1. Nonetheless if we shift the center of the curve to the point $(2, 0)$ then the resulting circle will no longer be in the admissible class of curves as it no longer can be represented in the original coordinate system. However, by shifting the polar center accordingly, i.e. to $(2, 0)$, we obtain a coordinate system, which provides a well-defined parameterization for the new contour. (See Figure 2.2).

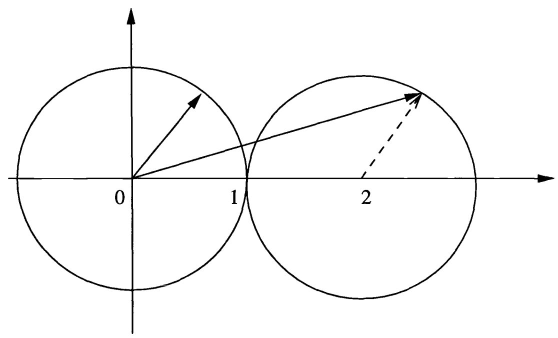

Figure 3: Any circle is representable in polar coordinates if the center of coordinates is placed appropriately

DEFINITION 2.4. *By a parameter-dependent coordinate system, $\bar{\mathscr{L}}$, we will understand a family of coordinate systems indexed by a (possibly vector-valued) parameter, i.e.*

$$\bar{\mathscr{L}} = \{(\mathscr{L}_p, O_p)\}_{p \in \mathscr{P}}, \tag{4}$$

where for all $p \in \mathscr{P}$, $\mathscr{L}_p$ is a curve, and $O_p \in \mathscr{L}_p$ is the "origin" belonging to the curve.

For example, a collection of circles $\bar{\mathscr{L}} = \{\mathscr{L}_{x_0,y_0}\}$ defined by

$$\mathscr{L}_{x_0,y_0} = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 = 1\},\ (x_0, y_0) \in \mathbb{R}^2 \tag{5}$$

together with "origins" $\{(x_0 + 1, y_0)\}$ defines a parameter-dependent polar coordinate system.

3. SAMPLING AND RECONSTRUCTION IN PARAMETER-DEPENDENT COORDINATE SYSTEMS

In this section we will assume that we are given a contour $\mathscr{C}$ and a parameter-based coordinate system $\bar{L} = \{(\mathscr{L}_p, O_p)\}$. Our goal is then to propose a sampling technique that would convert the contour into discrete samples and the reconstruction algorithm that would allow us to obtain the original curve from those samples.

The process of sampling can be split into several steps:

1. Find the parameter p that is appropriate for a given contour $\mathscr{C}$. That would provide us with a particular coordinate system;
2. Represent the contour as a 1D signal using the coordinate system $(\mathscr{L}_p, O_p)$;
3. Sample the resulting 1D function;
4. Convert the samples to points on the plane to produce the samples of the original contour.

The procedure of reconstruction of a curve from its samples could then be described by the following sequence of steps.

1. Given samples, recover the coordinate system that was used to produce them;
2. Represent the samples with respect to that coordinate system to obtain the samples of a 1D signal;
3. Reconstruct the 1D signal from its samples;
4. Convert the obtained 1D signal to a contour on the plane.

One cannot help but notice that in such a general setting several steps of these two algorithms present very non-trivial problems. However, as was shown in [3] those problems are feasible if one constraints himself to considering a particular class of contours and a coordinate system that best fits the application of interest.

4. CURVES REPRESENTABLE IN POLAR COORDINATES: SAMPLING AND RECONSTRUCTION

In this part of the paper we will demonstrate how the general sampling technique developed earlier can be applied to the problem of sampling shapes of particular kind. We will concentrate our attention on contours that can be represented in polar coordinates. Experiments with shapes that are common, for example, in medical imaging prove that this class is wide enough to encompass a variety of shapes of interest. At the same time the type of a coordinate system used to represent them is sufficiently simple to be theoretically tractable.

The following proposition constitutes the basis for the sampling algorithm that can be used to discretize shapes in polar coordinates (see [3]).

PROPOSITION 1. *Let C be a curve with the polar center γ_0. (For the sake of simplicity, assume $\gamma_0 = (0,0)$). If $r(\theta)$ is bandlimited then we can always choose sampling points $\{\theta_i\}_{i=1}^{N+1}$, where N is even, in such a way that*

$$\frac{1}{N+1}\sum_{i=1}^{N+1}\big(r(\theta_i)\cos(\theta_i), r(\theta_i)\sin(\theta_i)\big) = (0,0), \tag{6}$$

i.e. the orthocenter of the sampling points and the polar center are identical.

The last proposition means, as is spelled out next, that it is possible to choose samples of the curve in such a way that they encode the information about the coordinate system and the curve representation in those coordinates at the same time. It then follows that a unique reconstruction of the curve from its samples is possible without any additional information.

PROPOSITION 2. *Let C be a curve and $\gamma_0 \equiv (x_0, y_0)$ is its polar center. Suppose that $r(\theta)$ is a polar parameterization of C centered at γ_0. If $r(\theta)$ is bandlimited, then there exist points $\theta_1, \ldots, \theta_{N+1} \in [0, 2\pi]$, such that the curve C is uniquely defined by the points*

$$\big(r(\theta_i)\cos(\theta_i), r(\theta_i)\sin(\theta_i)\big) \in \mathbb{R}^2,\ i = 1, \ldots, N+1. \tag{7}$$

The Figure 4 shows the result of applying the described technique to a simple curve.

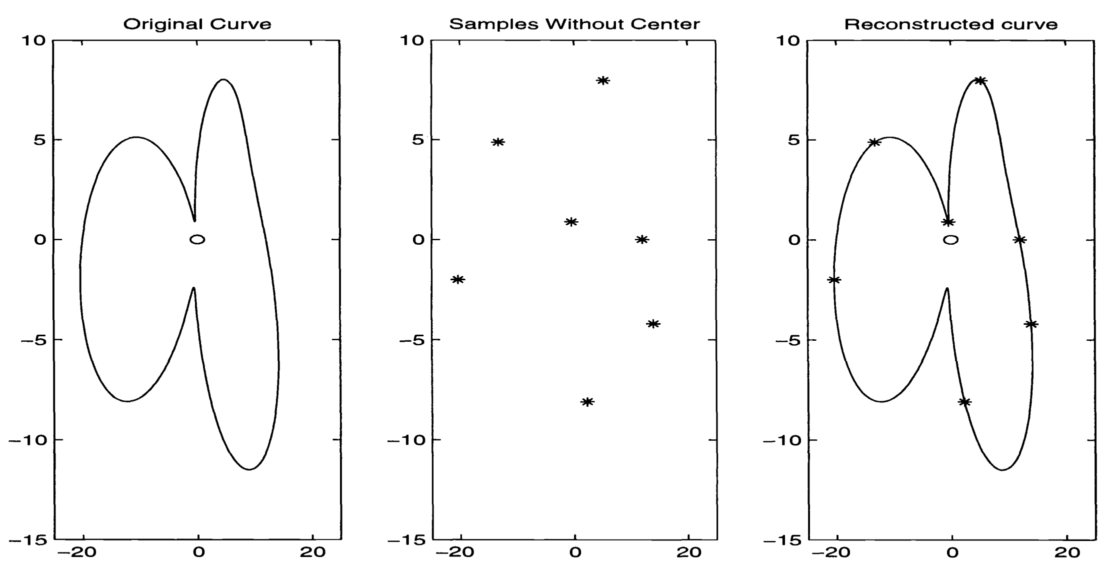

Figure 4: A curve is sampled and then perfectly reconstructed from the samples.

4.1. Sampling and Reconstruction of Curves in Polar Coordinates: Further Extensions

Curves that we encounter in real life normally fail to satisfy any conditions imposed a priori. This makes part of the above analysis non-applicable to such contours. In practice, a good approximation of a curve is often just as important as a perfect reconstruction. It is our goal then to generalize the above sampling technique so that for a curve, which lies outside the class of curves defined by the assumptions made in the statements of Proposition 1 and 2, we could still describe the sampling algorithm and a reconstruction procedure providing us with a good approximation of the original curve.

Recall, that if a curve is to be sampled and uniquely reconstructed, its sampling points are chosen so that the corresponding orthocenter coincides with the polar center. Denote the distance between the orthocenter of samples and the polar center by $d(\theta_1, \ldots, \theta_{N+1})$. The condition on sampling points can then be written as

$$d(\theta_1, \ldots, \theta_{N+1}) = 0. \tag{8}$$

This condition, as was shown earlier, is achievable for a bandlimited curve. It does not have to be so for a non-bandlimited curve. Thus we may only seek to minimize $d(\theta_1, \ldots, \theta_{N+1})$ while at the same time optimizing the distance between the curve and its reconstructed approximation.

4.1.1. Sampling By Optimized Approximation

Let C be a curve and γ_0 be its polar center. Assume $\{r(\theta), \theta \in [0, 2\pi]\}$ is a polar parameterization, which is not necessarily bandlimited. Towards formulating our generalized sampling define for each $(N+1)$-tuple (N even) $\theta_1, \ldots, \theta_{N+1} \in [0, 2\pi]$,

$$d^2(\theta_1, \ldots, \theta_{N+1}) = \left(\sum_{i=1}^{N+1} r(\theta_i)\cos(\theta_i)\right)^2 + \left(\sum_{i=1}^{N+1} r(\theta_i)\sin(\theta_i)\right)^2,$$

and

$$l^2(\theta_1, \ldots, \theta_{N+1}) = \int_0^{2\pi} \left|r(\vartheta) - \hat{r}_{\theta_1,\ldots,\theta_{N+1}}(\vartheta)\right|^2 d\vartheta,$$

where $\hat{r}_{\theta_1,\ldots,\theta_{N+1}}(\vartheta)$ is the reconstruction achieved by the samples defined at $\theta_1, \ldots, \theta_{N+1}$.

The function $d(\theta_1, \ldots, \theta_{N+1})$ is the measure of the quality of reconstruction of the coordinate system, and $l(\theta_1, \ldots, \theta_{N+1})$ is an indicator of how close the approximation of the curve is to the original.

Define $F : [0, 2\pi]^{N+1} \to \mathbb{R}^2$ as follows

$$F(\theta_1, \ldots, \theta_{N+1}) = \begin{pmatrix} d^2(\theta_1, \ldots, \theta_{N+1}) \\ l^2(\theta_1, \ldots, \theta_{N+1}) \end{pmatrix}^T. \tag{9}$$

DEFINITION 4.1. *A point $(\theta_1^0, \ldots, \theta_{N+1}^0) \in \mathbb{R}^{N+1}$ is called the Pareto optimal point of F, if $\forall(\theta_1, \ldots, \theta_{N+1}) \in \mathbb{R}^{N+1}$ we have*

$$\left\{ \begin{array}{lcl} d^2(\theta_1, \ldots, \theta_{N+1}) & \leq & d^2\left(\theta_1^0, \ldots, \theta_{N+1}^0\right) \\ l^2(\theta_1, \ldots, \theta_{N+1}) & \leq & l^2\left(\theta_1^0, \ldots, \theta_{N+1}^0\right) \end{array} \right. \Rightarrow (\theta_1, \ldots, \theta_{N+1}) = (\theta_1^0, \ldots, \theta_{N+1}^0).$$

The optimal point is essentially the one, from which one component of a two-dimensional objective function may not be decreased without increasing another. More information on multi-objective optimization can be found in [4].

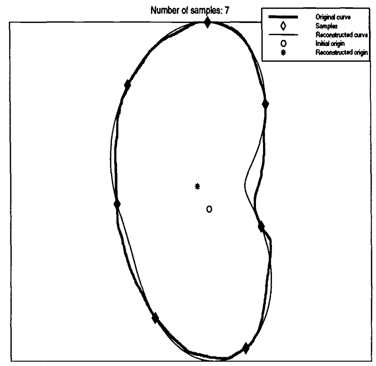

Figure 5: Human kidney is well defined by 7 samples.

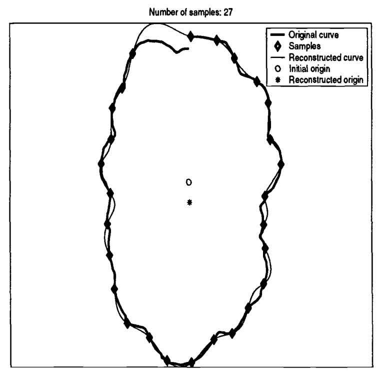

Figure 6: Shape of human kidney with cysts is described by 27 samples.

4.1.2. Implementation and Results

Here we present the results of implementing the aforementioned optimization by using the steepest-descent method, which is proven to achieve the global optimal point. For the sake of space we defer the description of this method to [5] and only present the simulation results. It is worth stressing out here that our examples clearly illustrate that the proposed sampling technique is implementable and a number of realistic shapes can be encoded with a very small number of samples. (See Figures 4.1.2, 4.1.2 and 4.1.2).

5. CONCLUSIONS

Sampling of curves and surfaces proves to be useful in a variety of applications. However, the problem of sampling a curve is much more complicated than discretizing 1D and 2D signals. The key distinction between a curve and a 1D signal lies in the fact, that a curve has no unique functional form attached to it. At the same time, to sample a curve one needs to have a unique way to represent it as a function of a parameter.

It has been shown that parameterizing a curve is identical to choosing a 1D coordinate system, by which we mean an *a priori* selected curve with the point on it, which we called the "origin" of the coordinate system. Once a 1D coordinate system is chosen, it automatically defines the class of contours, which can be represented as 1D functions with respect to that curve in a consistent manner.

It has been further demonstrated that having a coordinate system described by a single curve can be unnatural and can provide undesirable limitations in terms of the class of admissible curve. We have shown that by allowing the coordinate system itself depend on a vector-valued parameter, we can achieve an ability to represent a much wider class of curves.

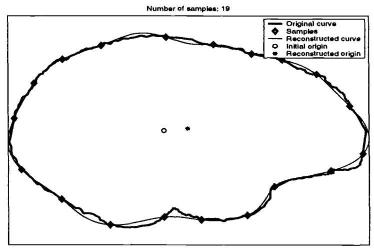

Figure 7: Human brain is well defined by 19 samples.

In light of these distinctions between a 1D signal and an arbitrary planar curve, sampling techniques used to discretize an otherwise continuous curve need to be reconsidered. It has been argued that for a curve representable in a parameterized coordinate system, samples must be chosen in such a way that they contain the information about the coordinate system as well as the curve itself. As was shown for a particular case of a polar coordinate system, such sampling can be implemented as an optimization technique which yields to minimize two objective functions at the same time, one being the error or reconstructing the coordinate system, and another one the error of the curve approximation. Numerical implementations of the proposed technique demonstrate that the proposed approach is feasible and capable of producing satisfactory results.

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