Quadratic Stability Analysis of Fuzzy-Model-Based Control Systems Using Staircase Membership Functions

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Abstract— **This paper presents the stability analysis of fuzzy-model-based control systems. Staircase membership functions are introduced to facilitate the stability analysis. Through the staircase membership functions approximating those of the fuzzy model and fuzzy controller, the information of the membership functions can be brought into the stability analysis. Based on the Lyapunov stability theory, stability conditions in terms of linear matrix inequalities are derived in a simple and easy-to-understand manner to guarantee the system stability. The proposed stability analysis approach offers a nice property that includes the membership functions of both fuzzy model and fuzzy controller in the LMI-based stability conditions for a dedicated fuzzy-model-based control system. Furthermore, the proposed stability analysis approach can be applied to the fuzzy-model-based control systems of which the membership functions of both fuzzy model and fuzzy controller are not necessarily the same. Greater design flexibility is allowed for choosing the membership functions during the design of fuzzy controllers. By employing membership functions with simple structure, it is possible to lower the structural complexity and the implementation cost. Simulation examples are given to illustrate the merits of the proposed approach.**

Index Terms— **Fuzzy Control, Linear Matrix Inequality, Stability Analysis, Staircase Membership Functions, T-S Fuzzy Model**

I. INTRODUCTION

FUZZY-model-based (FMB) control approach [1] offers a systematic and effective way to handle nonlinear control problems. With the powerful Takagi-Sugeno (T-S) fuzzy model [2]-[3], a nonlinear plant can be generally and systematically represented as an average weighted sum of some local linear state-space models. The T-S fuzzy model separates the linear and nonlinear dynamics of the nonlinear plant. This semi-linear property of the T-S fuzzy model allows that some linear analysis and control approaches can be applied to facilitate stability analysis and controller synthesis. The FMB control approach has been applied successfully in various applications such as tracking control [4], chaotic synchronization and communication [5]-[6], regulation of DC-DC switching converters [7] and stabilization of inverted pendulum [8].

Based on the T-S fuzzy model, a fuzzy controller [9] was proposed to close the feedback loop to form a FMB control system. It was shown in [9]-[10] that the FMB control system is guaranteed to be asymptotically stable by a set of linear matrix inequalities (LMIs) [11]. The solution to the LMIs can be found numerically by using convex programming techniques. As the stability analysis in [9]-[10] did not consider the membership functions of both T-S fuzzy model and fuzzy controller, the stability conditions are valid for any arbitrary membership functions and thus very conservative. However, for the same reason, there is no restriction on the design of the membership functions of fuzzy controller. As a result, the implementation cost of the fuzzy controller can be lower by using some simple membership functions. For relaxation of stability analysis, a parallel compensation distribution (PDC) design approach was proposed [12] that the fuzzy controller shares the membership functions of the T-S fuzzy model. Although the conservativeness of the stability analysis can be relaxed compared with that in [9]-[10], the structural complexity of the fuzzy controller may be increased under such a design criterion when the membership functions of the T-S fuzzy model are complex. Under the PDC design, further relaxed stability conditions were achieved in [13]-[20]. Various analysis approaches based on the T-S fuzzy model can also be found in the literature. Stability analysis of the FMB control systems was studied using circle criteria in [21]-[22]. Switching/sliding mode control techniques were employed to analyze the system stability and controller synthesis in [23]-[24]. In [25]-[26], adaptive control technique was combined with the fuzzy logic theory to come up with an adaptive fuzzy control scheme. The parameter values of the fuzzy controller are updated in an online manner to stabilize the nonlinear plant.

In this paper the focus is on the stability analysis of FMB control systems [9]-[10] with state-feedback fuzzy controller. It was revealed in [27]-[31] that the information of membership functions plays an important role for relaxation of stability analysis result. Some constraints on membership functions were proposed [27]-[31] to carry the boundary information of the membership functions into the stability analysis. As a result, the system stability of the FMB control system is guaranteed by the stability conditions for certain sets [27]-[31] of membership functions subject to the

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membership-function constraints, under the PDC design approach. As the membership functions are not included in the stability conditions [13]-[20], [27]-[31], the information of the membership functions is not fully utilized in the stability analysis. In this paper, we aim at offering LMI-based stability conditions including the membership functions of both fuzzy model and fuzzy controller to aid the stable design of a dedicated FMB control system in a simple and easy-to-understand manner. The difficulty on bringing the membership functions into the stability conditions is mainly due to the continuity of the membership functions. When the membership functions are considered, it can be found that the number of LMIs becomes infinity that the solution cannot be found practically using convex programming techniques. In this paper, in order to include the membership functions to the stability conditions, staircase membership functions are employed to approximate the continuous membership functions of the T-S fuzzy model and fuzzy controller. *It is worth mentioning that the staircase membership functions are for stability analysis only and not necessarily implemented physically*. As the staircase membership functions have finite number of discrete values, it circumvents the difficulty by converting the infinite number of LMIs into a finite one. Furthermore, unlike the stability analysis approaches in [13]-[20], [30]-[31], the stability analysis proposed in this paper does not require that the T-S fuzzy model and fuzzy controller shares the same premise membership functions. Consequently, it offers a greater design flexibility for the membership functions of the fuzzy controller. By employing some simple membership functions, the structural complexity and implementation cost of the fuzzy controller can be lower. Based on the Lyapunov stability theory, stability conditions in terms of LMIs are derived to achieve a stable FMB control system.

The organization of this paper is as follows. In section II, the T-S fuzzy model and fuzzy controller are briefly presented. In section III, the system stability of the FMB control system is investigated based on the Lyapunov stability theory through the proposed staircase membership functions. In section IV, simulation examples are given to illustrate the merits of the proposed fuzzy control approach. In section V, a conclusion is drawn.

II. FUZZY MODEL AND FUZZY CONTROLLER

The fuzzy model [2]-[3] and the fuzzy controller are briefly presented in this section. The fuzzy model systematically represents the nonlinear plant in a general framework to facilitate the stability analysis and controller synthesis. A fuzzy controller designed based on the fuzzy model is employed to close the feedback loop to form a FMB control system.

A. Fuzzy Model

Let p be the number of fuzzy rules describing the nonlinear plant. The i-th rule is of the following format:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)$ (1)

where M_α^i is a fuzzy term of rule i corresponding to the known function $f_\alpha(\mathbf{x}(t))$, α = 1, 2, ..., Ψ; i = 1, 2, ..., p; Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices, respectively; $\mathbf{x}(t) \in \Re^n$ is the system state vector and $\mathbf{u}(t) \in \Re^m$ is the input vector. The system dynamics is described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\big) \tag{2}$$

where

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\ w_i(\mathbf{x}(t)) \ge 0 \text{ for all } i \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\Big)} \tag{4}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, α = 1, 2, …, Ψ, are the grades of membership corresponding to the fuzzy terms of M_α^i.

B. Fuzzy Controller

A fuzzy controller with c fuzzy rules is to be designed for the nonlinear plant. The j-th rule of the fuzzy controller is of the following format.

Rule j: IF $g_1(\mathbf{x}(t))$ is N_1^j AND … AND $g_\Omega(\mathbf{x}(t))$ is N_Ω^j

THEN $\mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t)$ (5)

where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, β = 1, 2, ..., Ω; j = 1, 2, ..., c; c is the number of rules; Ω is a positive integer; $\mathbf{G}_j \in \Re^{m\times n}$ is the feedback gain of rule j to be designed. The inferred output of the fuzzy controller is given by,

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{6}$$

where

$$\sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1,\ m_j(\mathbf{x}(t)) \ge 0 \text{ for all } j, \tag{7}$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{\mathrm{N}_1^j}(g_1(\mathbf{x}(t)))\times\mu_{\mathrm{N}_2^j}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{N}_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^{c}\Big(\mu_{\mathrm{N}_1^k}(g_1(\mathbf{x}(t)))\times\mu_{\mathrm{N}_2^k}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{N}_\Omega^k}(g_\Omega(\mathbf{x}(t)))\Big)} \tag{8}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{N}_\beta^j}(g_\beta(\mathbf{x}(t)))$, j = 1, 2, ..., c, are the grades of membership corresponding to the fuzzy terms N_β^j.

III. STABILITY ANALYSIS

The system stability of the FMB control system is investigated using the Lyapunov stability theorem in this section. It can be seen that the stability analysis is very simple and easy-to-understand through the staircase membership functions. Considering the fuzzy model of (2) and the fuzzy controller of (6) connected in a closed loop, the FMB control system is obtained as follows.

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\left(\sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t)\right)\right)$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t) \quad (9)$$

Remark 1: The system stability of the FMB control system is guaranteed by the LMI-based stability conditions in [9]-[10], [27]-[29]. Under a particular case (the PDC design [12]), when $c = p$ and $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ for all i, relaxed stability conditions can be found in [12]-[20], [30]-[31]. Under the PDC design approach, the membership functions of the fuzzy model are required for implementation of the fuzzy controller. As a result, the design flexibility of the membership functions for the fuzzy controller vanishes. Furthermore, it may increase the implementation cost when the membership functions are complex.

The system stability of the FMB control system of (9) is investigated by the following quadratic Lyapunov function candidate.

$$V(t) = \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \quad (10)$$

where $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n}$ and $\mathbf{P} > 0$. It can be shown below that $\dot{V}(t) \le 0$ (equality holds when $\mathbf{x}(t) = \mathbf{0}$) is guaranteed by satisfaction of some LMIs which implies the asymptotic stability of the FMB control system (i.e., $\mathbf{x}(t) \to \mathbf{0}$ as time $t \to \infty$). For brevity, $w_i(\mathbf{x}(t))$ and $m_j(\mathbf{x}(t))$ are denoted as w_i and m_j, respectively. In the following analysis, the equality of $\sum_{i=1}^{p} w_i = \sum_{j=1}^{c} m_j = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j = 1$ given by the properties of the membership functions in (3) and (7) is utilized to facilitate the stability analysis. From (9) and (10), we have,

$$\dot{V}(t) = \dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}(t)$$
$$= \left(\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t)\right)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t)$$
$$+\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t)$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{x}(t)^{\mathrm{T}}\left(\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\right)\mathbf{x}(t) \quad (11)$$

Denote $\mathbf{X} = \mathbf{P}^{-1}$, $\mathbf{z}(t) = \mathbf{X}^{-1}\mathbf{x}(t)$ and $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-1}$ where $\mathbf{N}_j \in \Re^{m\times n}$, j = 1, 2, ..., c, are arbitrary matrices to be determined. From (11), we have,

$$\dot{V}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{A}_i\mathbf{X} + \mathbf{X}\mathbf{A}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{N}_j + \mathbf{N}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}\right)\mathbf{z}(t)$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{c} \overline{w}_i\overline{m}_j \mathbf{z}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i\overline{m}_j\right)\mathbf{z}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\mathbf{z}(t) \quad (12)$$

where $\mathbf{Q}_{ij} = \mathbf{A}_i\mathbf{X} + \mathbf{X}\mathbf{A}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{N}_j + \mathbf{N}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}$; $\overline{w}_i$ and $\overline{m}_j$ are staircase membership functions approximating the continuous w_i and m_j, respectively, to facilitate the stability analysis. The staircase membership functions $\overline{w}_i$ and $\overline{m}_j$ are proposed in a way that they satisfy the properties of membership functions in (3) and (7), respectively, namely, $\overline{w}_i \in [0,\ 1]$, $\overline{m}_j \in [0,\ 1]$, $\sum_{i=1}^{p}\overline{w}_i = 1$ and $\sum_{j=1}^{c}\overline{m}_j = 1$ which lead to $\sum_{i=1}^{p}\sum_{j=1}^{c}\overline{w}_i\overline{m}_j = 1$. *It should be noted that $\overline{w}_i$ and $\overline{m}_j$ are introduced for stability analysis only and not necessarily implemented physically.* An example of the continuous and staircase membership functions is shown in Fig. 1. It can be seen that the continuous membership function is approximated by a staircase membership function with finite number of levels.

Remark 2: It can be seen from (12) that the inequality of $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{Q}_{ij} < 0$ is the necessary and sufficient stability condition to ensure $\dot{V}(t) \le 0$ (equality holds when $\mathbf{z}(t) = \mathbf{0}$) which implies the asymptotic stability of the FMB control system of (9). However, as w_i and m_j are continuous membership functions, the inequality of $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{Q}_{ij} < 0$ contains an infinite number of LMIs (each LMI is corresponding to a single value of w_i and m_j) that the solution cannot be solved practically using convex programming techniques. Instead of investigating $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{Q}_{ij} < 0$ and to make sure that the inequality is satisfied, it was proposed in [9]-[10] that $\mathbf{Q}_{ij} < 0$ for all i and j are the stability conditions. Indeed, $\mathbf{Q}_{ij} < 0$ for all i and j will satisfy $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{Q}_{ij} < 0$, however, for any shapes of membership functions. As the membership functions of w_i and m_j are not considered, the stability conditions [9]-[10] are thus very conservative.

Remark 3: The staircase membership functions of $\overline{w}_i$ and $\overline{m}_j$ are chosen in a way that they consist of finite number of discrete values to approximate the continuous membership functions w_i and m_j. Consequently, referring to (12), the staircase membership functions $\overline{w}_i$ and $\overline{m}_j$ can be regarded as some sampled points of the continuous membership functions of w_i and m_j. It can be seen from (12) that if $w_i m_j - \overline{w}_i\overline{m}_j \approx 0$, $\dot{V}(t) \le 0$ is mainly determined by $\sum_{i=1}^{p}\sum_{j=1}^{c}\overline{w}_i\overline{m}_j\mathbf{Q}_{ij} < 0$ which can be regarded as the approximation of $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{Q}_{ij} < 0$. As the staircase membership functions of $\overline{w}_i$ and $\overline{m}_j$ consist of finite number of discrete values, the inequality of $\sum_{i=1}^{p}\sum_{j=1}^{c} \overline{w}_i \overline{m}_j \mathbf{Q}_{ij} < 0$ contains finite number of LMIs. Furthermore, the inequality of $\sum_{i=1}^{p}\sum_{j=1}^{c} \overline{w}_i \overline{m}_j \mathbf{Q}_{ij} < 0$ consists of the staircase membership functions dedicated to the fuzzy model and fuzzy controller but not for any shapes of membership functions. It can be seen that the idea is simple and the novelty of the proposed analysis approach is to investigate $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{Q}_{ij} < 0$ using the staircase membership functions which turn the infinite LMI stability conditions into finite ones. The above presents the concept and motivation of the proposed stability analysis approach through the staircase membership functions. In the following, stability analysis is carried out mathematically based on the Lyapunov stability theory.

To facilitate the stability analysis, the property of the membership functions is utilized to introduce some slack matrices. Based on the property of the membership functions in (3) and (7), we have the following equality.

$$\sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i \overline{m}_j\right)\mathbf{M} = \mathbf{0}, \tag{13}$$

where $\mathbf{M} = \mathbf{M}^{\mathrm{T}} \in \Re^{n \times n}$ is an arbitrary matrix.

Proof: Expanding the terms in (13) and utilizing the equality of $\sum_{i=1}^{p} w_i = \sum_{i=1}^{p} \overline{w}_i = 1$ and $\sum_{j=1}^{c} m_j = \sum_{j=1}^{c} \overline{m}_j = 1$, we have $\sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i \overline{m}_j\right)\mathbf{M} = \left(\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j - \sum_{i=1}^{p}\sum_{j=1}^{c} \overline{w}_i \overline{m}_j\right)\mathbf{M} = (1-1)\mathbf{M} = \mathbf{0}$. **QED**

Furthermore, we consider $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{W}_{ij} \geq 0$ where $\mathbf{W}_{ij} = \mathbf{W}_{ij}^{\mathrm{T}} \in \Re^{n \times n}$ and $\mathbf{W}_{ij} \geq 0$, and $w_i m_j - \overline{w}_i \overline{m}_j - \gamma_{ij} \geq 0$ for all *i* and *j* where γ_{ij} is a scalar to be determined. From (12) and (13), and with the just mentioned inequalities, we have,

$$\begin{aligned}
\dot{V}(t) \leq\ & \sum_{i=1}^{p}\sum_{j=1}^{c} \overline{w}_i \overline{m}_j \mathbf{z}(t)^{\mathrm{T}} \mathbf{Q}_{ij} \mathbf{z}(t) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i \overline{m}_j\right)\mathbf{z}(t)^{\mathrm{T}} \mathbf{Q}_{ij} \mathbf{z}(t) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i \overline{m}_j\right)\mathbf{z}(t)^{\mathrm{T}} \mathbf{M} \mathbf{z}(t) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{z}(t)^{\mathrm{T}} \mathbf{W}_{ij} \mathbf{z}(t) \\
=\ & \sum_{i=1}^{p}\sum_{j=1}^{c} \overline{w}_i \overline{m}_j \mathbf{z}(t)^{\mathrm{T}} \mathbf{Q}_{ij} \mathbf{z}(t) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i \overline{m}_j\right)\mathbf{z}(t)^{\mathrm{T}} \left(\mathbf{Q}_{ij} + \mathbf{M}\right)\mathbf{z}(t) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i \overline{m}_j + \overline{w}_i \overline{m}_j\right)\mathbf{z}(t)^{\mathrm{T}} \mathbf{W}_{ij} \mathbf{z}(t) \\
=\ & \sum_{i=1}^{p}\sum_{j=1}^{c} \overline{w}_i \overline{m}_j \mathbf{z}(t)^{\mathrm{T}} \left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right)\mathbf{z}(t) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i \overline{m}_j\right)\mathbf{z}(t)^{\mathrm{T}} \left(\mathbf{Q}_{ij} + \mathbf{W}_{ij} + \mathbf{M}\right)\mathbf{z}(t) \\
=\ & \sum_{i=1}^{p}\sum_{j=1}^{c} \overline{w}_i \overline{m}_j \mathbf{z}(t)^{\mathrm{T}} \left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right)\mathbf{z}(t) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i \overline{m}_j\right)\mathbf{z}(t)^{\mathrm{T}} \left(\mathbf{Q}_{ij} + \mathbf{W}_{ij} + \mathbf{M}\right)\mathbf{z}(t) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c}\left(\gamma_{ij} - \gamma_{ij}\right)\mathbf{z}(t)^{\mathrm{T}} \left(\mathbf{Q}_{ij} + \mathbf{W}_{ij} + \mathbf{M}\right)\mathbf{z}(t) \\
=\ & \sum_{i=1}^{p}\sum_{j=1}^{c} \mathbf{z}(t)^{\mathrm{T}} \left(\left(\overline{w}_i \overline{m}_j + \gamma_{ij}\right)\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right) + \gamma_{ij}\mathbf{M}\right)\mathbf{z}(t) \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{c}\left(w_i m_j - \overline{w}_i \overline{m}_j - \gamma_{ij}\right)\mathbf{z}(t)^{\mathrm{T}} \left(\mathbf{Q}_{ij} + \mathbf{W}_{ij} + \mathbf{M}\right)\mathbf{z}(t)
\end{aligned} \tag{14}$$

It can be seen from (14) that $\dot{V}(t) \leq 0$ (equality holds for $\mathbf{z}(t) = \mathbf{x}(t) = \mathbf{0}$) can be achieved if $\sum_{i=1}^{p}\sum_{j=1}^{c}\left(\left(\overline{w}_i \overline{m}_j + \gamma_{ij}\right)\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right) + \gamma_{ij}\mathbf{M}\right) < 0$ for all discrete values of $\overline{w}_i$ and $\overline{m}_j$ and $\mathbf{Q}_{ij} + \mathbf{W}_{ij} + \mathbf{M} < 0$ for all *i* and *j*. It implies that the FMB control system of (9) is guaranteed to be asymptotically stable, i.e., $\mathbf{x}(t) \to \mathbf{0}$ when time $t \to \infty$. The stability analysis result is summarized in the following theorem.

Theorem 1: *The fuzzy-model-based control system of* (9), *formed by the nonlinear plant represented by the fuzzy model in the form of* (2) *and the fuzzy controller in the form of* (6) *connected in a closed loop, is guaranteed to be asymptotically stable if there exist pre-defined scalars* γ_{ij} *satisfying* $w_i(\mathbf{x}(t))m_j(\mathbf{x}(t)) - \overline{w}_i(\mathbf{x}(t))\overline{m}_j(\mathbf{x}(t)) - \gamma_{ij} \geq 0$ *and matrices* $\mathbf{M} = \mathbf{M}^{\mathrm{T}} \in \Re^{n \times n}$, $\mathbf{N}_j \in \Re^{m \times n}$, $\mathbf{W}_{ij} = \mathbf{W}_{ij}^{\mathrm{T}} \in \Re^{n \times n}$ *and* $\mathbf{X} = \mathbf{X}^{\mathrm{T}} \in \Re^{n \times n}$ *such that the following LMIs are satisfied.*

$\mathbf{X} > 0$; $\mathbf{W}_{ij} \geq 0$, *i = 1, 2, ..., p; j =1, 2, ..., c;*

$\sum_{i=1}^{p}\sum_{j=1}^{c}\left(\left(\overline{w}_i \overline{m}_j + \gamma_{ij}\right)\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right) + \gamma_{ij}\mathbf{M}\right) < 0$ *for all valid discrete values of* $\overline{w}_i$ *and* $\overline{m}_j$;

$\mathbf{Q}_{ij} + \mathbf{W}_{ij} + \mathbf{M} < 0$, *i = 1, 2, ..., p; j =1, 2, ..., c;*

where the feedback gains are defined as $\mathbf{G}_j = \mathbf{N}_j \mathbf{X}^{-1}$, *j =1, 2, ..., c.*

Remark 4: There must exist the values of γ_{ij} satisfying the inequalities of $w_i m_j - \overline{w}_i\overline{m}_j - \gamma_{ij} \geq 0$ for all i and j. When the forms of w_i, m_j, $\overline{w}_i$ and $\overline{m}_j$ are known, the values of γ_{ij} can be found numerically or analytically.

Remark 5: It can be shown that the solution ($\mathbf{X}$ and $\mathbf{N}_j$) in [9]-[10] is also the solution of the stability conditions in Theorem 1. It was reported in [9]-[10] that a stable FMB control system is asymptotically stable if there exist $\mathbf{X}$ and $\mathbf{N}_j$ such that $\mathbf{X} > 0$ and $\mathbf{Q}_{ij} = \mathbf{A}_i\mathbf{X} + \mathbf{X}\mathbf{A}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{N}_j + \mathbf{N}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} < 0$ for all i and j under the case that the fuzzy model and fuzzy controller do not share the same premise membership functions. From Theorem 1, choosing $\mathbf{W}_{ij} = \mathbf{0}$ for all i and j and $\mathbf{M} = -\varepsilon\mathbf{I} < 0$ where $\varepsilon > 0$ is a scalar, the LMIs in Theorem 1 become $\sum_{i=1}^{p}\sum_{j=1}^{c}\overline{w}_i\overline{m}_j\mathbf{Q}_{ij} + \sum_{i=1}^{p}\sum_{j=1}^{c}\gamma_{ij}\left(\mathbf{Q}_{ij} - \varepsilon\mathbf{I}\right) < 0$ and $\mathbf{Q}_{ij} - \varepsilon\mathbf{I} < 0$ for all i and j. As $\mathbf{Q}_{ij} < 0$ for all i and j, the second LMI is satisfied and there must exist a sufficiently small value of γ_{ij} such that the first LMI is satisfied. As the staircase membership functions of $\overline{w}_i$ and $\overline{m}_j$ can be chosen arbitrarily, they can be chosen such that $w_i m_j - \overline{w}_i\overline{m}_j \geq \gamma_{ij}$ is satisfied by a sufficiently small value of γ_{ij}. Hence, it can be seen that the solution of the stability conditions in [9]-[10] is also the solution in Theorem 1. However, the solution of stability conditions in Theorem 1 might not be that in [9]-[10]. Furthermore, we consider the stability conditions in [27]-[29] which require that $c = p$. If there exists a solution to the stability conditions in [27]-[29], it implies that $\sum_{i=1}^{p}\sum_{j=1}^{p}w_i m_j\mathbf{Q}_{ij} < 0$. In this case, considering the stability conditions in Theorem 1, we have $\sum_{i=1}^{p}\sum_{j=1}^{p}\overline{w}_i\overline{m}_j\mathbf{Q}_{ij} < 0$ as $\overline{w}_i$ and $\overline{m}_j$ can be regarded as the sampled points of w_i and m_j, respectively. Choosing $\mathbf{W}_{ij} = 0$ for all i and j, and $\mathbf{M} = -\varepsilon\mathbf{I} < 0$ where $\varepsilon > 0$ is a scalar, the LMI of $\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\left(\overline{w}_i\overline{m}_j + \gamma_{ij}\right)\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right) + \gamma_{ij}\mathbf{M}\right) < 0$ in Theorem 1 becomes $\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\left(\overline{w}_i\overline{m}_j + \gamma_{ij}\right)\mathbf{Q}_{ij} - \gamma_{ij}\varepsilon\mathbf{I}\right) = \sum_{i=1}^{p}\sum_{j=1}^{p}\overline{w}_i\overline{m}_j\mathbf{Q}_{ij} + \sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_{ij}\left(\mathbf{Q}_{ij} - \varepsilon\mathbf{I}\right) < 0$ which is satisfied with a sufficiently small value of γ_{ij} subject to $\sum_{i=1}^{p}\sum_{j=1}^{p}\overline{w}_i\overline{m}_j\mathbf{Q}_{ij} < 0$. As the staircase membership functions of $\overline{w}_i$ and $\overline{m}_j$ can be chosen arbitrarily, they can be chosen such that $w_i m_j - \overline{w}_i\overline{m}_j \geq \gamma_{ij}$ is satisfied by a sufficiently small value of γ_{ij}. Similarly, the LMI of $\mathbf{Q}_{ij} + \mathbf{W}_{ij} + \mathbf{M} < 0$ in Theorem 1 becomes $\mathbf{Q}_{ij} - \varepsilon\mathbf{I} < 0$ which is satisfied by choosing a sufficiently large positive value of ε. Hence, it can be seen that the solution of the stability conditions in [27]-[29] is also the solution of the proposed ones but may not be the other way round.

In the following, we consider the PDC design [12] to further relax the stability analysis. Under the PDC design, we choose $c = p$ and $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$ for all i. To investigate the system stability of (9) under the PDC design approach, we proceed from (14) with $\mathbf{W}_{ij} = \mathbf{W}_{ji}^{\mathrm{T}}$ and $\gamma_{ij} = \gamma_{ji}$ for all i and j and rewrite (14) as follows.

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{p}\sum_{j=1}^{p}\mathbf{z}(t)^{\mathrm{T}}\left(\left(\overline{w}_i\overline{w}_j + \gamma_{ij}\right)\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right) + \gamma_{ij}\mathbf{M}\right)\mathbf{z}(t) \\
&\quad + \sum_{i=1}^{p}\sum_{j=1}^{p}\left(w_i w_j - \overline{w}_i\overline{w}_j - \gamma_{ij}\right)\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij} + \mathbf{M}\right)\mathbf{z}(t) \\
&= \sum_{i=1}^{p}\sum_{j=1}^{p}\mathbf{z}(t)^{\mathrm{T}}\left(\left(\overline{w}_i\overline{w}_j + \gamma_{ij}\right)\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right) + \gamma_{ij}\mathbf{M}\right)\mathbf{z}(t) \\
&\quad + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\left(w_i w_j - \overline{w}_i\overline{w}_j - \gamma_{ij}\right) \\
&\quad \times\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij} + 2\mathbf{M} + \mathbf{Q}_{ji} + \mathbf{W}_{ji}\right)\mathbf{z}(t)
\end{aligned} \tag{15}$$

It is required that the inequality of $\sum_{i=1}^{p}\sum_{j=1}^{p}w_i w_j\mathbf{W}_{ij} \geq 0$ holds which can be written as $\begin{bmatrix} w_1\mathbf{I} \\ w_2\mathbf{I} \\ \vdots \\ w_p\mathbf{I}\end{bmatrix}^{\mathrm{T}}\mathbf{W}\begin{bmatrix} w_1\mathbf{I} \\ w_2\mathbf{I} \\ \vdots \\ w_p\mathbf{I}\end{bmatrix} \geq 0$ where $\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} & \cdots & \mathbf{W}_{1p} \\ \mathbf{W}_{21} & \mathbf{W}_{22} & \cdots & \mathbf{W}_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{W}_{p1} & \mathbf{W}_{p2} & \cdots & \mathbf{W}_{pp}\end{bmatrix}$. It can be seen that $\mathbf{W} \geq 0$ implies $\sum_{i=1}^{p}\sum_{j=1}^{p}w_i w_j\mathbf{W}_{ij} \geq 0$. From (15), we have $\dot{V}(t) \leq 0$ (equality holds for $\mathbf{z}(t) = \mathbf{x}(t) = \mathbf{0}$)) when $\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\left(\overline{w}_i\overline{w}_j + \gamma_{ij}\right)\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right) + \gamma_{ij}\mathbf{M}\right) < 0$ for all discrete values of $\overline{w}_i$ and $\mathbf{Q}_{ij} + \mathbf{W}_{ij} + 2\mathbf{M} + \mathbf{Q}_{ji} + \mathbf{W}_{ji} < 0$ for all i and j. The stability analysis result under the PDC design is summarized in the following theorem.

Theorem 2: *The fuzzy-model-based control system of* (9), *formed by the nonlinear plant represented by the fuzzy model in the form of* (2) *and the fuzzy controller in the form of* (6) *sharing the same premise membership functions (i.e., $c = p$, $m_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t))$, $i = 1, 2, \ldots, p$) connected in a closed loop, is guaranteed to be asymptotically stable if there exist pre-defined scalars $\gamma_{ij} = \gamma_{ji}$ satisfying $w_i(\mathbf{x}(t))w_j(\mathbf{x}(t)) - \overline{w}_i(\mathbf{x}(t))\overline{w}_j(\mathbf{x}(t)) - \gamma_{ij} \geq 0$ and matrices $\mathbf{M} = \mathbf{M}^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{N}_j \in \Re^{m\times n}$, $\mathbf{W}_{ij} = \mathbf{W}_{ji}^{\mathrm{T}} \in \Re^{n\times n}$ and $\mathbf{X} = \mathbf{X}^{\mathrm{T}} \in \Re^{n\times n}$ such that the following LMIs are satisfied.*

$\mathbf{X} > 0$; $\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} & \cdots & \mathbf{W}_{1p} \\ \mathbf{W}_{21} & \mathbf{W}_{22} & \cdots & \mathbf{W}_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{W}_{p1} & \mathbf{W}_{p2} & \cdots & \mathbf{W}_{pp} \end{bmatrix} \geq 0$;

$\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\left(\overline{w}_i\overline{w}_j + \gamma_{ij}\right)\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right) + \gamma_{ij}\mathbf{M}\right) < 0$ *for all valid discrete values of* $\overline{w}_i$;

$\mathbf{Q}_{ij} + \mathbf{W}_{ij} + 2\mathbf{M} + \mathbf{Q}_{ji} + \mathbf{W}_{ji} < 0$, *i, j = 1, 2, ..., p;*

where the feedback gains are defined as $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-1}$, *j =1, 2, ..., p.*

Remark 6: By following the same line of logic in Remark 5, it can be shown that the solution of the existing stability conditions in [12]-[20], [30]-[31] is also that of Theorem 2. If there exists a solution for the stability conditions in [12]-[20], [30]-[31] it implies that $\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{Q}_{ij} < 0$. As $\overline{w}_i$ can be regarded as the sampled points of w_i, it is obvious that $\sum_{i=1}^{p}\sum_{j=1}^{p}\overline{w}_i\overline{w}_j\mathbf{Q}_{ij} < 0$. We choose $\mathbf{W}_{ij} = \mathbf{0}$ for all i and j, $\mathbf{M} = -\varepsilon\mathbf{I} < 0$ where $\varepsilon > 0$ is scalar, and $\overline{w}_i$ such that $w_i w_j - \overline{w}_i\overline{w}_j \geq \gamma_{ij}$ with a sufficiently small value of γ_{ij}. It can be seen that the LMI of $\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\left(\overline{w}_i\overline{w}_j + \gamma_{ij}\right)\left(\mathbf{Q}_{ij} + \mathbf{W}_{ij}\right) + \gamma_{ij}\mathbf{M}\right) < 0$ in Theorem 2 becomes $\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\left(\overline{w}_i\overline{w}_j + \gamma_{ij}\right)\mathbf{Q}_{ij} - \gamma_{ij}\varepsilon\mathbf{I}\right) = \sum_{i=1}^{p}\sum_{j=1}^{p}\overline{w}_i\overline{w}_j\mathbf{Q}_{ij} + \sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_{ij}\left(\mathbf{Q}_{ij} - \varepsilon\mathbf{I}\right) < 0$ which is satisfied by a sufficiently small value of γ_{ij} subject to $\sum_{i=1}^{p}\sum_{j=1}^{p}\overline{w}_i\overline{w}_j\mathbf{Q}_{ij} < 0$. Similarly, the LMI stability condition of $\mathbf{Q}_{ij} + \mathbf{W}_{ij} + 2\mathbf{M} + \mathbf{Q}_{ji} + \mathbf{W}_{ji} < 0$ in Theorem 2 becomes $\mathbf{Q}_{ij} - 2\varepsilon\mathbf{I} + \mathbf{Q}_{ji} < 0$ which is satisfied by choosing a sufficiently large positive value of ε. Hence, it can be seen that the solution of the stability conditions in [12]-[20], [30]-[31] is also that of the proposed ones. However, the solution of stability conditions in Theorem 2 might not be those in [12]-[20], [30]-[31].

Remark 7: In this paper, only the system stability is considered. The system performance can be realized by employing the LMI-based performance conditions in [28]. In [28], the scalar cost function $J = \int_0^\infty \begin{bmatrix}\mathbf{x}(t) \\ \mathbf{u}(t)\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2\end{bmatrix} \begin{bmatrix}\mathbf{x}(t) \\ \mathbf{u}(t)\end{bmatrix} dt$, where $\mathbf{J}_1 = \mathbf{J}_1^{\mathrm{T}} \in \Re^{n\times n} > 0$ and $\mathbf{J}_2 = \mathbf{J}_2^{\mathrm{T}} \in \Re^{m\times m} > 0$ are predefined weighting matrices, is employed to measure quantitatively the system performance. LMI-based performance conditions [28] were derived to attenuate the scalar performance index J to a prescribed level of η.

IV. SIMULATION EXAMPLES

Three simulation examples are given to illustrate the merits of the proposed stability conditions.

A. Simulation Example 1

In this simulation example, a 3-rule fuzzy model in the form of (2) is considered and a 3-rule fuzzy controller in the form of (6) is employed to close the feedback loop. The membership functions of the fuzzy model and fuzzy controller are considered to be different. Consider a fuzzy model in the form of (2) with the following 3 rules [19].

Rule i: IF $x_1(t)$ is M_1^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$, $i = 1, 2, 3$ (16)

where $\mathbf{A}_1 = \begin{bmatrix}1.59 & -7.29 \\ 0.01 & 0\end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix}0.02 & -4.64 \\ 0.35 & 0.21\end{bmatrix}$, $\mathbf{A}_3 = \begin{bmatrix}-a & -4.33 \\ 0 & 0.05\end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix}1 \\ 0\end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix}8 \\ 0\end{bmatrix}$, $\mathbf{B}_3 = \begin{bmatrix}-b+6 \\ -1\end{bmatrix}$, $2 \leq a \leq 9$, $2 \leq b \leq 22$. The membership functions of the fuzzy model are defined as follows and shown graphically in Fig. 2.

$$w_1(x_1(t)) = \begin{cases} 1 & \text{for } x_1(t) < -10 \\ \dfrac{-x_1(t)+2}{12} & \text{for } -10 \leq x_1(t) \leq 2 \\ 0 & \text{for } x_1(t) > 2 \end{cases} \quad (17)$$

$$w_2(x_1(t)) = 1 - w_1(x_1(t)) - w_3(x_1(t)) \quad (18)$$

$$w_3(x_1(t)) = \begin{cases} 0 & \text{for } x_1(t) < -2 \\ \dfrac{x_1(t)+2}{12} & \text{for } -2 \leq x_1(t) \leq 10 \\ 1 & \text{for } x_1(t) > 10 \end{cases} \quad (19)$$

A fuzzy controller in the form of (6) with the following 3 rules is employed to stabilize the fuzzy model of (16).

Rule j: IF $x_1(t)$ is N_1^j

THEN $u(t) = \mathbf{G}_j\mathbf{x}(t)$, $i = 1, 2, 3$ (20)

The membership functions of the fuzzy controller are defined as follows and shown graphically in Fig. 2.

$$m_1(x_1(t)) = 1 - \frac{1}{1 + e^{-\frac{x_1(t)+4}{2}}} \quad (21)$$

$$m_2(x_1(t)) = 1 - m_1(x_1(t)) - m_3(x_1(t)) \quad (22)$$

$$m_3(x_1(t)) = \frac{1}{1 + e^{-\frac{x_1(t)-4}{2}}} \quad (23)$$

The staircase membership functions are employed to approximate the continuous membership functions of both fuzzy model and fuzzy controller. The staircase membership functions are chosen as $\overline{w}_i(x_1(t)) = w_i(h\delta)$ and $\overline{m}_i(x_1(t)) = m_i(h\delta)$ for $(h - 0.5)\delta < x_1(t) \leq (h + 0.5)\delta$ where $i = 1, 2, 3$ and $h = -\infty, \ldots -10, -9, \ldots 10 \ldots, \infty$. As the grades of membership for $x_1(t) > 10$ or $x_1(t) < -10$ keep constant (for example, $w_1(x_1(t)) = w_1(10)$ for $x_1(t) > 10$), we only need to consider $h\delta = -10, -9, \ldots 10$.

The system stability of the FMB control system is examined using the stability conditions in Theorem 1 with the help of MATLAB LMI toolbox. For demonstration purposes, choosing $\delta = 0.1$ and $\delta = 0.05$, the stability regions are shown in Fig. 3 indicated by 'o' and '×', respectively. It can be found numerically that $\gamma_{11} = -0.006327$, $\gamma_{12} = -0.003277$, $\gamma_{13} = -0.001006$, $\gamma_{21} = -0.003955$, $\gamma_{22} = -0.005526$, $\gamma_{23} = -0.003955$, $\gamma_{31} = -0.001060$, $\gamma_{32} = -0.003279$ and $\gamma_{33} = -0.006326$ for $\delta = 0.1$ and $\gamma_{11} = -0.003164$, $\gamma_{12} = -0.001642$, $\gamma_{13} = -0.000545$, $\gamma_{21} = -0.001981$, $\gamma_{22} = -0.002763$, $\gamma_{23} = -0.001966$, $\gamma_{31} = -0.000545$, $\gamma_{32} = -0.001642$ and $\gamma_{33} = -0.003164$ for $\delta = 0.05$ which satisfy the inequalities of $w_i(x_1(t))m_j(x_1(t)) - \overline{w}_i(x_1(t))\overline{m}_j(x_1(t)) - \gamma_{ij} \geq 0$ for all i and j. It is revealed from Fig. 3 that a smaller value of δ is able to produce a larger stability region as the staircase membership functions are able to better approximate their corresponding continuous membership with smaller difference. *It should be noted that the staircase membership functions are not necessarily implemented physically and for stability analysis only.* For comparison purposes, the stability conditions [9]-[10], [27]-[29] for FMB control systems with fuzzy model and fuzzy controller not sharing the same membership functions are employed to check for the system stability. However, there is no stability region found with the stability conditions in [9]-[10], [27]-[29]. It can be seen from Fig. 3 that the stability conditions in Theorem 1 are more relaxed comparatively in terms of larger stability region. The simulation result also complies with Remark 5 that the solution of the stability conditions in [9]-[10], [27]-[29] is also that of the stability conditions in Theorem 1. It should be noted that the stability conditions in [12]-[20] (which require that both fuzzy model and fuzzy controller sharing the same premise membership functions) are not applicable to the FMB control system considered in this simulation example.

B. Simulation Example 2

In order to apply the stability conditions in [12]-[20], we consider the same fuzzy model of (16) and the fuzzy controller of (20), and both of them share the same membership functions defined in (17)-(19). The staircase membership functions are chosen as $\overline{w}_i(x_1(t)) = w_i(h\delta)$ for $(h - 0.5)\delta < x_1(t) \leq (h + 0.5)\delta$ where $i = 1, 2, 3$ and $h = -\infty, \ldots -10, -9, \ldots 10 \ldots, \infty$.

Choosing $\delta = 0.1$ and $\delta = 0.05$, the stability regions given by the stability conditions in Theorem 2 are shown in Fig. 4 indicated by 'o' and '×', respectively. It can be found numerically that $\gamma_{11} = -0.008177$, $\gamma_{12} = \gamma_{21} = -0.004115$, $\gamma_{13} = \gamma_{31} = -0.001337$, $\gamma_{22} = -0.005538$, $\gamma_{23} = \gamma_{32} = -0.004115$ and $\gamma_{33} = -0.008247$ for $\delta = 0.1$ and $\gamma_{11} = -0.004145$, $\gamma_{12} = \gamma_{21} = -0.002070$, $\gamma_{13} = \gamma_{31} = -0.000681$, $\gamma_{22} = -0.002773$, $\gamma_{23} = \gamma_{32} = -0.002018$ and $\gamma_{33} = -0.004162$ for $\delta = 0.05$ which satisfy the inequalities of $w_i(x_1(t))w_j(x_1(t)) - \overline{w}_i(x_1(t))\overline{w}_j(x_1(t)) - \gamma_{ij} \geq 0$ for all i and j. It can be seen that a smaller value of δ is able to produce a larger stability region. For comparison purposes, the stability conditions in [20] (with the parameter $d = 4$) and [30]-[31] (the upper bounds of the products of membership functions are used) are employed to check for the stability region for the same FMB control system and the stability regions given by different stability conditions are shown in Fig. 5. It was reported in [20] that the stability conditions in [20] are superior to those in [12]-[19]. Hence, only the stability region given by [20] is shown but not those by [12]-[19]. It can be seen from Fig. 4 and Fig. 5 that the proposed stability conditions in Theorem 2 with $\delta = 0.05$ are able to produce a larger stability region. The simulation result also complies with Remark 6 that the solution of the stability conditions in [12]-[20], [30]-[31] is also that of the stability conditions in Theorem 2 with a proper design of the staircase membership functions.

C. Simulation Example 3

In this example, we consider the stabilization of an inverted pendulum on a cart to illustrate the stabilizability of the fuzzy controller with the support of the stability conditions in Theorem 1 and Theorem 2. The inverted pendulum on a cart is described by the following dynamic equations [8].

$$\dot{x}_1(t) = x_2(t) \tag{24}$$

$$\dot{x}_2(t) = \frac{\begin{pmatrix} -F_1(M+m)x_2(t) - m^2l^2x_2(t)^2\sin x_1(t)\cos x_1(t) \\ +F_0mlx_4(t)\cos x_1(t) \\ +(M+m)mgl\sin x_1(t) - ml\cos x_1(t)u(t) \end{pmatrix}}{(M+m)(J+ml^2) - m^2l^2(\cos x_1(t))^2} \tag{25}$$

$$\dot{x}_3(t) = x_4(t) \tag{26}$$

$$\dot{x}_4(t) = \frac{\begin{pmatrix} F_1mlx_2(t)\cos x_1(t) \\ +(J+ml^2)mlx_2(t)^2\sin x_1(t) - F_0(J+ml^2)x_4(t) \\ -m^2gl^2\sin x_1(t)\cos x_1(t) + (J+ml^2)u(t) \end{pmatrix}}{(M+m)(J+ml^2) - m^2l^2(\cos x_1(t))^2} \tag{27}$$

where $x_1(t)$ and $x_2(t)$ denote the angular displacement (rad) and the angular velocity (rad/s) of the pendulum from vertical respectively, $x_3(t)$ and $x_4(t)$ denote the displacement (m) and the velocity (m/s) of the cart respectively, $g = 9.8$ m/s^2 is the acceleration due to gravity, $m = 0.22$ kg is the mass of the pendulum, $M = 1.3282$ kg is the mass of the cart, $l = 0.304$ m is the length from the center of mass of the pendulum to the shaft axis, $J = ml^2/3$ kgm^2 is the moment of inertia of the pendulum around the center of mass, $F_0 = 22.915$ N/m/s and $F_1 = 0.007056$ N/rad/s are the friction factors of the cart and the pendulum respectively, and u(t) is the force (N) applied to the cart.

The control objective is to balance the pole and drive the cart to the origin, i.e., $x_k(t) \to 0$, $k = 1, 2, 3, 4$, as time $t \to \infty$. It was reported in [8] that the nonlinear plant can be modelled by the fuzzy model in the form of (2) with the following 2 rules.

Rule i: IF $x_1(t)$ is M_1^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_iu(t)$, $i = 1, 2$ (28)

where $\mathbf{x}(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]^T$;

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (M+m)mgl/a_1 & -F_1(M+m)/a_1 & 0 & F_0 ml/a_1 \\ 0 & 0 & 0 & 1 \\ -m^2gl^2/a_1 & F_1 ml/a_1 & 0 & -F_0(J+ml^2)/a_1 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{3\sqrt{3}}{2\pi}(M+m)mgl/a_2 & -F_1(M+m)/a_2 & 0 & F_0 ml\cos(\pi/3)/a_2 \\ 0 & 0 & 0 & 1 \\ -\frac{3\sqrt{3}}{2\pi}m^2gl^2\cos(\pi/3)/a_2 & F_1 ml\cos(\pi/3)/a_2 & 0 & -F_0(J+ml^2)/a_2 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ -ml/a_1 \\ 0 \\ (J+ml^2)/a_1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ -ml\cos(\pi/3)/a_2 \\ 0 \\ (J+ml^2)/a_2 \end{bmatrix};$$

$a_1 = (M+m)(J+ml^2) - m^2l^2$, $a_2 = (M+m)(J+ml^2) - m^2l^2\cos(\pi/3)^2$ and the membership functions are defined as $w_1(x_1(t)) = \mu_{M_1^1}(x_1(t)) = \left(1 - \frac{1}{1+e^{-7(x_1(t)-\pi/6)}}\right)\frac{1}{1+e^{-7(x_1(t)+\pi/6)}}$ and $w_2(x_1(t)) = \mu_{M_1^2}(x_1(t)) = 1 - \mu_{M_1^1}(x_1(t))$ which are shown as the Gaussian shape in Fig. 6.

A fuzzy controller in the form of (6) with the following 2 rules is employed to stabilize the nonlinear plant.

Rule j: IF $x_1(t)$ is N_1^i

THEN $u(t) = \mathbf{G}_j\mathbf{x}(t)$, $j = 1, 2$ (29)

There are two cases to be considered, i.e., the membership functions of the fuzzy model and fuzzy controller are the same or not.

1. Mismatched Membership Functions

The membership functions of the fuzzy controller are considered to be different from those of the fuzzy model. The membership functions of the fuzzy controller are chosen as follows and shown graphically as the trapezoids in Fig. 6.

$$m_1(x_1(t)) = \mu_{N_1^1}(x_1(t)) = \begin{cases} 0 & \text{for } x_1(t) < -0.9 \\ \frac{37}{28}(x+0.9) & \text{for } -0.9 \le x_1(t) \le -0.2 \\ 1 & \text{for } -0.2 < x_1(t) < 0.2 \\ -\frac{37}{28}(x-0.9) & \text{for } 0.2 \le x_1(t) \le 0.9 \\ 0 & \text{for } x_1(t) > 0.9 \end{cases} \quad (30)$$

$$m_2(x_1(t)) = \mu_{N_1^2}(x_1(t)) = 1 - \mu_{N_1^1}(x_1(t)) \quad (31)$$

The staircase membership functions are chosen as $\overline{w}_i(x_1(t)) = w_i(h\delta)$ and $\overline{m}_i(x_1(t)) = m_i(h\delta)$ for $(h - 0.5)\delta < x_1(t) \le (h + 0.5)\delta$ where $i = 1, 2$ and $h = -\infty, \ldots -10, -9, \ldots 10 \ldots, \infty$. It can be found numerically that $\gamma_{11} = -0.177336$, $\gamma_{12} = -0.066804$, $\gamma_{21} = -0.114347$ and $\gamma_{22} = -0.186624$ which satisfy the inequalities of $w_i(x_1(t))m_j(x_1(t)) - \overline{w}_i(x_1(t))\overline{m}_j(x_1(t)) - \gamma_{ij} \ge 0$ for all i and j.

Considering the stability conditions in Theorem 1, with the help of MATLAB LMI toolbox, the feedback gains are obtained as $\mathbf{G}_1$ = [837.7968 57.4587 3.9187 57.8283] and $\mathbf{G}_2$ = [858.3272 59.0134 3.9928 58.5750]. The fuzzy controller is employed to control the nonlinear plant with the initial system state conditions of $\mathbf{x}(0) = \begin{bmatrix} \frac{4}{9}\pi & 0 & 0 & 0 \end{bmatrix}^T$ and $\mathbf{x}(0) = \begin{bmatrix} \frac{2}{9}\pi & 0 & 0 & 0 \end{bmatrix}^T$, respectively. The system responses and control signal are shown in Fig. 7. It can be seen from the figure that the fuzzy controller can successfully stabilize the nonlinear plant.

2. Matched Membership Functions

The fuzzy controller sharing the same membership functions as those of the fuzzy model is employed to stabilize the nonlinear plant. In this case, it can be found numerically that $\gamma_{11} = -0.199564$, $\gamma_{12} = \gamma_{21} = -0.063664$ and $\gamma_{22} = -0.185515$ which satisfy the inequalities of $w_i(x_1(t))w_j(x_1(t)) - \overline{w}_i(x_1(t))\overline{w}_j(x_1(t)) - \gamma_{ij} \ge 0$ for all i and j. With the help of the MATLAB LMI toolbox to find the solution of the stability conditions in Theorem 2, the feedback gains are found as $\mathbf{G}_1$ = [531.4760 36.2053 2.7739 45.9703] and $\mathbf{G}_2$ = [805.0605 45.3275 3.4105 50.7430]. The system responses and control signal for the FMB control system with the initial system state conditions of $\mathbf{x}(0) = \begin{bmatrix} \frac{4}{9}\pi & 0 & 0 & 0 \end{bmatrix}^T$ and $\mathbf{x}(0) = \begin{bmatrix} \frac{2}{9}\pi & 0 & 0 & 0 \end{bmatrix}^T$, respectively, are shown in Fig 7. It can be seen that the nonlinear plant can be stabilized by the fuzzy controller.

In this example, both fuzzy controllers using the same or different membership functions between fuzzy model and fuzzy controller are able to stabilize the nonlinear plant. It can be shown in the previous two examples that the stability conditions in Theorem 1 are able to offer larger stability regions compared with Theorem 2. To apply the stability conditions in Theorem 1, it is require that the fuzzy controller must share the same set of membership functions as those of the fuzzy model. However, this restriction does not apply to the stability conditions in Theorem 1. Hence, simple membership functions can be employed to implement the fuzzy controller to lower the implementation cost. Both Theorem 1 and Theorem 2 have its own advantages for the design of stable FMB control systems. In general, at the beginning of the design, it is suggested to apply Theorem 1 with simple membership functions to achieve a lower-cost fuzzy controller. When stable design cannot be achieved, the membership functions of the fuzzy model are employed for the design of the fuzzy controller. By taking the advantage of matched membership functions, Theorem 2 is employed to achieve a stable design of fuzzy controller for the nonlinear plant.

V. CONCLUSION

The system stability of the fuzzy-model-based control systems has been investigated based on the Lyapunov stability theory in a simple and easy-to-understand manner. Staircase membership functions have been introduced to approximate the continuous membership functions of both fuzzy model and fuzzy controller. It allows the membership functions to be considered in the stability analysis for relaxation of stability conditions in terms of linear matrix inequalities. Furthermore, unlike the traditional analysis approach, the proposed stability analysis does not require that both fuzzy model and fuzzy controller share the same membership functions. Some simple membership functions can be employed for the fuzzy controller to lower the implementation cost. Simulation examples have been given to illustrate the merits of the proposed fuzzy control approach.

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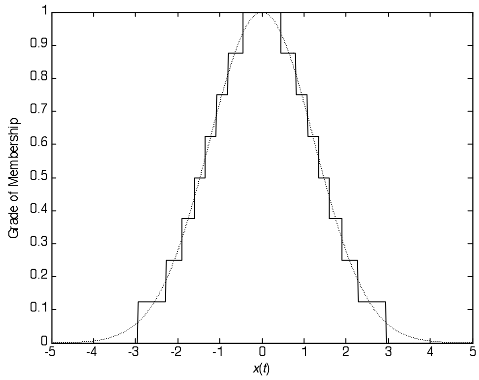

Fig. 1. Example of continuous and staircase membership functions.

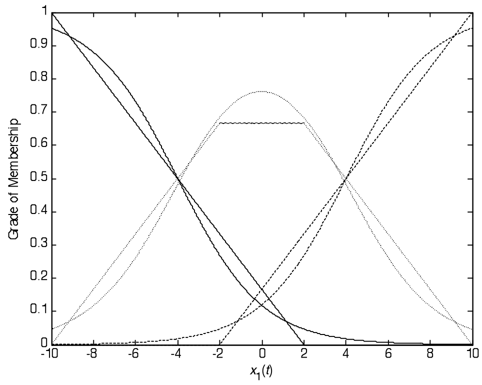

Fig. 2. Membership functions of the fuzzy model: $w_1(x_1(t))$ (left triangle in solid line), $w_2(x_1(t))$ (trapezoid in dotted line) and $w_3(x_1(t))$ (right triangle in dash line). Membership functions of the fuzzy controller: $m_1(x_1(t))$ (left z shape in solid line), $m_2(x_1(t))$ (Gaussian shape in dotted line) and $m_3(x_1(t))$ (right s shape in dash line).

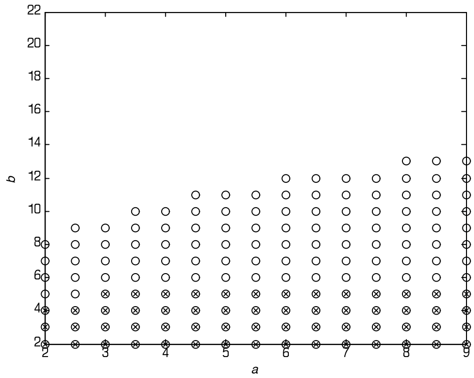

Fig. 3(a). Stability regions given by the stability conditions in Theorem 1 with $\delta = 0.1$ ('×') and $\delta = 0.05$ ('o').

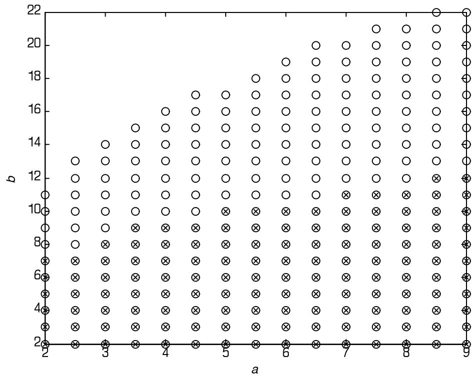

Fig. 4. Stability regions given by the stability conditions in Theorem 2 with $\delta = 0.1$ ('×') and $\delta = 0.05$ ('o').

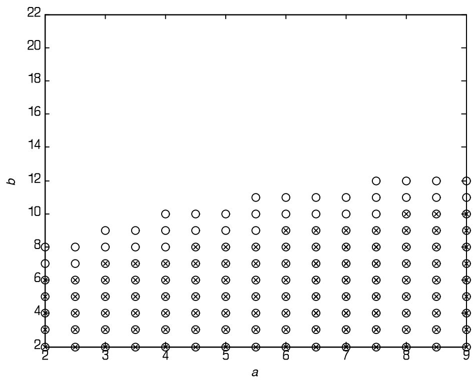

Fig. 5. Stability regions given by the stability conditions in [20] ('×') and [30]-[31] ('o').

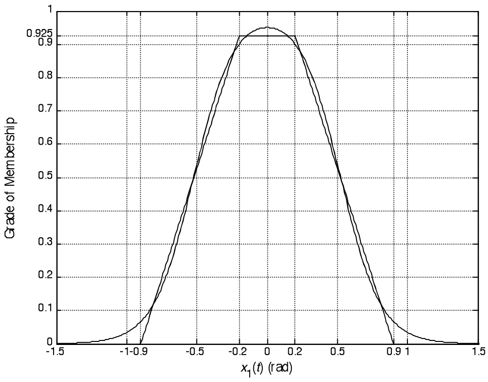

Fig. 6(a). $\mu_{\mathrm{M}_1^1}(x_1(t))$ (Gaussian) and $\mu_{\mathrm{N}_1^1}(x_1(t))$ (trapezoid).

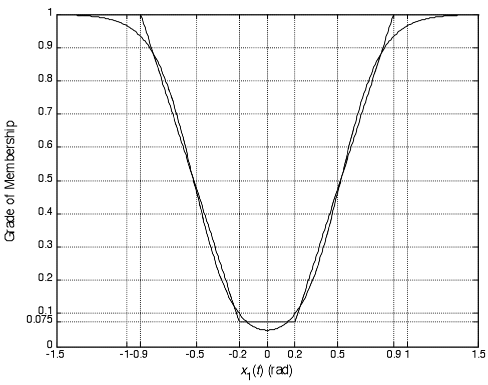

Fig. 6(b). $\mu_{\mathrm{M}_1^2}(x_1(t))$ (Gaussian) and $\mu_{\mathrm{N}_1^2}(x_1(t))$ (trapezoid).

Fig. 6. Membership functions of fuzzy model and fuzzy controller for the inverted pendulum on a cart.

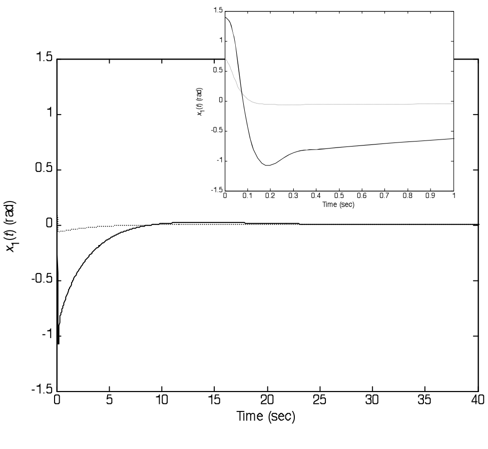

Fig. 7(a). $x_1(t)$.

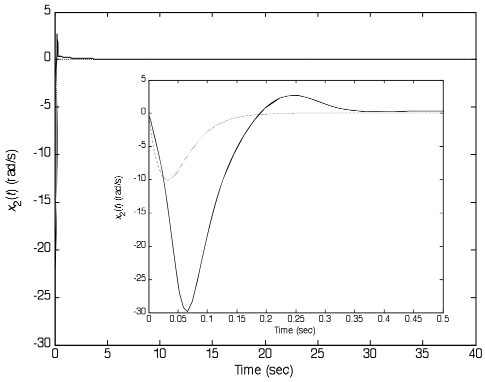

Fig. 7(b). $x_2(t)$.

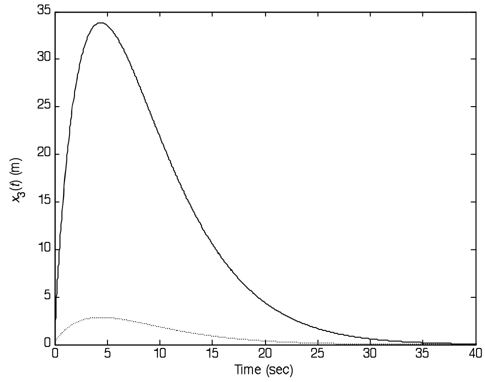

Fig. 7(c). $x_3(t)$.

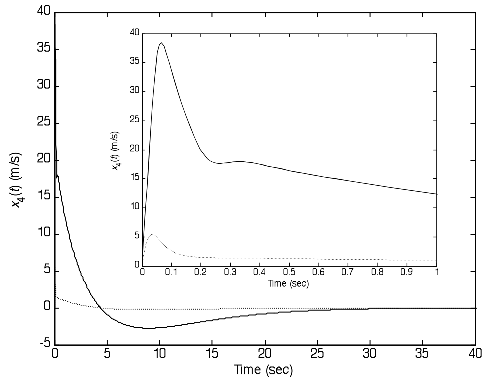

Fig. 7(d). $x_4(t)$.

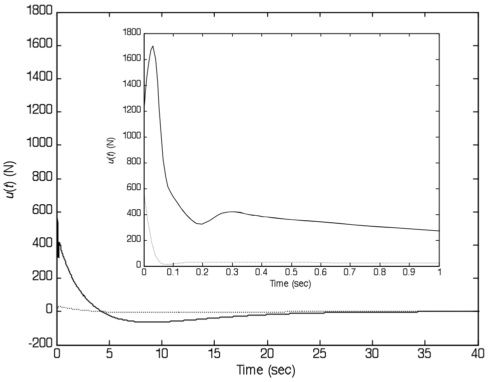

Fig. 7(e). $u(t)$.

Fig. 7. System responses and control signals of the inverted pendulum with fuzzy model and fuzzy controller not sharing the same membership functions. Solid lines: $\mathbf{x}(0)=\begin{bmatrix}\frac{4}{9}\pi & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$. Dotted lines: $\mathbf{x}(0)=\begin{bmatrix}\frac{2}{9}\pi & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$.

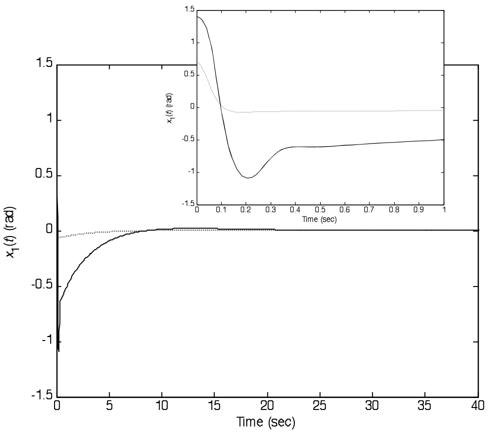

Fig. 8(a). $x_1(t)$.

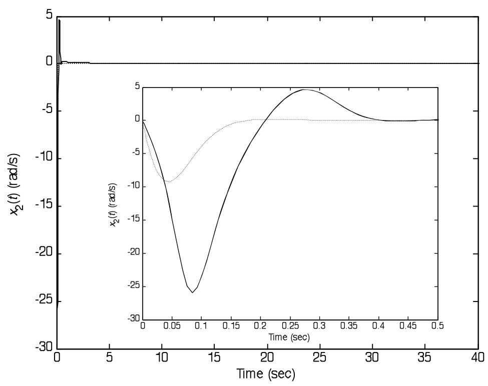

Fig. 8(b). $x_2(t)$.

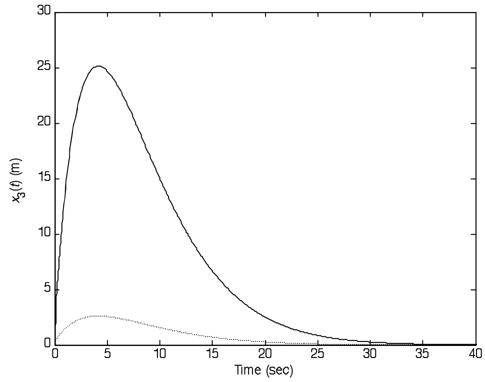

Fig. 8(c). $x_3(t)$.

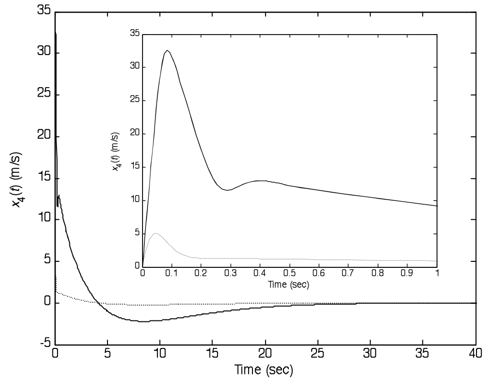

Fig. 8(d). $x_4(t)$.

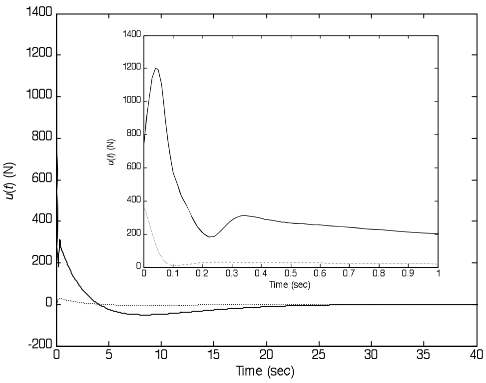

Fig. 8(e). $u(t)$.

Fig. 8. System responses and control signals of the inverted pendulum with fuzzy model and fuzzy controller sharing the same membership functions. Solid lines: $\mathbf{x}(0)=\begin{bmatrix}\frac{4}{9}\pi & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$. Dotted lines: $\mathbf{x}(0)=\begin{bmatrix}\frac{2}{9}\pi & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$.