Recent developments in total colouring

H.R. Hind
Department of Mathematics, University of Reading, Reading, RG6 2AX, UK

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Abstract
This paper gives a number of recent results concerning total colourings and suggests that recolouring schemes which are somewhat more complex than those currently being considered, need to be developed. Some examples of graphs and colourings which support this assertion are presented.

The aim of this paper is firstly to publicise a number of recent results concerning total colourings and secondly to suggest that recolouring schemes which are somewhat more complex than those currently being considered, need to be developed. Some examples which support this assertion will be presented. All graphs considered in this paper will be finite, simple graphs. As is conventional when discussing colourings of finite graphs, the sets of colours considered in this paper will be subsets of the set $\mathbb{N}$ of positive integers.

1. Introduction

If $G$ is a graph, a total colouring of $G$ is a function

$$\psi : V(G) \cup E(G) \to \mathbb{N}$$

such that no two adjacent vertices are assigned the same colour, no two adjacent edges are assigned the same colour and no incident vertex and edge are assigned the same colour.

The total chromatic number of $G$, denoted by $\chi''(G)$, is the smallest positive integer $k$ for which there exists a total colouring

$$\psi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}.$$
The concept of a total colouring was introduced by Behzad in 1965 [1]. In his thesis, he made the following conjecture, which has come to be known as the total colouring conjecture or TCC. It is maintained by some authors that the conjecture was also made independently by Vizing at some time between 1964 and 1968. Vizing mentions the conjecture in [16], but makes no comment as to its origin.

**Conjecture 1.1.** If \( G \) is a graph with maximum degree \( \Delta(G) \), then

\[
\Delta(G) + 1 < \chi''(G) < \Delta(G) + 2.
\]

The lower bound follows immediately from the observation that a vertex and its incident edges must each be assigned a distinct colour. A trivial upper bound, namely \( \chi''(G) < 2\Delta(G) + 1 \), follows from the greedy algorithm.

### 2. Some recent results

In this section of the paper a selection of recent total colouring results will be mentioned. A more comprehensive survey, including many earlier results, can be found in [5].

**Theorem 2.1** (Hind [12]). *If \( G \) is a graph with maximum degree \( \Delta(G) \), then

\[
\chi''(G) \leq \Delta(G) + 1 + 2\lceil \sqrt{\Delta(G)} \rceil.
\]

This result establishes that the total chromatic number of a graph has the form \( \chi''(G) = \Delta(G) + o(\Delta(G)) \). The proof uses a partitioning argument and a lemma showing that any 'suitable' vertex-colouring of an independent set \( W \) of vertices of a graph \( G \) can always be extended to a colouring of \( E(G) \cup W \) where at most the edge-chromatic number plus one colours are used, the colouring of \( E(G) \) is proper and the colour (pre)assigned to a vertex in \( W \) is not used to colour any of the edges incident with that vertex. The term 'suitable' simply means that the original vertex-colouring should not use more than the edge-chromatic number of colours.

**Theorem 2.2** (Chetwynd et al. [8]). *Let \( G \) be a graph with minimum degree \( \delta(G) \). If

\[
\delta(G) \geq 5/6(|V(G)| + 1),
\]

then

\[
\chi''(G) \leq \Delta(G) + 2.
\]

Furthermore, if \(|V(G)|\) is even then 5/6 can be replaced by 3/4.

The significance of this result is that it establishes that the TCC is true for a reasonably large class of graphs, including many special classes of graphs for which the TCC had previously been proved (see, e.g. [2, 3]). The proof of this result essentially involves finding a vertex-colouring which assigns each available colour to
one or two vertices of the graph. The well-known theorem of Dirac on the existence of Hamiltonian cycles is then used to find suitable matchings for the edge-colouring.

**Theorem 2.3** (Sánchez-Arroyo [15]). _Total colouring graphs is NP-complete._

This result shows that it is unlikely that a simple (polynomial time) algorithmic proof can be found which proves the TCC by establishing the total chromatic number of the graph. The proof of this result is obtained by showing that the existence of a polynomial time algorithm for establishing the total chromatic number of a graph would imply a polynomial time algorithm for establishing the edge-chromatic number of an r-regular graph. In particular the result is shown for cubic bipartite graphs. The result does not, however, rule out the possibility of finding a simple algorithm which colours any graph with at most the maximum degree plus two colours.

Within the last year, a number of significant results have been established. Once again, in the interests of brevity, consideration of these results will be restricted to a statement of the results and a few brief comments about their significance and the proof techniques used.

**Theorem 2.4** (Chetwynd and Häggkvist [7], McDiarmid and Reed [14]). _Let G be a graph with maximum degree Δ(G). If t is an integer such that t! > |V(G)|, then_

\[ \chi''(G) \leq \chi'(G) + t. \]

This result establishes a good upper bound for the total chromatic number for those graphs having relatively large maximum degree compared to the order of the graph. The main significance of the result lies in the rather elegant enumeration (or probabilistic) technique used in the proof. Strictly, it should be mentioned that in McDiarmid and Reed’s statement of the result the edge-chromatic number \( \chi'(G) \), was replaced by \( \chi'(G) + 1 \).

As usual, let \( G_{n,p} \) be the space of random graphs on \( V = \{1, 2, \ldots, n\} \) where each unordered pair of distinct vertices is chosen to be an edge with probability \( p \) and independently of the choice for any other pair. In this paper, we assume that \( p \) is a constant independent of \( n \).

**Theorem 2.5** (McDiarmid and Reed [14]). _Almost every graph G in \( G_{n,p} \) satisfies \( \chi''(G) = \Delta(G) + 1 \)._

This result is pleasing in that not only does it show that the TCC is correct for most graphs, it also shows that this is true in the strongest possible way. The authors provide upper bounds for the proportion of graphs failing to have \( \chi'(G) = \Delta(G) + 1 \) and the proportion of graphs failing to have \( \chi''(G) \leq \Delta(G) + 2 \). The proof of this result involves showing that almost every graph in \( G_{n,p} \) has a particular structure, followed by a reasonably simple lemma showing that a graph having this structure has total
chromatic number $\Delta(G) + 1$. It is thus unlikely that the techniques used in this proof will be able to be extended to a proof of the TCC for all graphs.

**Theorem 2.6** (Hind [13]). If $G$ is a graph with maximum degree $\Delta(G)$, then

$$\chi''(G) = \Delta(G) + 2 \left\lfloor \frac{|V(G)|}{\Delta(G)} \right\rfloor + 1.$$ 

The above theorem provides an upper bound for the total chromatic number which shows that the TCC is 'almost' correct for graphs having large maximum degree relative to their order. For example, if the maximum degree of a graph $G$ is at least half the order of $G$, then $\chi''(G) \leq \Delta(G) + 5$ and if the maximum degree is at least one quarter the order, then $\chi''(G) \leq \Delta(G) + 9$. The result is not, however, particularly surprising as a graph having a large maximum degree relative to its order should possess a vertex-colouring using each colour on relatively few vertices. Given such a vertex-colouring, it should be possible to use an adapted version of the proof of Vizing's Theorem to find the edge-colouring; this was essentially the technique used to prove the theorem. Following suggestions from numerous people, the author has shown that the proof of this result can be adapted to show that almost every graph in $\mathcal{G}_{n,p}$ has total chromatic number at most $\Delta(G) + 2$. It should be noted, however, that the author was aware of Theorem 2.5 before proving Theorem 2.6.

**Theorem 2.7** (Hilton and Hind [10]). Let $G$ be a graph with maximum degree $\Delta(G)$. If $\Delta(G) > \frac{3}{4}|V(G)|$, then

$$\chi''(G) \leq \Delta(G) + 2.$$ 

This result combines the techniques used in the proof of Theorems 2.2 and 2.6 above. The result is satisfying in that the statement of Theorem 2.7 involves the maximum degree of the graph rather than the minimum degree as in Theorem 2.2; there does not appear to be a simple relationship between the total chromatic number and the minimum degree of a graph.

3. Examples

In this section it will be shown that standard edge recolouring argument, used in numerous papers proving results about the total chromatic number, needs to be generalized if there is to be any hope of using it to prove the TCC. In many papers (see e.g. [7, 10, 12, 13]) theorems proving results establishing upper bounds for the total chromatic number begin with (essentially) the following statement: "Let $\varphi$ be a fixed proper vertex-colouring of $G$ ...". The central idea used in all these proofs being to find an edge-colouring for the graph $G$ under consideration which is proper and has the additional property that no edge is assigned either of the
Recent developments in total colouring

Fig. 1. Graph constructed from three copies of $K_{n,n}$ each with one edge subdivided

colours assigned to its endvertices by the (fixed) vertex-colouring. These results may thus be viewed as results concerning the following statement which is, of course, stronger than the TCC.

Statement $A$: If $G$ is a graph having maximum degree $\Delta(G)$, then every vertex-colouring $\phi : V(G) \rightarrow \{1, 2, \ldots, \Delta(G) + 2\}$ can be extended to a proper total colouring, $\psi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, \Delta(G) + 2\}$, of $G$ such that $\psi|_{V(G)} = \phi$.

The above statement is not, however, true for every graph $G$ and every vertex-colouring $\phi$ of $G$. In particular, the statement is false for the graph and
vertex-colouring shown in Fig. 1. The graph is composed of three blocks, each isomorphic to $K_{6,6}$ with one subdivided edge. Let us initially consider one such block. A well-known edge-colouring result states that any graph formed by subdividing a single edge of an even order regular graph is a class two graph (i.e., has edge-chromatic number equal to the maximum degree plus one). It follows that at least seven of the colours in $\{1, 2, \ldots, 8\}$ would have to be used to colour the edges of the block if there were to be a total colouring of the block only using colours from the set $\{1, 2, \ldots, 8\}$. Considering the assignment of colours to the vertices (which is the same for each block), it follows that one of the colours 1 or 2 would have to be assigned to one of the edges of the block incident with the central cutvertex: that is, the vertex of the subdivided edge of the block. Since this statement must be true for each of the three blocks, it follows that there is no proper total colouring for the graph in Fig. 1 having the indicated assignment of colours to the vertices.

By considering similar graphs where the blocks are isomorphic to $K_{d,d}$ with one subdivided edge, it follows that the statement is not true, in this general form, for graphs $G$ having $\Delta(G) > 6$. At present, no counterexample for statement A is known for graphs having maximum degree three, four or five. The statement is true for graphs having maximum degree two. There is, perhaps, some chance of proving the following weaker statement having first established some formal definition of what is meant by the term ’most’.

Statement B: If $G$ is a graph having maximum degree $\Delta(G)$, then ’most’ vertex-colourings $\varphi: V(G) \to \{1, 2, \ldots, \Delta(G) + 2\}$ can be extended to a proper total colouring $\psi: V(G) \cup E(G) \to \{1, 2, \ldots, \Delta(G) + 2\}$ of $G$ such that $\psi|_{V(G)} = \varphi$.

A counterexample to statement B would provide considerable insight into the problem of totally colouring graphs.

In summary, the above discussion suggests that if a recolouring algorithm, which attempts to extend a fixed vertex-colouring of a graph to a total colouring of the graph, is to be used in a proof of the TCC, then some care will have to be taken to choose a suitable fixed vertex-colouring. More disturbingly, if a counterexample to statement B can be found then it is likely that the recolouring algorithm will have to allow changes in the vertex-colouring itself.

4. Open problems

To conclude this paper, an open problem and a conjecture relating to the ideas discussed in this paper will be mentioned.

Problem 4.1. It is true that for every graph $G$ with maximum degree three and every (proper) vertex-colouring $\varphi: V(G) \to \{1, 2, 3, 4, 5\}$ of $G$, there exists a total colouring, $\psi: V(G) \cup E(G) \to \{1, 2, 3, 4, 5\}$ of $G$ such that $\psi|_{V(G)} = \varphi$?
The author suspects that the answer to this question is no, but has not yet found a graph and vertex-colouring for which a total colouring cannot be found.

The conjecture below is a special case of a conjecture Häggkvist (see [6]). It may be worth concentrating on obtaining a proof for this conjecture, before attempting to prove the TCC itself.

Conjecture 4.2 (Häggkvist [6]). Let $G$ be a graph with maximum degree $\Delta(G)$. If $\varphi : V(G) \to \{1, 2, 3, \ldots, \Delta(G) + 3\}$ is a (possibly improper) vertex-colouring of $G$, then there exist a proper edge-colouring $\theta : E(G) \to \{1, 2, 3, \ldots, \Delta(G) + 3\}$ such that the colour of each edge $xy \in E(G)$ satisfies $\theta(xy) \neq \varphi(x)$ and $\theta(xy) \neq \varphi(y)$.

It should be noted that the above conjecture has been shown to be correct for bipartite graphs and complete graphs [6] and for graphs having maximum degree at most three [9]. Furthermore, the graph in Fig. 2 shows that the above conjecture, if correct, is sharp.
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