The Prometheus Orthonormal Set for Wideband CDMA

Hakan Delić  
Department of Electrical and Electronics Engineering  
Boğaziçi University  
Bebek 34342 Istanbul, Turkey  
E-mail: delic@boun.edu.tr

James S. Byrnes and Gerald Ostheimer  
Prometheus Inc.  
21 Arnold Avenue  
Newport, Rhode Island 02840, USA  
E-mail: {jim, gerald}@prometheus-inc.com

Abstract—The Prometheus orthonormal set (PONS) represents a Hadamard-type sequence construction with advantageous autocorrelation, spectrum and peak-to-average power ratio properties. The PONS-based orthogonal variable spreading factor (OVSF) sequences are proposed for channelization in wideband code-division multiple access. A duocode version is shown to yield the same code capacity as Walsh-based OVSF. The duocode OVSF-PONS channelization offers a strong alternative to Walsh-based constructions in next generation wireless communications, due its superior sequence properties which can be implemented at the expense of slightly increased complexity.

I. INTRODUCTION

The third-generation Universal Mobile Telecommunication Systems (UMTS), which is based on the wideband CDMA (WCDMA) concept, envisions data transfers at a nominal rate of 384 kbps, aside from the traditional voice services. All users spread their data to a 5 MHz bandwidth by replacing each information bit with a unique code, which is a sequence of shorter duration chips, and simultaneously transmit over the same wideband channel. The codes are designed to be orthogonal so that the corresponding despreading at the receiver recovers the desired message, while canceling other users’ signals and spreading any narrowband interference to a low amplitude.

The deployment of WCDMA in third-generation UMTS relies on orthogonal variable spreading factor (OVSF) sequences for channelization in both uplink and downlink [6], [8]. High-rate services are assigned codes with low spreading factors. Starting with the $2 \times 2$ Hadamard matrix, a tree-structured method generates $2^m$ spreading codes of length $2^m$ chips from the $2^{m-1}$ parent codes of length $2^{m-1}$ by concatenating each row of the parent code matrix with itself and its complement [1], [8].

The UMTS standard utilizes OVSF codes with $m = 2, 3, \ldots , 8$ for the uplink and $m = 2, 3, \ldots , 9$ for the downlink transmissions. Despreading through multiplication by the appropriate code at the receiver recovers the desired data, due to the orthogonality among the OVSF codes, provided that chip synchronization is maintained [8].

The performance of next generation cellular systems, OVSF- or multicode-based, will depend on the correlation properties of the code-set that is employed. Due to multipath effects, non-negligible timing offsets causing asynchronism and hence non-zero cross-correlations will occur at the receiver. While the resulting multiple access interference (MAI) and intersymbol interference can be combated by equipping the receiver with complex equalization and detection devices, efforts should be undertaken to design OVSF sequences with more suitable correlation functions.

The OVSF code construction proposed by [1] (henceforth called OVSF-Walsh) inherits the shortcomings of Walsh sequences. In particular, the autocorrelation function is nonzero around the maximum, leading to synchronization difficulties; energy spreading is inefficient with a concentrated mainlobe in the frequency spectrum; and the peak-to-average power ratio is high.

In this paper, it is shown that the Prometheus orthonormal set (PONS) development inherently possesses an OVSF structure, making it suitable for WCDMA with the promise of better performance. Because it is possible to obtain PONS from the Walsh-Hadamard matrices with a straightforward transformation, existing hardware and software can accommodate PONS with minimal additional cost. The advantages of PONS-based OVSF codes come at the expense of a minor reduction in the number of available codewords in certain cases, which can be overcome by fusing the OVSF and multicode CDMA concepts, as will be explained in the sequel.

II. THE PROMETHEUS ORTHONORMAL SET

The original development of PONS is based on the Shapiro polynomials, which have coefficients $\pm 1$ [15]. The recursive construction that leads to the Shapiro polynomials consists of the equations

$$P_0(z) = Q_0(z) = 1,$$

$$P_{n+1}(z) = P_n(z) + z^{2^n} Q_n(z),$$

$$Q_{n+1}(z) = P_n(z) - z^{2^n} Q_n(z),$$

where $z$ is a complex number of modulus one and $n$ is a nonnegative integer. Thus, $P_n$ and $Q_n$ are polynomials of degree $2^n - 1$, and

$$|P_n(z)|^2 + |Q_n(z)|^2 = 2^{n+1}.$$

It then follows that

$$\frac{||P_n(z)||_{\infty}}{||P_n(z)||_2} \leq \sqrt{2}, z \in \mathbf{C}, |z| = 1,$$  \hspace{1cm} (1)
where \( || \cdot ||_\infty \) and \( || \cdot ||_2 \) stand for the \( L_\infty \) and \( L_2 \) norms, respectively. The peak-to-average ratio depicted by equation (1) corresponds to the crest factor in antenna design. The Shapiro polynomials have crest factors that are uniformly bounded by \( \sqrt{2} \), for all \( n \), which implies that the energy is spread evenly around the unit circle [3].

Let \( \epsilon_i \) and \( \delta_i \) denote the coefficient of \( P_n(z) \) and \( Q_n(z) \), respectively. To prove that a certain global uncertainty bound is satisfied, Byrnes expanded the Shapiro polynomials via a concatenation rule such that the coefficients of \( P_{n+1}(z) \) and \( Q_{n+1}(z) \) are \( \{ \epsilon_0, \epsilon_1, \ldots, \epsilon_{2^n-1}, \delta_0, \delta_1, \ldots, \delta_{2^n-1} \} \) and \( \{ \epsilon_0, \epsilon_1, \ldots, \epsilon_{2^n-1}, -\delta_0, -\delta_1, \ldots, -\delta_{2^n-1} \} \), respectively [3]. In polynomial terms, this concatenation leads to the complementary pair

\[
P_{n+1}(z) = \sum_{i=0}^{2^n-1} \epsilon_i z^i + z^{2^n} \sum_{i=0}^{2^n-1} \delta_i z^i, \tag{2}
\]

\[
Q_{n+1}(z) = \sum_{i=0}^{2^n-1} \epsilon_i z^i - z^{2^n} \sum_{i=0}^{2^n-1} \delta_i z^i. \tag{3}
\]

In a similar fashion, reversing the concatenations in (2) and (3) produces the complementary polynomials

\[
R_{n+1}(z) = \sum_{i=0}^{2^n-1} \delta_i z^i + z^{2^n} \sum_{i=0}^{2^n-1} \epsilon_i z^i,
\]

\[
S_{n+1}(z) = -\sum_{i=0}^{2^n-1} \delta_i z^i + z^{2^n} \sum_{i=0}^{2^n-1} \epsilon_i z^i.
\]

The PONS sequences, which are the coefficients of the polynomials \( P, Q, R, S \), make up the rows of the PONS matrices. The construction starts with

\[
P_1 = \begin{bmatrix} P_{1,1} \\ Q_{1,1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
\]

Concatenation leads to

\[
P_2 = \begin{bmatrix} P_{2,1} \\ Q_{2,1} \\ P_{2,2} \\ Q_{2,2} \end{bmatrix} = \begin{bmatrix} P_{1,1} & Q_{1,1} \\ P_{1,1} & -Q_{1,1} \\ Q_{1,1} & P_{1,1} \\ Q_{1,1} & -P_{1,1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix},
\]

and letting

\[
P_{m-1} = \begin{bmatrix} P_{m-1,1} \\ Q_{m-1,1} \\ \vdots \\ P_{m-1,2^m-2} \\ Q_{m-1,2^m-2} \end{bmatrix},
\]

which is of dimension \( 2^{m-1} \times 2^{m-1} \) with each row being one of the \( 2^{m-1} \) PONS sequences, the \( 2^m \times 2^m \) PONS matrix is obtained as

\[
P_m =
\]

Fig. 1. Magnitude plots of PONS correlation averages for \( m = 6 \). The x-axis shows the amount of shift.

In addition to leading directly to an “energy spreading” basis for \( L_2(\mathbb{R}) \), PONS functions possess the following qualities which are absent in Walsh functions [3], [4].

1) Global uncertainty bounds are satisfied.
2) Crest factor of \( \sqrt{2} \) is attained for all \( m \) (see also [2]).
3) \( P_{k,\ell} \) and \( Q_{k,\ell} \) satisfy the quadrature mirror filter (QMF) definition and are complementary pairs for \( k = 1, 2, \ldots, 10 \).

The significance of the advantages that a PONS-based OVSF code-set offers is better understood when the autocorrelation and spectral properties of the sequences are investigated. In particular, PONS has (i) an impulse-like autocorrelation making it much more amenable for synchronization (see Fig. 1); (ii) uniformly spread frequency spectrum making it more resistant to frequency-selective fading and narrowband interference (see Fig. 2 with a comparison to Walsh spreading); (iii) peak-to-average power ratio (PAPR) that is less than or equal to 2 making it energy- and hence battery-efficient. Spreading with PONS rather than Walsh sequences yields signals requiring lower short-term peak power to maintain a specified average transmission power. This is advantageous in view of the power control issues involved in CDMA systems to address the near-far effect.

Note that the good behavior of PONS is inherited from the structure of Shapiro polynomials and the associated sequences, recognized as the Golay complementary sequences in the engineering community [10]. Indeed, the low PAPR feature of complementary sequences has
inspired researchers to employ them in orthogonal frequency division multiplexing (OFDM) systems [7], [13].

Within the framework of multicarrier CDMA (MC-CDMA), Popović has demonstrated that the Walsh sequences have the worst performance in terms of crest factor, dynamic range of the complex signal envelope, and the average bit error probability among various spreading sequences [14]. In contrast, complementary sequences display the lowest average bit error probability.

III. OVSF-PONS

Based on the discussion so far, PONS is a powerful candidate for OVSF coding in next generation mobile communications due to its low PAPR; uniform energy spreading in the frequency domain; potential for the best average bit error probability performance; and the structure that naturally lends itself to OVSF construction.

Let layer $m$ correspond to the OVSF codes of length $2^m$. At layer $m$, there are $2^m$ mutually orthonormal PONS codes. Higher layer codes are assigned to low-rate users. The construction of $P_m$ from the parent rows of $P_{m-1}$ presents itself naturally as a foundation for OVSF code design.

Note that while all $P_k$, $k = 1, 2, \ldots$, are Hadamard, the simultaneous utilization of any code at layer $m$, and either of its parents at layer $m - 1$ (through repetition) for channelization will result in loss of orthogonality between $2^{m-1}$ chips. Hence, the whole bit of layer $m$ and half the bit of layer $m-1$ will collide with maximum interference, if perfect synchronization is assumed at the receiver.

Define the admissible OVSF set to be the collection of codes that are mutually orthogonal. Orthogonality between codes that belong to different layers is established through repetition of the shorter code. Considering equation (4) at layer $m$, define further the $k$th quadruplet as

$$C_m(k) = \{P_{m-1,k}, Q_{m-1,k}, -Q_{m-1,k}, P_{m-1,k}, Q_{m-1,k}, -Q_{m-1,k}, P_{m-1,k}\},$$

$k = 1, 2, \ldots, 2^{m-2}$. If any code in $C_m(k)$ is assigned to a user, the parent codes $P_{m-1,k}$ and $Q_{m-1,k}$ of layer $m - 1$ can no longer be employed for spreading due to lack of orthogonality with $C_m(k)$. Thus, the use of any code at layer $m$ results in the exclusion of two codes from the admissible OVSF set at the higher-rate layer $m - 1$. Likewise, if either $P_{m-1,k}$ or $Q_{m-1,k}$ is occupied, the entire quadruplet $C_m(k)$ has to be dropped from the admissible OVSF code-set.

In contrast, the OVSF-Walsh construction is such that when a code from layer $m$ is assigned, only one code from layer $m - 1$ is dropped out of the admissible OVSF set, because each sequence at the former layer is generated from a single parent. Similarly, the use of a code at layer $m - 1$ implies the loss of two codes at layer $m$.

Based on the observations above, one can see that when a single parent code is in use, the OVSF-PONS construction produces two fewer admissible sequences at the next layer down. This fact is true when any odd number of parent codes are employed as long as complementary sequences are exhausted first in code assignment. That is, the code allocation strategy should be such that if $P_{m-1,k}$ is already in use, the next assignment should be $Q_{m-1,k}$ so that premature code blocking is avoided. Moreover, for even number of layer-$k$ code allocations, the code capacity of OVSF-PONS is the same as OVSF-Walsh, again provided that complementary pairs are given out together. Generalizing from the preceding discussion, we have the following result.

Proposition: The number of inadmissible OVSF-PONS sequences at layer $j$ per complementary pair of occupied codes (regardless of whether only one or both are in use) at layer $i$, $i < j$, is $2^{j-i+1}$. □

On the other hand, the inadmissible OVSF-Walsh sequences at layer $j$ amount to $2^{j-2}$ per allocated code at layer $i < j$. Depending on the depth of the low-rate codes and whether an odd or even number of high-rate codes are in demand, the capacity difference between OVSF-Walsh and OVSF-PONS may be significant. Next, we propose the duocode OVSF-PONS scheme, which overcomes this problem.

IV. DUOCODE OVSF-PONS

Multicode CDMA satisfies the variable bit rate demands by allocating multiple spreading sequences to each user [11]. The data are converted from high-rate serial to low-rate parallel bit streams, following channel coding and interleaving. Each parallel bit subsequence is spread by a distinct code. The higher the rate or priority required by a transmission, the more codes are assigned to the corresponding unit. Due to its strong suitability for accommodating multimedia communications, multicode WCDMA is under consideration for next generation wireless networking [9].

Considering the OVSF-PONS framework described in the previous section, suppose that a user requires a code from layer $m - 1$. If either $P_{m-1,k}$ or $Q_{m-1,k}$, $k = 1, 2, \ldots, 2^{m-2}$, is occupied, then all codes in $C_m(k)$ and their descendants become inadmissible. On the other
hand, if the user’s data are spread by any of the two complementary layer-$m$ codes in $C_{m+1}(k)$ instead, then the remaining two (or their descendants) can still be employed by others. Therefore, the OVSF-PONS scheme is modified such that any user that requires a layer-$(m-1)$ code is given two layer-$m$ PONS codes depending on the code capacity constraints at the time of signalization. The integrated duocode-OVSF mechanism will henceforth be called duocode OVSF-PONS.

**Lemma 1:** The code capacity of duocode OVSF-PONS equals that of OVSF-Walsh.

**Proof:** For some arbitrary $k$, assume that two complementary layer-$(i+1)$ codes from the set $C_{i+1}(k)$ are assigned to a user instead of a layer-$i$ code, as dictated by the duocode OVSF-PONS paradigm, so that the remaining complementary pair in $C_{i+1}(k)$ is freed. The assigned pair is responsible for the generation of a quadruplet of codes at layer-$(i+2)$, which now become inadmissible. Continuing in this fashion, the number of inadmissible codes at layer $j, j > i+1$, is $2^{j-i}$ per allocated code at layer $i$, which is the same as that of OVSF-Walsh.

In duocode OVSF-PONS with layer-$(m-1)$ spreading, the bits are converted from serial to two parallel streams, each spread by a distinct layer-$m$ complementary code. After summation, there is the option to apply inverse fast Fourier transform for transmission over two subcarriers as in [5]. For binary phase shift-keying (BPSK), it can be easily shown that the low crest factor of PONS is preserved with the duocode structure.

**Lemma 2:** The crest factor of the BPSK-modulated duocode OVSF-PONS signal is bounded by $\sqrt{2}$.

**Proof:** Without loss of generality, suppose that the parallel bits $b_0$ and $b_1$ are spread by the layer-$m$ codes $P_{m,1}$ and $Q_{m,1}$. Then, the duocode OVSF-PONS output is $S(z) = b_0P_{m,1}(z) + b_1Q_{m,1}(z)$, and

\[
||S(z)||_\infty = |b_0P_{m,1}(z) + b_1Q_{m,1}(z)|^2 \\
\leq 2(|P_{m,1}(z)|^2 + |Q_{m,1}(z)|^2) = 2^{m+2}, \quad (5)
\]

\[
||S(z)||_2^2 = 2 \cdot 2^m. \quad (6)
\]

The square-rooted ratio of (5) to (6) gives the crest factor. Note that a similar result is presented in [5] for multicode MC-CDMA with complementary sequences.

The bit rate quality of service (QoS) satisfaction is handled by OVSF coding. The duocode mechanism introduced here is just a part of the OVSF-PONS implementation, and does not serve QoS control purposes (unlike, e.g., the multicode scheme in [16]).

By spreading the two parallel data streams with longer codes, the processing gain is doubled. Hence, duocode OVSF-PONS provides a higher degree of multipath and multiple access interference (MAI) resistance, in addition to the statistical qualities that are inherent in PONS sequences. It is certainly possible to extend duocode OVSF-PONS to a multicore OVSF-PONS structure by replacing a layer-$(m-1)$ code with four codes from layer $m + 1$. In fact, the duocode OVSF-PONS multiuser spread spectrum system becomes a PONS-based multicore CDMA in the limit.

**V. CONCLUSION**

The duocode extension of OVSF-PONS does not compromise the code capacity, while it promises superior performance and power efficiency. The performance gains offered by the duocode OVSF-PONS channelization come at the expense of added complexity due to the serial-to-parallel conversion (and vice versa) and multiple simultaneous spreading, as well as slightly increased PAPR when compared to basic PONS coding.

Protocols for efficient scheduling of the codes among users (e.g., see [12]) are not directly applicable to the duocode OVSF-PONS system. Thus, novel scheduling algorithms that accompany the physical layer design will have to be developed, as well.

**REFERENCES**


