Vortex solid with frozen undulation

Hajime Yoshino[1], Tomoaki Nogawa[2], Bongsoo Kim[3]
[1] *Department of Earth and Space Science, Faculty of Science, Osaka University, Toyonaka 560-0043, Japan*
[2] *Department of Applied Physics. School of Engineering, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan*
[3] *Department of Physics, Changwon National University, Changwon 641-773, Korea.*

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A vortex solid with self-generated randomness is found theoretically in a frustrated Josephson junction array (JJA) under external magnetic field with anisotropic Josephson couplings. Vortexes induced by magnetic field develop stripes parallel to the direction of weaker Josephson coupling. It is shown analytically that transverse undulation of the stripes brings about a gapless band of low lying metastable states. The vortex solid with the frozen undulation in a metastable state freely slides along the direction of stronger coupling, destroying ordering of phases even at zero temperature, but is jammed along the direction of weaker coupling.

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Solids are systems with rigidity ranging from crystals, quasi-crystals to glasses and jammed granular matters [1]. Generally it becomes more challenging to understand mechanism of the formations of *less* periodic solids distingushing them from liquids [2]. A useful guiding concept to study non-crystalline solids is *frustration* which inhibits crystalizations [3, 4]. Imagine that there is a trick to "inject" dislocations artificially into a crystal from outside and that their density can be controlled at will. Such a system will provide a very interesting ground to study consequences of geometrical frustration. Quite remarkably the Josephson junction array (JJA) under external magnetic field realizes such an ideal situation [5–9]. Furthermore, transport properties of the JJA under external current can be regarded as "rheology" of a frustrated system under external shear [10].

JJA is a network of superconducting islands connected with each other by Josephson junction in the form, say, of a square lattice of size $N = L \times L$ as shown in Fig. 1 a) [5, 6]. The phases θ_i of the superconducting order parameter on the islands $i = 1, 2, \ldots, N$ interact with each other via Josephson coupling. Under magnetic field B, which can be varied at will in experiments, the number density $f = Ba^2/\phi_0$ of vortexes (dislocations) can be forced into the configuration of the phases. Here a^2 is the area of a plaquette and ϕ_0 is the flux quantum.

A somewhat unexpected connection to the problem of *frictions* provides valuable insights. The frustrated JJA becomes essentially equivalent to the Frenkel-Kontorova (FK) model [11] and the two-chain model of Matsukawa and Fukuyama (MF) [12] in a one dimensional limit. FK and MF models are known to exhibit a kind of jamming or frictional transition at zero temperature $T = 0$, known as the Aubry's transition [13, 14] at a critical value of the strength of coupling λ between two surfaces which are *incommensurate* with respect to each other. Then one would naturally be led to consider *irrationally frustrated anisotropic* JJA [10] with 1) *irrational* vortex density f [7, 8] and 2) *anisotropic* Josephson couplings in to x and y directions, say 1 into x direction and λ into y direction.

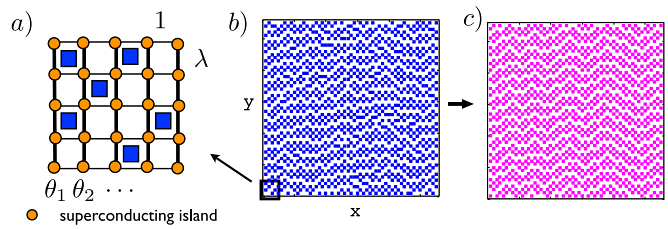

FIG. 1: Vortex patterns in an irrationally frustrated Josephson junction array (JJA) under magnetif field with anisotropic Josephson coupling. Here $\lambda = 1.5$ so that the Josephson coupling is stronger into y direction. Such an anisotropic JJA can be fabricated in laboratory by the lithography technique [9]. Panel a) displays the JJA on a square lattice. A fraction of $f = 21/55$, which approximates an irrational number $(3-\sqrt{5})/2 = 0.381966..$, of the plaquettes are occupied by a vortex with charge $1 - f$ represented by filled squares. Panel b) displays vortex pattern in a snapshot in equilibrium at $T = 0.2$ and Panel c) displays that at a nearby energy minimum reached via an energy descent algorithm.

In this Letter we study the ground state and low-lying states of the irrationally frustrated anisotropic JJA. Numerically we found vortexes spontaneously develop many metastable states with undulated stripes of vortexes such as the one shown in Fig. 1. By a perturbative analysis in series of $1/\lambda$ starting from infinite anisotropy limit $\lambda = \infty$, we are able to reproduce such metastable states analytically. The coexistence of sliding and jamming of the system [10] is proved on the analytically constructed ground state and the low-lying states. Because of the sliding, the phases remain disordered even at $T = 0$ for irrational f, in sharp contrast to JJA with rational f where not only vortexes (chiralities for $f = 1/2$) but also phases exhibit (quasi-)long ranged order at $T > 0$ [15].

Model To simplify notations we label the vertexes (superconducting islands) as $i = 1, 2, \ldots, N$ whose position in the real space is given by (n_i, m_i). The properties of the JJA under the transverse magnetic field are known to be described by an effective Hamiltonian [5],

$$H = -\sum_{<i,j>\|x\text{-axis}} \cos(\psi_{ij}) - \lambda \sum_{<i,j>\|y\text{-axis}} \cos(\psi_{ij}) \quad (1)$$

with the gauge-invariant phase difference, $\psi_{ij} \equiv \theta_i - \theta_j - A_{ij}$. We measure temperature T in a unit with $k_{\rm B} = 1$. For the anisotropy of the Josephson coupling λ, we need to consider only $\lambda > 1$ by symmetry. The vector potential A_{ij} is defined such that directed sum of them is $2\pi f$.

Charge or vorticity v_i of the vortex at the cell associated with the i-th vertex is defined by taking directed sum of $(\psi_{ij}/2\pi - [\psi_{ij}/2\pi]_n)$ on the junctions around the cell. Here $[x]_n$ denotes the nearest integer of the real variable x. It takes values $\ldots, -1-f, -f, 1-f, \ldots$. We use periodic boundary conditions so that the total vorticity is enforced to be zero (charge neutrality condition).

It has been proposed that *superconducting glass* may be realized if f is *irrational* [8]. While JJAs with rational f develop periodic vortex lattices [15, 16], such simple orderings may be avoided with irrational f. Indeed equilibrium relaxations were similar to the primary relaxations observed in typical fragile supercooled liquids [17]. Such a system is called as *irrationally frustrated* JJA.

One would worry that a system with, for instance, $f = 1/2 + \epsilon$ with *infinitesimal irrational* ϵ should behave as a system with $f = 1/2$ in some sense. Although this is a very interesting issue [7], we focus on properties of systems with irrational f in $N \to \infty$ limit.

Numerical methods In numerical simulations, we used a series of rational numbers p/q = 5/13, 8/21, 13/34, 21/55, 34/89, 55/144, 89/233 for the filling f, which approximate an irrational number $f = (3 - \sqrt{5})/2 = 0.38196601...$ We worked on systems of size $L \times L$ with $L = q$ so that in the limit $L \to \infty$ the value of p/q converges to the target irrational number f.

To generate the equilibrium ensemble at finite temperatures, we used Monte Carlo (MC) simulations. We performed MC simulations on $L = 13-89$ using $20-120$ temperatures in the temperature range $T = 0.2 - 0.4$ by the exchange MC method convined with the over-relaxation method [18]. We used $10^5 - 10^6$ MC steps for the equilibration and observations.

Stacked undulating vortex stripes As shown in Fig.1, the vortexes develop undulated stripes parallel to the direction of weaker coupling at low temperatures. The formation of the stripes is reasonable because the repulsive interactions between vortexes are anisotropic. A remarkable feature is that the stripes are stacked quite regularly along the stronger coupling as shown in Fig.1 c) in the nearby energy minimum which we obtained via a simple energy descent algorithm. Starting from different thermalized configurations we obtained numerous energy minima similar to the one shown in Fig.1 c) but

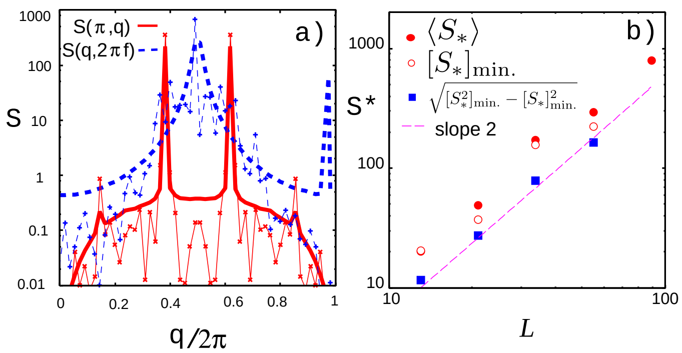

FIG. 2: Structure factor of vortexes. a) displays the cross-sections of the structure factor $S(q_x, q_y)$ ($L = 55$) with thermal average (thick lines) and at the energy minimum shown in Fig. 1 c) (thin lines). b) displays the amplitude of the peak of the structure factor $S_* = S(\pi, 2\pi f)$ with thermal average $\langle S_* \rangle$, average over the energy minima $[S_*]_{\rm min.}$ and variance of the minima-to-minima fluctuation $\sqrt{[S_*^2]_{\rm min.} - [S_*]^2_{\rm min.}}$. Here the average over minima $[\ldots]_{\rm min}$ is taken over 100 energy-minima obtained by independent initial conditions.

with different realization of the transverse undulation. The nearly perfect stacking of the stripes strongly suggest that the energy barrier to go from one to another realization of significantly different undulation of vortexes (dislocations), which necessarily involve large number of plastic deformations, diverges with the system size so that the ergordicity is broken. This feature is markedly different from usual undulations found, for example, in liquid crystals which are fluctuating dynamically [19].

The stacked undulation is manifested in the structure factor. As shown in Fig. 2, the structure factor $S(q_x, q_y)$ exhibits prominent peaks at $(q_x, q_y) = (\pi, 2\pi f)$ and $(\pi, 2\pi(1-f))$ whose height scales with the system size as $N = L^2$ as usual Bragg peaks. However the profile of the peak is peculiar: it decays sharply along q_y reflecting the stacking but decays slowly by a power law $|\pi - q_x|^{-2}$ along q_x reflecting the transverse undulation. Here we emphasize again that the transverse undulation is frozen in time. The frozen-in randomness is manifested in the minimum-to-minimum fluctuations of the structure factor shown in Fig.2 a) and b). Note that the variance of the fluctuation as well as the average grows linearly with the system size N meaning that the structure factor is *not* self-averaging.

Analytic construction of the ground state Let us now turn to explicit construction of low-lying states by an analytical approach. To this end we propose a non-trivial ansatz for the ground state using the notion of the so called hull functions developed in the studies of the FK and MF models [11–14]. We propose that the gauge invariant phase differences ψ_{ij} across the Josephson junctions, in the ground states of the anisotropic JJA

($\lambda > 1$) can be represented as,

$$\psi_{(x,y)(x+1,y)} = \phi_x[y+\alpha(x)], \qquad \psi_{(x,y)(x,y+1)} = \phi_y[y+\alpha(x)] \quad (2)$$

Here $\phi_x[y]$ and $\phi_y[y]$ are functions defined on the "folded coordinate" which takes values limited in the range $0 < [y] \leq 1$. Such a function is called as a hull function [13]. Note that if f is *irrational*, the vertexes of the JJA uniformly fill the entire range of the folded coordinate $[y]$ in the limit $N \to \infty$. Thus we treat $[y]$ as a *continuous* variable. This assumption plays crucial roles below.

An obvious constraint is that the directed sum over ψ_{ij} around each plaquette must be $-2\pi f$. In addition, the Josephson currents must be conserved at each vertex (force balance condition) in each energy minimum. Thus the following two conditions should hold,

$$\phi_x[y] + \phi_y[y+\delta] - \phi_x[y+1] - \phi_y[y] = -2\pi f \quad (3)$$

$$\frac{1}{\lambda}\sin\phi_x[y] + \sin\phi_y[y] = \frac{1}{\lambda}\sin\phi_x[y-\delta] + \sin\phi_y[y-1] \quad (4)$$

where δ is a phase difference.

Now our task is to look for the hull functions $\phi_x[y]$, $\phi_y[y]$ and phase differences δ which satisfy the conditions on the plaquettes Eq. (3) and vertexes Eq. (4). We solve this problem by performing a $1/\lambda$ expansion [14, 20]around the infinite anisotropy limit $\lambda = \infty$.

In the infinite anisotropic limit $\lambda \to \infty$ the weaker couplings can be neglected so that we easily find $\phi_x[y] = (2[y]-1)\pi + O(1/\lambda)$ and $\phi_y[y] = O(1/\lambda)$ which trivially satisfies Eq. (3) and Eq. (4) (with $\lambda = \infty$). As such, the phase difference δ is not fixed at this stage.

Then using the above results in Eq. (4) we find $1/\lambda$ correction term of $\phi_y[y]$, which in turn allows us to find $1/\lambda$ correction term of $\phi_x[y]$ through Eq. (3). In this way we obtained analytic form of the hull functions of the ground state up to $O(\lambda^{-3})$ as,

$$\phi_x[y] = (2fy-1)\pi + \frac{|a_1|^2}{\lambda}s_x(1,y)$$
$$+\frac{|a_1|^4}{8\lambda^3}\left(|a_3|^2 s_x(3,y) - 3|a_1|^2 s_x(1,y)\right)$$
$$\phi_y[y] = \frac{|a_1|}{\lambda}s_y(1,y) + \frac{|a_1|^4}{8\lambda^3}\left(|a_3| s_y(3,y) - 3|a_1| s_y(1,y)\right)$$

with $|a_n| \equiv \sqrt{2/(1-\cos(2n\pi f))}$, $s_x(n,y) \equiv \sin(2n\pi fy)$ and $s_y(n,y) \equiv \sin(2n\pi fy + \pi(nf-1/2))$. By minimizing the energy for *irrational* f, the ground state energy is obtained as $\frac{E/\lambda}{N} = -1 - \frac{|a_1|^2}{4\lambda^2} + \frac{|a_1|^4}{16\lambda^4}\left(\frac{1}{4} - |a_1|^2\right) + O(\lambda^{-6})$ and the phase difference is fixed as $\delta = 1/(2f)$.

Band of undulated metastable states Next let us construct the low lying states with transverse undulation of vortex stripes shown in Fig. 1. We assume that such a state has phase shifts with respect to the ground state parametrized as $\delta(x) = 1/(2f) + \Delta(x)$. Somewhat surprisingly, we can solve Eq. (3) and Eq. (4) with *arbitrary* $\Delta(x)$ finding,

$$\phi_x[y] = (2fy-1)\pi + \frac{|a_1|^2}{\lambda}s_x(1,y)$$
$$-\frac{|a_1|^2}{4\lambda}\left[C_1(\Delta)\cos(2\pi fy) + C_2(\Delta)\sin(2\pi fy)\right] + O(1/\lambda^2)$$
$$\phi_y[y] = \frac{|a_1|}{2\lambda}\left(s_y(1, y-\Delta(x)) + s_y(1,y)\right) + O(\lambda^{-2}) \quad (5)$$

with $C_1(\Delta) = \sin(2\pi f\Delta(x+1)) + \sin(2\pi f\Delta(x))$ and $C_2(\Delta) = 2 - \cos(2\pi f\Delta(x+1)) - \cos(2\pi f\Delta(x))$.

As the result the energy becomes, for *irrational* f,

$$\frac{E}{\lambda} = -N\left(1 + \frac{|a_1|^2}{4\lambda^2}\right) + L\frac{|a_1|^2}{8\lambda^2}\sum_{x=1}^{L}(1-\cos(2\pi f\Delta(x))) + O(\lambda^{-4})$$

Here it is evident that there is a gapless, continuous spectrum of low lying states each of which is parametrized by a function $\Delta(x)$. Assuming $\Delta(x) \ll 1$, we obtain an one-dimensional *elastic* Hamiltonian with an unusual elastic constant which grows linearly with the system size L.

Let us emphasize here that the undulated states with the arbitraly $\Delta(x)$ are constructed by *imposing the force balance condition* Eq. (4), which is not possible unless the hull functions are continuous, i. e. f is irrational and $N \to \infty$. Thus the system trapped in an undulated state with non-zero $\Delta(x)$ *cannot* relax *spontaneously* down to the ground state $\Delta(x) = 0$. Thus they are *metastable*. Note that undulation is distinct from "phonons" by which vortexes do not move.

Now the unusual structure factor in Fig. 2 can be understood as follows. The configuration of the ground state is a function of the folded coordinate $[y]$ so that its Fourier transform must have peaks along the q_y-axis at $q_y = 2\pi f$ and $2\pi(1-f)$. On the other hand the phase shift of the hull function $\delta = 1/(2f)$ along x-direction means that the Fourier transforms must have a single peak along the q_x axis at $q_x = \pi$. The power law tail $|q_x - \pi|^{-2}$ naturally follows from the effective one-dimensional elastic Hamiltonian obtained above.

In Fig. 3 we plot the phase differences across the Josephson junctions in energy minima obtained numerically and compare them with the hull functions obtained above. For simplicity we show here the hull function for the ground state disregarding small differences due to the undulation $\Delta(x)$. The perturbative result grasps well the overall features. The agreement will be improved by taking into account higher order terms in the $1/\lambda$ expansion.

A remarkable consequence of the analytic hull function is that the undulated vortex solid can *freely slide* into the y direction: Given an energy minimum described by the hull functions $\phi_x[y]$ and $\phi_y[y]$, a family of solutions with exactly the same energy can be obtained through the operation $[y] \to [y+\alpha]$ with varying phase shift parameter α. Consequently the phases must remain disordered even at $T = 0$ due to the sliding. As shown in Fig. 3

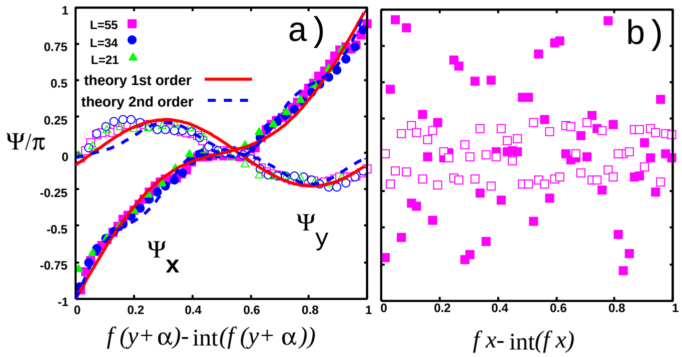

FIG. 3: Configuration of the gauge invariant phase differences across Josephson junctions in energy minima. Here the anisotropy is $\lambda = 1.5$. In panel a) original y axis is "folded" to $[y] = fy - \mathrm{int}(fy)$. The symbols are data of the gauge invariant phase differences $\psi_x = \psi_{(x,y)(x+1,y)}$ (filled symbols) and $\psi_y = \psi_{(x,y)(x,y+1)}$ (open symbols) across Josephson junctions parallel to x and y axis. Each data set consists of data points on a 'column' $(x, 1), (x, 2), \ldots, (x, L)$ at an arbitrary chosen x $(1 \leq x \leq L)$ in an arbitrary chosen energy minima. Each data set is shifted globally by some α so that different data sets collapse on top of each other. The lines are analytically obtained hull functions of the ground state. In panel b) the data on the same energy minimum ($L = 55$) are plotted against "folded" x-axis.

a) the phase differences across Josephson junctions parallel to the x-axis take all possible values in the range $-\pi < \phi_x[y] < \pi$ meaning that the system can be *sheared* indefinitely along the x-axis without changing the energy. The phase differences along y-axis is bounded in a narrow range meaning that each 'column' parallel to y-axis behaves as a quasi-rigid body which deforms only mildly. These variation of the phase differences amount to unidirectional sliding of the undulated vortex solid into the y-direction without changing its pattern.

In contrast to Fig. 3 a), the plot against "folded" x-axis shown in Fig. 3 b) exhibits no hint of regular hull function. Thus the vortex solid cannot slide along x-axis, i. e., it exhibits a jamming along x-axis.

It is instructive to compare the above results with the FK model. In the FK model the hull function is rigorously proved to be an analytic function in the sliding phase $\lambda < \lambda_c$ but becomes non-analytic in the jammed phase $\lambda > \lambda_c$ [13]. In the anisotropic JJA, sliding and jamming coexist taking place along different axes. Indeed in [10] it was found numerically that the shear-modulus is zero/finite along the direction of weaker/stronger coupling at zero temperature. Furthermore it was suggested that the symmetric point $\lambda = 1$ is a critical point at zero temperature similar to the point-J (jamming point) in granular matters [1]. Quite interestingly recent studies at finite temperatures suggest $T_c(\lambda = 1) = 0$ [21, 22]. On the other hand our present study suggests $T_c(\lambda) > 0$ at least sufficiently away from $\lambda = 1$.

To conclude we found undulated vortex stripes in irrationally frustrated Josephson junction array with anisotropic Josephson coupling theoretically. It will be very interesting to study critical properties of the system closer to the symmetric point where the perturbative approach should break down.

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