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Solving MHD Falkner-Skan Boundary-Layer Equation Using Collocation Method Based On Rational Legendre Function With Transformed Hermite-Gauss Nodes

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Abstract: The Falkner-Skan equation arises in the study of laminar boundary layers exhibiting similarity. The MHD systems are used effectively in many applications including power generators, pumps, accelerators, electrostatic filters, droplet filters, the design of heat exchangers, the cooling of reactors, etc. For the MHD Falkner-Skan equation, we have developed a new numerical technique transforming the governing partial differential equation into a nonlinear third-order boundary value problem by similarity variables and then solve it by the rational Legendre collocation method. In this paper we use transformed Hermite-Gauss nodes as interpolation points. The solutions obtained thus are in excellent agreement with those obtained by previous papers. In this work we concentrated on the boundary conditions of $f(0) = f'(0) = 0$, $f'(+\infty) = 1$, which is corresponding to a fixed and impermeable wedge flow. The physical quantities of interest which is represented by the value of $f''(0)$ is the skin friction coefficient.

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INTRODUCTION

Many problems arising in the field of mathematical physics and fluids mechanic can be modeled by differential boundary value problems defined in semi-infinite domains. While spectral approximations for Ordinary Differential Equations (ODEs) in bounded domains have achieved great success and popularity in recent years, spectral approximations for ODEs in unbounded domains have only received limited attention. Yet it is also a very difficult subject since the unboundedness of the domain Several introduce considerable theoretical and practical challenges which are not present in bounded domains. spectral methods for approximating of semi-infinite domains are known, a first approach is using Laguerre or Hermite polynomials/functions [1-5]; a second approach is reformulating the original problem in semi-infinite domain to a singular problem in bounded domain by variable transformation and then using the Jacobi polynomials to approximate the resulting singular problem [6-8]; another effective approach for problems in unbounded domains is based on rational approximations, for example, Christov [9] and Boyd [10, 11] developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Authors of [12-17] applied the spectral method to solve the nonlinear ordinary differential equations on semi-infinite intervals. Their approach is based on the rational Tau and collocation methods. The fourth approach is replacing the semi-infinite domain with [0,L] interval by choosing L, sufficiently large, this method is as the domain truncation [18]. In this paper, we investigate the rational Legendre collocation method based on transformed Hermite-Gauss nodes which is another approach for solving MHD Falkner-Skan boundary-layer equation.

This equation has been the focus of many studies [19-30] and all these approaches have mainly used shooting and invariant imbedding. The mathematical treatments of MHD Falkner-Skan equation due to Rosenhead [21] and Weyl [22] have focused on obtaining existence and uniqueness results. Finite-difference methods for this problem are presented in [23, 24]. A differential transformation method, which obtains a series solution of the Falkner-Skan equation, is presented in [25]. Liao [26, 27] has applied a new analytical method, namely homotopy analysis method, which is independent of small or large physical

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parameters, to solve the Falkner-Skan equation and gave an explicit, totally analytical solution with the boundary conditions. Asaithambi [28] used a coordinate transformation to convert the physical problem on a semi-infinite physical domain to a problem on a fixed computational domain and solved the nonlinear equation using a shooting technique. Parand *et al*. [29, 30] proposed the Sinc and Hermite type spectral method for solving three order nonlinear ordinary differential laminar boundary layer equation. They introduced a general presentation of the method and solved the problem on the semi-infinite domain without truncating it to a finite domain and transforming domain of the problem to a finite domain.

This paper is arranged as follows: in Section 2, we explain the properties of MHD Falkner-Skan boundary-layer equation and in Section 3, we explain the properties of rational Legendre polynomials and Hermite functions. In Section 4, the proposed method is applied to solve this equation and compare it with the existing methods in the literature like homotopy-padè analysis and shooting method. Finally we give a brief conclusion in the last section.

MHD FALKNER-SKAN BOUNDARY-LAYER EQUATION

Let us consider the problem of the effects of viscous dissipation and stress work on the steady two-dimensional laminar magneto-hydrodynamic flow. All the fluid properties are assumed to be constant, introducing the boundary layer approximation, the governing equations for the continuity and momentum can be written as follows [19]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2}{\rho}(u-U)$$

$$\begin{cases} u=0, v=0, & y=0 \\ u=U(x), & y=\infty \end{cases} \qquad (1)$$

Here, u and ν are velocity components, U is the inherent characteristic velocity, ν is a kinematic viscosity, σ is the electrical conductivity, B is the magnetic field and ρ is the fluid density. It was proved that the similar solution exists when the velocity of the potential flow is proportional to a power of the length coordinate, x, that is $U(x) = ax^m$ [20]. By using dimensionless and the newly defined variables, we have the following equation

$$\eta=\sqrt{\frac{m+1}{2}}\sqrt{\frac{U}{\nu x}}y,\quad \phi=\sqrt{\frac{2}{m+1}}\sqrt{\nu\nu U}f(\eta) \qquad (2)$$

$$u=U f'(\eta),\quad v=-\sqrt{\frac{m+1}{2}}\sqrt{\frac{U\nu}{x}}\left(f+\frac{m-1}{m+1}\eta f'\right) \qquad (3)$$

$$m=\frac{\beta}{2-\beta}\Rightarrow\beta=\frac{2m}{m+1} \qquad (4)$$

where

$$f'''(\eta)+f(\eta)f''(\eta)+\beta[1-f'^2(\eta)]-M^2(f'(\eta)-1) \qquad (5)$$

subject to

$$f(0)=f'(0)=0,\quad f'(+\infty)=1 \qquad (6)$$

in above equation

$$M^2=\frac{2\sigma B_0^2}{\rho a(1+m)}$$

Our interest now is to find the solution of Eq (5) for wedge in the accelerated flow (m>0, β>0) and decelerated flow (m<0, β<0) with separation.

RATIONAL LEGENDRE FUNCTIONS AND HERMITE FUNCTIONS PROPERTIES

This section is devoted to the introduction of the basic notions and working tools concerning orthogonal rational Legendre functions and later we present some properties of Hermite functions.

Properties of rational legendre functions: The well-known Legendre polynomials are orthogonal in the interval [-1,1] with respect to the weight function $\rho(y) = 1$ and can be determined with the aid of the following recurrence formula:

$$P_0(y)=1,\quad P_1(y)=y$$

$$P_{n+1}(y)=\left(\frac{2n+1}{n+1}\right)yP_n(y)-\left(\frac{n}{n+1}\right)P_{n-1}(y),\quad n\geq 1 \qquad (7)$$

The new basis functions, "Rational Legendre" denoted by $R_n(x)$, are defined by

$$R_n(x)=P_n\left(\frac{x-L}{x+L}\right) \qquad (8)$$

where L is a constant parameter and Boyd [31] offered guidelines for optimizing the map parameter L for rational Chebyshev functions, which is useful for rational Legendre functions, too. $R_n(x)$ is the nth eigenfunction of the singular Sturm-Liouville problem

$$\frac{(x+L)^2}{L}\frac{d}{dx}(x\frac{d}{dx}R_n(x))+n(n+1)R_n(x)=0$$

$$x\in[0,\infty),\quad n=0,1,2,\ldots \tag{9}$$

and by Eq. (7) satisfied in the following recurrence relation:

$$R_0(x)=1,\quad R_1(x)=\frac{x-L}{x+L}$$

$$R_{n+1}(x)=\left(\frac{2n+1}{n+1}\right)\left(\frac{x-L}{x+L}\right)R_n(x)-\left(\frac{n}{n+1}\right)R_{n-1}(x), n\ge 1 \tag{10}$$

Let

$$w(x)=\frac{2L}{(x+1)^2}$$

denotes a non-negative, integrable, real-valued function over the interval $(0,\infty)$, we define

$$L_w^2=\{v:(0,\infty)\to R\ ,\ \|v\|_w<\infty\} \tag{11}$$

where

$$\|v\|_w=\left(\int_0^\infty |v(x)|^2\, w(x)dx\right)^{\frac{1}{2}} \tag{12}$$

is the norm induced by the inner product of the space L_w^2,

$$<u,v>_w=\int_0^\infty u(x)v(x)w(x)dx \tag{13}$$

$\{R_n(x)\}_{n\ge 0}$ functions are orthogonal in $(0,\infty)$ under Eq. (13), i.e.

$$<R_n,R_m>_w=\frac{2}{2n+1}\delta_{nm} \tag{14}$$

where δ_{nm} is the Kronecker delta function. This system is complete in L_w^2. For any function $f\in L_w^2$ the following expansion holds:

$$f(x)=\sum_{j=0}^{\infty}c_jR_j(x) \tag{15}$$

with

$$c_j=\frac{<f,R_j>_w}{\|R_j\|_w^2} \tag{16}$$

The c_j's are the discrete expansion coefficients associated with the family $\{R_j\}$. Another important property of rational Legendre functions is that

$$\lim_{x\to\infty}R'(x)=0 \tag{17}$$

Properties of hermite functions: The Hermite function is defined for all $x\in R$ and can be written in recursive formula as follows [32, 33]:

$$H_{n+1}(x)=x\sqrt{\frac{2}{n+1}}H_n(x)-\sqrt{\frac{n}{n+1}}H_{n-1}(x),\quad n\ge 1$$

$$H_0(x)=e^{-\frac{x^2}{2}},\quad H_1(x)=\sqrt{2}xe^{-\frac{x^2}{2}} \tag{18}$$

$\{H_n\}$ is an orthogonal system in $L^2(R)$, i.e.,

$$\int_{-\infty}^{+\infty}H_n(x)H_m(x)dx=\sqrt{\pi}\delta_{n,m} \tag{19}$$

where $\delta_{n,m}$ is the Kronecker delta function. Using the recurrence relation of Hermite polynomials and the above formula leads to

$$H_n'(x)=\sqrt{2n}H_{n-1}(x)-xH_n(x)$$

$$\sqrt{\frac{n}{2}}H_{n-1}(x)-\sqrt{\frac{n+1}{2}}H_{n+1}(x)$$

Let us define

$$\tilde{P}_N:=\{u:u=e^{-\frac{x^2}{2}}v,\forall v\in P_N\} \tag{20}$$

where P_N is the set of all Hermite polynomials of degree at most N. We now introduce the Gauss quadrature associated with the Hermite functions approach. Let $\{x_j\}_{j=0}^N$ be the Hermite-Gauss nodes which can be N+1 roots of H_{N+1} and define the weights

$$w_j=\frac{\sqrt{\pi}}{(n+1)H_n^2(x_j)},\quad 0\le j\le N \tag{21}$$

Then we have :

$$\int_R p(x)dx=\sum_{j=0}^{N}p(x_j)w_j,\quad \forall p\in \tilde{P}_{2N+1}$$

SOLVING FALKNER-SKAN EQUATION

To apply rational Legendre collocation method to the Falkner-Skan equation introduced in Eq. (5)

with boundary conditions Eq. (6), it is expected to approximate f as the truncated series

$$\hat{\varsigma}_N f(\eta) = \sum_{j=0}^{N} c_j R_j(\eta)$$

But by Eq. (17) we have

$$\lim_{n\to\infty} \frac{d}{dx} \hat{\varsigma}_N f(\eta) = 0$$

so we define a polynomial $p(\eta)$ that satisfy Eq. (6). This polynomial is given by

$$p(\eta) = \frac{\eta^3}{(\eta + L)^2} \quad (22)$$

The approximate solution for $f(\eta)$, in Eq. (5) with boundary conditions in Eq. (6) is represented by

$$\hat{\varsigma}_N f(\eta) = p(\eta) + \sum_{j=0}^{N} c_j R_j(\eta) \quad (23)$$

It is noted that the approximate solution $\hat{\varsigma}_N f(\eta)$, satisfy boundary conditions in Eq. (6), since

$$\lim_{\eta\to\infty} \frac{d}{d\eta} \hat{\varsigma}_N f(\eta) = 1 \quad (24)$$

We construct the residual function by substituting $f(\eta)$ by $\hat{\varsigma}_N f(\eta)$ in the Falkner-Skan equation (5):

$$\begin{aligned} Res(\eta) &= \frac{d^3}{d\eta^3} \hat{\varsigma}_N f(\eta) + \hat{\varsigma}_N f(\eta) \frac{d^2}{d\eta^2} \hat{\varsigma}_N f(\eta) \\ &+ \frac{2m}{m+1}\left(1 - \left(\frac{d}{d\eta} \hat{\varsigma}_N f(\eta)\right)^2\right) - M^2 \left(\frac{d}{d\eta} \hat{\varsigma}_N f(\eta) - 1\right) \end{aligned} \quad (25)$$

To find the unknown coefficients $\{c_j\}_{j=0}^{N}$'s, we equalize $Res(\eta)$ to zero at N-1 Hermite-Gauss nodes $\{x_j\}_{j=0}^{N-2}$ and we equalize boundary conditions in Eq. (6) too, therefore we have:

$$Res(x_j) = 0, \quad j = 0 \ldots N-2$$

$$\hat{\varsigma}_N f(0) = 1, \quad \frac{d\hat{\varsigma}_N f(\eta)}{d\eta}\Big|_{\eta=0} = 0 \quad (26)$$

But as mentioned before Falkner-Skan equation are defined on the interval $(0,\infty)$ and we know properties of Hermite functions are derived in the infinite domain $(-\infty,\infty)$. Also we know approximations can be constructed for infinite, semi-infinite and finite intervals. One of the approaches to construct Hermite-Gauss nodes on the interval $(0,\infty)$ which is used in the current paper, is to use a mapping, that is a change of variable of the form :

$$z_j = \phi(x_j) = e^{x_j} \quad (27)$$

where is the inverse map of

$$x_j = \phi^{-1}(z_j) = \ln(z_j) \quad (28)$$

Finally to find the unknown coefficients $\{c_j\}_{j=0}^{N}$'s, we have:

$$Res(z_j) = 0, \quad j = 0..N-2 \quad (29)$$

$$\hat{\varsigma}_N f(0) = 1, \quad \frac{d\hat{\varsigma}_N f(\eta)}{d\eta}\Big|_{\eta=0} = 0 \quad (30)$$

where z_j is transformed root of $H_{N-1}(x)$. Eqs. (29),(30) gives N+1 nonlinear algebraic equations which can be solved for the unknown coefficients c_j by using the well known Newton's method by Maple programming and we use

$$c_j = 0, \quad j = 0 \ldots N$$

Table 1: Numerical results obtained by the present method for m = 2 and N = 11,22

	N = 11		N = 22		Other methods	
M	L	$f''(0)$	L	$f''(0)$	HAM [19]	Reference [28]
1	1.43400	1.719478923	1.3891900	1.71946540	1.71947219	1.719465400
2	1.07800	2.439410244	0.7155296	2.43949833	2.43949870	2.439498330
5	1.29207	5.219603250	1.0255630	5.19685669	5.19095980	5.190959450
10	1.38000	10.09672313	0.9709005	10.09677545	10.09677575	10.096775450
50	1.50620	50.01941138	1.1249460	50.01944071	50.01944084	50.019440710
100	1.67936	100.0097098	1.1651240	100.00972170	100.00972177	100.009721700

Table 2: Numerical results obtained by the present method for m = -3/5 and N = 11,22

	N = 11		N = 22		Other Methods	
M	L	$f''(0)$	L	$f''(0)$	HAM [19]	Reference [28]
3	1.61000	2.273391298	0.79558190	2.27338836	2.27338419	2.273388360
4	1.33600	3.488148356	0.89454900	3.48814857	3.48814572	3.488148570
5	1.83910	4.600709426	0.79900000	4.60075494	4.60075228	4.600754940
10	1.76600	9.806436078	0.97735700	9.80646420	9.80646300	9.806464200
15	1.49750	14.87160371	0.99130540	14.87167484	14.87167401	14.87167484
20	1.65275	19.90395230	1.04999270	19.90393701	19.90393626	19.90393701
50	1.69745	49.96169672	1.36031925	49.96165233	49.96165198	49.96165233

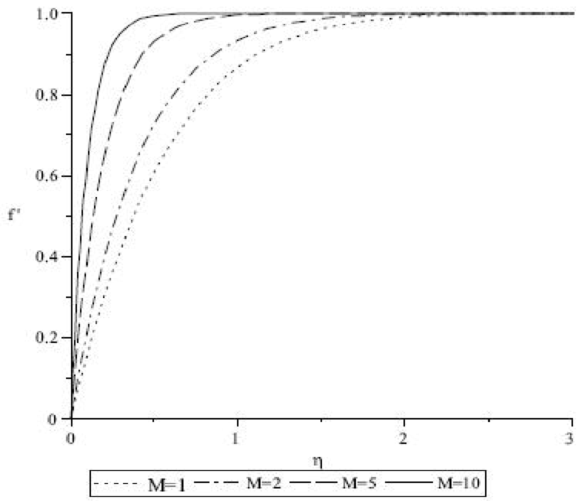

Fig. 1: Graph of the approximations of $f'(\eta)$ for m = 2 and N = 22

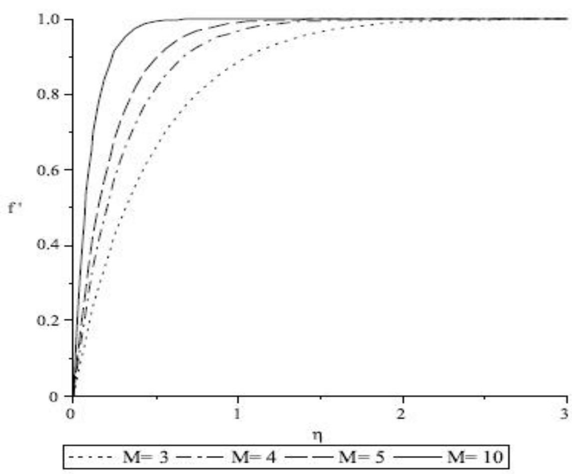

Fig. 2: Graph of the approximations of $f'(\eta)$ for and N = 22

as starting points to obtain convergence of the method, consequently, $f(\eta)$ given in Eq. (5) can be calculated.

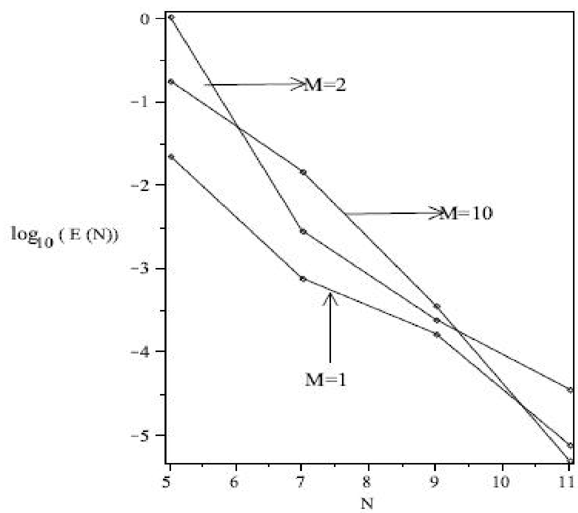

Fig. 3: Graph of the base-10 logarithmic of relative error E(N) versus N with respect to the Asaithambi results [28] for m = 2 and M = 1,2,10

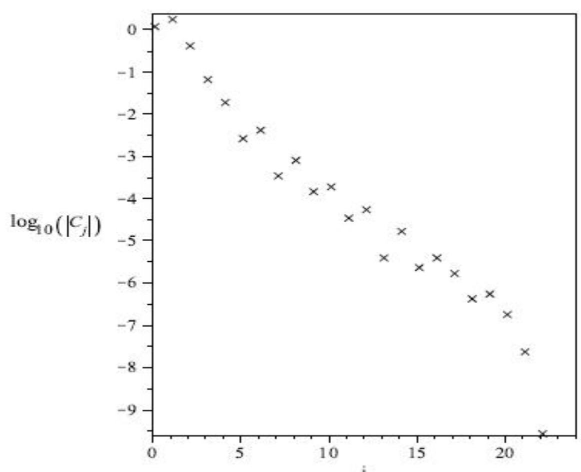

Fig. 4: Graph of the base-10 logarithmic of absolute values of coefficients, $\lg_{10}(|c_j|)$ versus j for N = 22

RESULTS

The values of $f''(0)$ obtained by the present method and the comparison with method in [28] are shown in Table 1 and 2 for m = 2, -3/5 and various M.

Graph of the approximations of $f'(\eta)$ for Falkner-Skan equation with M = 1,2,5,10 and m = 2 is shown in Fig. 1.

Graph of the approximations of $f'(\eta)$ for Falkner-Skan equation with M = 3,4,510 and m = -3/5 is shown in Fig. 2.

Figure 3 shows the base-10 logarithmic of relative error E(N) versus N (number of collocation points) with respect to the Asaithambi results [28] where

$$E(N)=\left|\frac{f''_N(0)-f''_{exact}(0)}{f''_{exact}(0)}\right|$$

for m = 2 and M = 1,2,10.

The base-10 logarithmic of absolute values of coefficients, $lg_{10}(|c_j|)$ for the rational scaled generalized Laguerre function for N = 22, m = 2 and M = 2 are shown in Fig. 4.

CONCLUSIONS

There are two significant points deserved further inform in this paper. The first is Falkner-Skan equation which is defined in semi-infinite domain was solved by collocation method and employed Hermite-Gauss nodes with e^x mapping that transformed these nodes from $(-\infty, +\infty)$ to semi-infinite domain. Another one is, the new algorithm has a fast convergence speed because by increasing collocation points the relative error of $f''(0)$ tends to zero and logarithmic scale for absolute values of coefficients c_j delivers a solution with very good accuracy and rapid convergence when it is compared with other methods of solution found in the literature.

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