Non-matchable distributive lattices

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\textbf{A B S T R A C T}

Based on an acyclic orientation of the Z-transformation graph, a finite distributive lattice (FDL for short) $\mathbf{M}(G)$ is established on the set of all 1-factors of a plane (weakly) elementary bipartite graph $G$. For an FDL $L$, if there exists a plane bipartite graph $G$ such that $L$ is isomorphic to $\mathbf{M}(G)$, then $L$ is called a matchable FDL. A natural question arises: Is every FDL a matchable FDL? In this paper we give a negative answer to this question. Further, we obtain a series of non-matchable FDLs by characterizing sub-structures of matchable FDLs with cut-elements.

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1. Introduction

In this paper we take graph terminologies from [1]. As combinatorial structures, finite distributive lattices (FDLs for short) have been established on many combinatorial objects, such as the stable matchings of a bipartite graph [13], the set of flows of a planar graph [9] and the set of c-orientations with fixed flow difference on a plane graph [17,18] or in the dual setting [3]. A result of Propp [18] establishes some FDLs on the sets of $d$-factors, spanning trees, and Eulerian orientations in a plane (bipartite) graph.

The $Z$-transformation graph $Z(G)$ of a plane bipartite graph $G$ having a 1-factor (that is, a perfect matching) is a simple graph on the set of all 1-factors of $G$: two 1-factors are adjacent if their symmetric difference is a cycle that is the boundary of a bounded face of $G$. This concept originates from Zhang et al. [25] for benzenoid graphs. In fact, this graph has been introduced independently several times under different names. For example, Gründler [6] introduced it, under the name resonance graph, on the set of Kekulé structures of benzenoid graphs. Randić [19] showed that the leading eigenvalue of the resonance graph correlates with the resonance energy of benzenoid by giving a quite satisfactory regression formula. Fournier [4] re-introduced this concept under the name perfect matching graph in domino tiling space. For more mathematical properties and chemical applications about $Z$-transformation graphs, the interested reader is referred to [14,33] and a recent survey [23] and references therein.

By distinguishing all alternating cycles with respect to some 1-factor of a plane bipartite graph into two classes, Zhang and Zhang [30] gave an orientation $\tilde{Z}(G)$ on the $Z$-transformation graph $Z(G)$. The property that $\tilde{Z}(G)$ is acyclic [31] yields a natural poset, denoted by $\mathbf{M}(G)$, on the set of 1-factors of $G$. In general, $\tilde{Z}(G)$ is the Hasse diagram of $\mathbf{M}(G)$, and $Z(G)$ is the cover graph (also called undirected Hasse diagram) of $\mathbf{M}(G)$. For a plane (weakly) elementary bipartite graph $G$ (its definition

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An FDL is a matchable FDL [28] if there exists a plane bipartite graph \( G \) such that \( L \cong M(G) \). So it is natural to ask whether every FDL is a matchable FDL. In fact, in this paper we will show that non-matchable FDLs exist. Thus, characterizing the matchable FDLs is a further problem. Zhang et al. [28] showed that for a plane elementary bipartite graph \( G \), \( M(G) \) is irreducible. From this, a decomposition theorem is obtained: an FDL \( L \) is matchable if and only if for any direct product decomposition of \( L \), every factor is matchable.

The remainder of this paper consists of three sections. Some basic results about matchable FDLs are given in Section 2. In Section 3, we give some results and structures of Fibonacci cubes and Lucas cubes related to matchable FDLs. We show that some of these cubes can be and some cannot be the 2-transformation graphs of planar bipartite graphs. Thus sequences of matchable and non-matchable FDLs are obtained. In the last section, we construct a sequence of non-matchable FDLs by characterizing sub-structures of matchable FDLs with cut-elements, where a cut-element of an FDL is a cut-vertex of its Hasse diagram, that is a cut-vertex of its cover graph, and a type \((m, n)\) cut-element is a cut-element which is covered exactly by \( m \) elements and covers exactly \( n \) elements. By the planarity of the duals for plane graphs, we show that if a matchable FDL has a type \((m, n)\) cut-element, then \( \min(m, n) \leq 2 \). We also show that a matchable FDL having a type \((2, n)\) cut-element with \( n \geq 2 \) must contain a special sublattice. Applying these results, we construct a sequence of non-matchable FDLs with cut-elements. By the decomposition theorem, some non-matchable FDLs without cut-elements are also obtained.

### 2. Matchable FDLs

Let \( G \) be a plane bipartite graph with a proper white/black coloring of its vertices, and let \( M \) be a 1-factor of \( G \). A cycle \( C \) of \( G \) is resonant or nice if \( G - V(C) \) has a 1-factor. A cell of \( G \) is a bounded face whose boundary is a cycle. In this paper we do not distinguish a cell from its boundary. We say that a face is resonant if its boundary is a resonant cycle. A bipartite graph is elementary [15] if it is connected and every edge belongs to a 1-factor. The complete graph \( K_2 \) with two vertices is the trivial elementary bipartite graph. Lovász and Plummer [15] showed that nontrivial elementary graphs are 2-connected. Also, a plane bipartite graph is a nontrivial elementary graph if and only if every face (including the outer face) is resonant (Zhang and Zhang [32]). A cycle or path of a graph is M-alternating if its edges are alternately in and out of \( M \). Furthermore, an M-alternating cycle in a plane bipartite graph with 1-factor \( M \) is proper if under the clockwise orientation of the cycle the edges in \( M \) are oriented from white vertices to black vertices; otherwise it is improper.

The symmetric difference \( A \oplus B \) of finite sets \( A \) and \( B \) is defined by \( A \oplus B = (A \setminus B) \cup (B \setminus A) \). If \( A \) and \( B \) are subgraphs of a graph, then \( A \oplus B \) is treated as the symmetric difference of their edge-sets.

**Definition 2.1** ([25,31]). Fix a white/black proper coloring of a plane bipartite graph \( G \) having a 1-factor, and let \( M(G) \) be the set of all 1-factors of \( G \). The Z-transformation digraph (or resonance digraph) of \( G \), denoted by \( \tilde{Z}(G) \), is defined as the digraph on \( M(G) \) such that there exists an arc from \( M \) to \( M' \) if and only if the symmetric difference \( M \oplus M' \) is a proper \( M^{-} \) alternating cell of \( G \). Ignoring all directions of arcs of \( \tilde{Z}(G) \), we get the usual Z-transformation graph or resonance graph \( Z(G) \) (see Fig. 1).

As noted in [31], the property that \( \tilde{Z}(G) \) is acyclic yields a partial ordering: for \( M, M' \in M(G) \), \( M' \leq M \) if and only if \( \tilde{Z}(G) \) has a directed path from \( M \) to \( M' \). As noted in [24], \( M \) covers \( M' \) if and only if \( M' \oplus M \) is a proper \( M^{-} \) (thus improper \( M^{-} \)) alternating cell. A change from \( M \) to \( M' \) on such a cell is a twist or Z-transformation on the cell. We denote this poset by \( M(G) \).
Let $G$ be a plane bipartite graph with a 1-factor. Elementary components of $G$ are components of the subgraph obtained from $G$ by removing all edges not contained in any 1-factors. Each elementary component having only one edge is a trivial elementary component. As introduced in [32], a graph $G$ is weakly elementary if $I(C)$ is elementary for every resonant cycle $C$ of $G$, where $I(C)$ denotes the subgraph of $G$ consisting of $C$ together with its interior. It is easy to check that every plane elementary bipartite graph is weakly elementary.

**Theorem 2.1** ([24]). If $G$ is a plane bipartite graph having a 1-factor, then $M(G)$ is an FDL if and only if $G$ is weakly elementary. Furthermore, if $G$ is weakly elementary, and $G_1, \ldots, G_n$ are the elementary components of $G$, then

$$M(G) \cong M(G_1) \times \cdots \times M(G_n),$$

where “$\times$” denotes the direct product of posets.

We now define matchable FDLs as follows.

**Definition 2.2** ([28]). An FDL $L$ is matchable if there is a plane weakly elementary bipartite graph $G$ such that $L \cong M(G)$; otherwise it is non-matchable.

For example, the $n$-element chain $n$ is matchable. So are $m \times n$ and $B_n$, the Boolean algebra of rank $n$. Furthermore, it was shown in [28] that $J(m \times n)$ and $J(T)$ are matchable, where $J(P)$ denotes the distributive lattice on all order ideals of poset $P$ ordered by inclusion and $T$ is a poset implied by any orientation of a tree $T$.

An FDL $L$ is nontrivial if it has at least two elements. The expression $L = \prod_{i=1}^k L_i$ is a direct product decomposition when each $L_i$ is an FDL, and then $L_1, \ldots, L_k$ are the factors of $L$. We say an FDL is irreducible if it cannot be decomposed into a direct product of two nontrivial FDLs. Zhang et al. [28] obtained some basic results about matchable FDLs.

**Theorem 2.2** ([28]). If $G$ is a plane elementary bipartite graph, then $M(G)$ is irreducible.

**Theorem 2.3** ([28]). If $L = \prod_{i=1}^k L_i$ is a direct product decomposition of an FDL $L$, then $L$ is matchable if and only if $L_1, \ldots, L_k$ are matchable.

**Remark 2.4.** It is immediate from the definitions that $Z(K_2) = K_1$ and that $H \times K_1 \cong H$ and $L \times 1 = L$ for any graph $H$ and any FDL $L$. By Theorem 2.1, we may assume that $G$ has no trivial elementary components when we look for some plane weakly bipartite graph $G$ such that $M(G) \cong L$ is a matchable FDL.

Let $P_0$ denote the dual poset of a poset $P$. Let $G^1$ be the 2-colored graph obtained from the 2-colored plane bipartite graph $G$ by interchanging the two color classes of $G$. By the definitions of proper and improper alternating cells, and $Z$-transformation digraph, the following proposition is obvious.

**Proposition 2.5.** $M(G^1) \cong (M(G))^*$. □

For an edge $e$ of a graph $G$, the operation of inserting an even number of new vertices of degree 2 on $e$ is an even subdividing of $e$. A graph $G'$ is an even subdivision of $G$ if $G'$ can be obtained from $G$ by even subdivisions of edges. When $G$ is a plane bipartite graph, an even subdivision $G'$ of $G$ is also a plane bipartite graph, and the incidence relations between the old vertices and faces are the same in $G'$ and $G$. For instance, in Fig. 4 the left three graphs are even subdivisions of the grid $I_5$.

**Lemma 2.6** ([15, Chapter 4]). Every even subdivision of a nontrivial elementary bipartite graph is also elementary.

**Lemma 2.7.** If $G'$ is an even subdivision of a plane elementary bipartite graph $G$, then $\bar{Z}(G') \cong \bar{Z}(G)$, and $M(G') \cong M(G)$.

**Proof.** It is sufficient to show that the claim is true when $G'$ is obtained from $G$ by subdividing an edge $e$ by introducing two new vertices. This replaces $e$ with three consecutive new edges, denoted by $e'_1, e'_2$ and $e'_3$. For any 1-factor $M$ of $G$, let $M' = (M \setminus \{e\}) \cup \{e'_1, e'_2\}$ if $e \in M$, and $M' = M \cup \{e'_2\}$ otherwise. Note that $M'$ is a 1-factor of $G$. This establishes a bijection from $M(G)$ to $M(G')$. Similarly, for any cycle $C$ of $G$ if $e \in C$, let $C' = (C \setminus \{e\}) \cup \{e'_1, e'_2, e'_3\}$ otherwise let $C' = C$. Note that $C$ is a proper (resp. improper) $M$-alternating cycle of $G$ if and only if $C'$ is a proper (resp. improper) $M'$-alternating cycle of $G'$. Hence, the bijection from $M(G)$ and $M(G')$ preserves the proper (resp. improper) alternating cycles (thus cells) in $G$ and those in $G'$; thus it is an isomorphism from $\bar{Z}(G)$ to $\bar{Z}(G')$. □

**Lemma 2.8** ([32]). Let $G$ be a plane elementary bipartite graph with a 1-factor $M$. If $C$ is an $M$-alternating cycle, then there is an $M$-alternating cell in $I(C)$. □

Let $G$ be a plane weakly elementary bipartite graph. The FDL $M(G)$ has a unique minimum element $M^0$, that is, $G$ has no proper $M^0$-alternating cycles. Therefore, $M^0$ is called the root 1-factor of $G$. Also $G$ has a unique source 1-factor $M^1$ such that $G$ has no improper $M^1$-alternating cycles.

**Lemma 2.9** ([28]). If $G$ is a nontrivial plane elementary bipartite graph, then the outer boundary of $G$ is a proper $M^1$ (resp. an improper $M^0$) alternating cycle. □

A path in a 2-connected graph is a thread if all its internal vertices have degree 2 and its endpoints have degree at least 3. An edge with both of its end-points have degree at least 3 is a thread of length 1. Given a 1-factor $M$ of $G$, an $M$-alternating
path $P$ with odd length is proper if both terminal edges of $P$ belong to $M$ and improper otherwise. Here are two simple lemmas on alternating cells and cycles. The first one is known (cf. [29, Lemma 2.6]).

**Lemma 2.10** ([29]). If $M$ is a 1-factor of a plane bipartite graph $G$, then the boundaries of all proper (resp. improper) $M$-alternating cells of $G$ are pairwise disjoint.

**Lemma 2.11.** Let $C_1$ and $C_2$ be distinct and intersecting $M$-alternating cycles, where $M$ is a 1-factor in a plane bipartite graph $G$. The maximum degree of $C_1 \cup C_2$ is 3, and each component of $C_1 \cap C_2$ is a proper $M$-alternating thread in $C_1 \cup C_2$. Thus each component of $C_1 - C_2$ and $C_2 - C_1$ is an improper $M$-alternating thread in $C_1 \cup C_2$, where $C_1 - C_2$ denotes the subgraph obtained from $C_1$ by deleting all edges together with interior vertices of each component of $C_1 \cap C_2$.

**Proof.** Since $C_1$ and $C_2$ are $M$-alternating, for any vertex $v$ of $C_1 \cup C_2$ there is exactly one edge in $M$, denoted by $e$, incident with $v$. If $v$ is a common vertex of $C_1$ and $C_2$, then $e \in E(C_1) \cap E(C_2)$. Hence $C_1 \cup C_2$ has the maximum degree at most 3, and a vertex with degree 3 must exist, since $C_1$ and $C_2$ are distinct. If $d(v) = 3$, then $v$ must be incident with exactly one edge not in $M$ in each $C_i$. If $d(v) = 2$, then $v$ must be incident with one edge in $M$ and one edge not in $M$ of either $C_1$ or $C_2$, or maybe both. This shows that each component of $C_1 \cap C_2$ is a proper $M$-alternating thread of $C_1 \cup C_2$. □

3. **Fibonacci cubes, Lucas cubes, and matchable FDLs**

The $n$-cube $Q_n$ is the graph whose vertex set is the set of binary $n$-tuple in which vertices are adjacent if they differ in one bit. A binary $n$-tuple is a Fibonacci string if it has no consecutive 1s; it is a Lucas string if also it does not begin and end with 1. For $n \geq 1$, the Fibonacci (resp. Lucas) cube $\Gamma_n$ (resp. $\Lambda_n$) [7, 10, 16], is the subgraph of $Q_n$ induced by the Fibonacci (resp. Lucas) strings with length $n$. Clearly $V(\Gamma_n) = \{0v : v \in V(\Gamma_{n-1})\} \cup \{10v : v \in V(\Gamma_{n-2})\}$ and $V(\Lambda_n) = \{0v : v \in V(\Gamma_{n-1})\} \cup \{10v0 : v \in V(\Gamma_{n-3})\}$. It is well known that $|V(\Gamma_n)| = f_{n+2}$ and $|V(\Lambda_n)| = f_{n+1} + f_{n+1} = l_n$. Here $\{f_n\}$ and $\{l_n\}$ are the Fibonacci sequence and Lucas sequence, respectively. They are defined by the same recurrence relation $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, with different initial conditions $f_0 = 0$, $f_1 = 1$ and $l_0 = 2$, $l_1 = 1$. Note that $\Gamma_1 = K_2$, $\Lambda_1 = K_1$, $\Gamma_2 = \Lambda_2 = P_3$, and $\Lambda_3 = K_{1,3}$. Some other Fibonacci and Lucas cubes are shown in Fig. 2.

Let $Z_n$ denote the “zigzag poset” or “fence” (cf. [8, 20] or [22, Chapter 3, ex23]): an $n$-element poset on $\{x_1, \ldots, x_n\}$ with cover relations $x_{2i} \prec x_{2i-1}$ and $x_{2i} \prec x_{2i+1}$; See Fig. 3(a) and (b). By adding one more cover relation $x_{2n} \prec x_1$ in poset $Z_{2n}$, we obtain a poset called a “crown” [8, 20], denoted by $\succeq_{2n}$ (see Fig. 3(c)). Further, by adjoining the minimum element 0 and the maximum element 1 to $\succeq_{2n}$, we obtain a poset $\Rightarrow_{2n}$; See Fig. 3(d).

Let $H$ and $G$ be two graphs. An edge-preserving map $\phi$ of $H$ to $G$ is a map of $V(H)$ to $V(G)$ such that $\phi(u)\phi(v) \in E(G)$ if $uv \in E(H)$. We say that $H$ is a retract of $G$ if there are two edge-preserving maps $\phi$ of $H$ to $G$ and $\psi$ of $G$ to $H$ such that $\psi \phi(v) = v$ for each $v \in V(H)$. Note that if $H$ is a retract of $G$ it is convenient to take $H$ as a subgraph of $G$ and $\phi$ to be an inclusion map.

**Lemma 3.1.** Letting $\Gamma_n = J(Z_n)$ and $\Lambda_{2n} = J(\Rightarrow_{2n})$ for positive integer $n$,

1. $\Lambda_{2n}$ is the cover graph of the FDL $\Lambda_{2n}$.
2. [5, 21] $\Gamma_n$ is the cover graph of the FDL $\Gamma_n$, and
3. $\Lambda_{2n-1}$ is not a cover graph of any FDL unless $n = 1$. 

![Fig. 2. The Fibonacci and Lucas cubes $\Gamma_3$, $\Gamma_4$, $\Gamma_5$, $\Lambda_4$, and $\Lambda_5$.](image)

![Fig. 3. Some posets: (a) fence $Z_{2n-1}$, (b) fence $Z_{2n}$, and (c) crown $\Rightarrow_{2n}$, (d) $\Rightarrow_{2n}$](image)
Proof. (1) If \( n = 1 \), then \( \triangleright\triangleright\triangleright\triangleright\triangleright \) degenerates into the 2-element chain. Both \( \Lambda_2 \) and \( \mathbf{J}(2) \) are isomorphic to the 3-element chain. Thus the claim holds for \( n = 1 \). For \( n \geq 2 \), we establish a map \( f \) from the Lucas sequences of length \( 2n \) to the order ideals of \( \triangleright\triangleright\triangleright\triangleright\triangleright \). For any binary string \( b = b_1 b_2 \cdots b_{2n} \in \Lambda_{2n} \), complement each even bit of \( b \) and denote the resulting binary string by \( b' = b_1' b_2' \cdots b_{2n}' \). Let \( f(b) \) be the subset of \( \triangleright\triangleright\triangleright\triangleright\triangleright \) whose elements are indexed by the positions that equal 1 in \( b' \). That is, \( x_i \in f(b) \) if and only if \( b_i' = 1 \). Now, if \( b_{2i-1}' = b_{2i-1} = 1 \), then \( x_{2i-1} \in f(b) \), \( b_{2i-2} = b_{2i} = 0 \) (taking the subscripts modulo \( 2n \)), and thus \( b_{2i-2}' = b_{2i}' = 1 \). This implies \( x_{2i-2}, x_{2i} \in f(b) \). Hence \( f(b) \) is an order ideal of \( \triangleright\triangleright\triangleright\triangleright\triangleright \).

Conversely, let \( I \) be an order ideal of \( \triangleright\triangleright\triangleright\triangleright\triangleright \). Let \( b_{2i-2}' = b_{2i}' = 1 \) or \( 0 \) according to whether \( x_{2i-1} \in I \) or not. Let \( b_{2i} = 0 \) or 1 according to whether \( x_{2i} \in I \) or not. Similarly we can see that \( b_1 b_2 \cdots b_{2n} \) is a Lucas string. Hence \( f \) is a bijection. Further we can see that \( f \) is an isomorphism from \( \Lambda_{2n} \) to the cover graph of the lattice \( \mathbf{J}(\triangleright\triangleright\triangleright\triangleright\triangleright) \).

Theorem 3.2. (1) [11] If \( F_n \) is a fibonacci with \( n \) hexagons, then \( Z(F_n) \cong \Gamma_n \).
(2) [27] If \( G \) is a plane bipartite graph with no trivial elementary component, then \( Z(G) \cong \Gamma_n \) if and only if \( G \) is an even subdivision of \( \Lambda_{2n} \).

Let \( O_{2n} \) be a plane embedding of graph \( C_{2n} \times K_2 \) with \( n \geq 2 \) such that the outer face of \( O_{2n} \) is bounded by \( C_{2n} \) (see Fig. 5). It is clear that \( O_{2n} \) is elementary. Note that \( O_4 \) is just the cube \( Q_3 \). Let \( \Lambda_{2n} \) be the graph obtained from \( \Lambda_{2n} \) by adding two new vertices with one adjacent to \( 1010 \cdots 10 \) and another adjacent to \( 0101 \cdots 01 \).

Lemma 3.3 ([27]). For positive integer \( n \).

(1) If \( n \geq 3 \), then \( \Lambda_n \) is not a \( Z \)-transformation graph of any plane bipartite graph, and

(2) If \( G \) is a plane bipartite graph with no trivial elementary component, then \( Z(G) \cong \Lambda_n \) if and only if \( G \) is an even subdivision of \( O_{2n} \).

Hence, by Theorem 2.1 and Lemma 3.3(2), we obtain two matchable FDLs \( M(L_n) \) and \( M(O_{2n}) \) determined by the orientations \( \hat{Z}(L_n) \) and \( \hat{Z}(O_{2n}) \) of \( \Gamma_n \) and \( \Lambda_{2n} \), respectively. The partial order determined by such an orientation of \( \Gamma_n \) is different from the natural order in the integer lattice \( \mathbf{Z} \). For examples, see Fig. 2(a), (b), and (c).

By Lemmas 3.1–3.3, we can show that these FDLs are matchable. The ordinal sum of two disjoint posets \( P \) and \( Q \) having unique maximal and minimal elements is the poset \( P \sqcup Q \) such that \( x \leq y \) in \( P \sqcup Q \) if \( (a) x, y \in P \) and \( x \leq y \in P \), or \( (b) x, y \in Q \) and \( x \leq y \) in \( Q \), or \( (c) x \in P \) and \( y \in Q \). Similarly, the vertical sum of \( P \) and \( Q \) is the poset \( P \sqcup Q \), where the only difference from the ordinal sum is that now the maximum element 1 of the lower summand \( P \) and the minimum element 0 of the upper summand \( Q \) are identified instead of becoming neighbors. Let \( \hat{P} \) denote the poset obtained from a poset \( P \) by adjoining a new 0 and 1 (in spite of an (old) 0) and 1 which \( P \) may already possess), i.e. \( \hat{P} = 1 \sqcup P \sqcup 1 \). See Fig. 3(c) and (d).

Theorem 3.4. For a positive integer \( n \), let \( \Lambda_{2n} = \mathbf{J}(\triangleright\triangleright\triangleright\triangleright\triangleright) \).

(1) \( \Gamma_n \cong M(L_n) \) is a matchable FDL;
(2) \( \Lambda_{2n} \cong M(O_{2n}) \) is a matchable FDL;
(3) If \( n \geq 2 \), then \( \Lambda_{2n} \) is a non-matchable FDL; and
(4) For any FDL \( L \), if \( n \geq 2 \), then the FDLs \( \Lambda_{2n} \sqcup L \), \( L \sqcup \Lambda_{2n} \), \( \Lambda_{2n} \sqcup \Lambda_{2n} \), \( \Lambda_{2n} \sqcup L \), \( L \sqcup \Lambda_{2n} \), and \( \Lambda_{2n} \sqcup \Lambda_{2n} \) are non-matchable.

Proof. (1) Let \( f_1, f_2, \ldots, f_n \) denote the squares of \( L_n \) from left to right (as drawn in Fig. 4). Recall that the outer face \( f_0 \) never twists in \( \hat{Z}(L_n) \). Since all cells of \( L_n \) are adjacent to \( f_0 \), each cell \( f_i \) twists at most once in \( \hat{Z}(L_n) \). On the other hand, each cell \( f_i \) of \( L_n \) is resonant, so there exists some 1-factor \( M \) of \( L_n \) such that \( f_i \) is (proper) \( M \)-alternating. Thus each cell \( f_i \) twists exactly once during the generation of \( \hat{Z}(L_n) \) from \( M \). Therefore, any 1-factor \( M \) is determined by the cells twisted from \( M \), which are the cells contained in some cycle of \( M \sqcup M \). By Lemma 2.9, \( f_1, f_2, \ldots, f_n \) are proper \( M \)-alternating (see Fig. 5(a)). So all
cells having odd subscripts, \(f_1, f_3, \ldots\), can twist in \(M^1\). However, each cell \(f_2\) with even subscript can twist in some 1-factor \(M\) of \(L_n\) after both \(f_{2i-1}\) and \(f_{2i+1}\) (if they exist) twist during the generation of \(M\) from \(M^1\). So any 1-factor \(M\) is determined by the cells twisted, and thus determined by the cells untwisted, from \(M^1\). On the other hand, for any given order ideal \(I\) of \(Z_n\), \(I \cup \{x_{2i}\}\) is also an order ideal, \(1 \leq i \leq n - 1\), and \(x_{2i-1} \in I\) or \(x_{2i+1} \in I\) imply \(x_{2i} \notin I\), but \(x_{2i} \in I\) does not imply \(x_{2i-1} \in I\) or \(x_{2i+1} \in I\) (see Fig. 3(a) and (b)). Thus, by mapping cell \(f_i\) of \(L_n\) to the element \(x_i\) of \(Z_n\), we can establish a bijection from the perfect matchings \(M = L_n\) to the order ideals \(I\) of \(Z_n\) (the set of untwisted cells from \(M^1\) to \(M\) corresponds to \(I\)), which is an isomorphism from \(\mathbf{M}(L_n)\) to \(\mathbf{J}(Z_n)\).

(2) Let \(f_0\) and \(f_1\) denote the outer face and the central cell of \(O_{2n}\), respectively. The other cells are denoted by \(g_1, h_1, g_2, \ldots, g_n, h_n\) in counterclockwise order. The 1-factor \(M^1\) consists of common edges of \(f_0\) and common edges of \(g_i\) and \(f_1\) and \(h_1\) (see Fig. 5(b)). As in (1), each \(g_i\) and \(h_i\) twists exactly once while generating \(Z(O_{2n})\) from \(M^1\). However, \(f_1\) twists twice since \(\text{Lemma 2.9}\), \(f_1\) is proper \(M^0 \oplus f_1\)-alternating and proper \(M^1\)-alternating. Also, each \(g_i\) is proper \(M^1 \oplus f_1\)-alternating. Moreover, for some 1-factor \(M^1\), \(h_i\) is proper \(M^\prime\)-alternating if and only if both \(g_i\) and \(g_{i+1}\) are improper \(M^\prime\)-alternating, where subscripts modulo 2n. After each \(h_i\) is twisted, \(f_1\) can be twisted again to reach the root \(M^0\) of \(\mathbf{M}(O_{2n})\). Hence, the poset, ordered by the order in which the cells are twisted, is just \(\approx_{2n}\), where \(\hat{1}\) and \(\hat{0}\) correspond to the first and final twists of \(f_1\), respectively. Hence the claim holds.

(3) This is implied by \(\text{Lemma 3.3}\).

(4) Suppose to the contrary that there exists some plane weakly elementary bipartite graph \(G\) such that \(A_{2n} \cup L = \mathbf{M}(G)\), where \(n \geq 2\). The graph \(G\) has a unique 1-factor \(M^1\) corresponding to the maximum element \(\approx_{2n}\) of \(A_{2n}\). Let \(\{x_1, x_2, \ldots, x_{2n}\}\) be the elements of \(\approx_{2n}\). Since \(\approx_{2n}\) covers exactly \(n\) order ideals \(\approx_{2n} \setminus \{x_{2i}\}\) of \(\approx_{2n}\), \(G\) has \(n\) proper \(M^\prime\)-alternating cells, say \(f_1, \ldots, f_n\), such that 1-factor \(M^1 \oplus f_i\) of \(G\) corresponds to the order ideal \(\approx_{2n} \setminus \{x_{2i}\}\), for \(1 \leq i \leq n\). By \(\text{Lemma 2.10}\), \(f_1, \ldots, f_n\) are pairwise disjoint. Letting \(M = M^1 \oplus (\oplus_{i=1}^n f_i)\), the 1-factor \(M\) corresponds to the order ideal \(\{x_2, x_4, \ldots, x_{2n}\}\). Note that \(G\) has exactly \(n\) 1-factors \(M \oplus g_i\) covered by \(M\) in \(\mathbf{M}(G)\), since \(\approx_{2n}\) has \(n\) order ideals \(\{x_2, x_4, \ldots, x_{2n}\} \setminus \{x_{2i}\}\) covered by \(\{x_2, x_4, \ldots, x_{2n}\}\) in \(\approx_{2n}\) for \(1 \leq i \leq n\), where \(g_1, \ldots, g_n\) are proper \(M^\prime\)-alternating cells of \(G\). Again by \(\text{Lemma 2.10}\), all \(g_1, \ldots, g_n\) are pairwise disjoint and different from any \(f_i\).

We claim that each \(g_i\) intersects only \(f_i\) and \(f_{i+1}\) (taking subscripts modulo \(n\)). Since \(M^1 \oplus f_i \oplus f_{i+1}\) corresponds to \(\approx_{2n} \setminus \{x_{2i-1}, x_{2i+1}\}\), in \(G\) there is a 1-factor \(M^1 \oplus f_i \oplus f_{i+1} \oplus g_i\) corresponding to \(\approx_{2n} \setminus \{x_{2i-1}, x_{2i+1}\}\), where \(g_i\) is a proper \((M^1 \oplus f_i \oplus f_{i+1})\)-alternating cell of \(G\). It is obvious that \(g_i\) is disjoint from all \(f_j\) except \(f_i\) and \(f_{i+1}\). Any two saturated chains of \(\mathbf{M}(G)\) from \(M^1\) to \(g_i\) passing through \(M^1 \oplus f_i \oplus f_{i+1} \oplus g_i\) respectively twist the same set of cells (cf. \(\text{Lemma 3.5}\) in [24]). So \(g_i = g_i\). If \(g_i\) is disjoint from \(f_i\), then \(M^1 \oplus f_i \oplus g_i\) is a 1-factor of \(G\), which does not correspond to an order ideal covered by \(\approx_{2n} \setminus \{x_{2i-1}\}\). Hence the claim holds.

Let \(M^\prime = M \cup \oplus_{i=1}^n g_i\). The 1-factor \(M^\prime\) corresponds to the order ideal \(\emptyset\) as the minimum element of \(I(\approx_{2n})\). That is, \(M^\prime\) is also the minimum element of \(\mathbf{M}(G)\). Let \(G^\prime = \cup_{i=1}^n f_i \cup (\oplus_{i=1}^n g_i)\) and \(G^\prime = \oplus_{i=1}^n (f_i \oplus g_i)\). By \(\text{Lemma 2.11}\), it follows that \(G^\prime\) consists of disjoint \(M^\prime\)-alternating cycles, and one must be a proper \(M^\prime\)-alternating cycle \(C\) whose interior does not contain any \(f_i\) or \(g_i\). By \(\text{Lemma 2.8}\), \(I(C)\) must contain a proper \(M^0\)-alternating cell \(f\) of \(G\) that does not equal any \(f_i\) or \(g_i\). Thus \(G\) must have another 1-factor \(M^\prime \oplus f\) covered by \(M^\prime\). This contradicts that \(M^\prime\) is the minimum element of \(\mathbf{M}(G)\).

In an analogous way, \(A_{2n} \oplus L\) is a non-matchable FDL. By the dual poset, the remaining results are true too.

**Example 3.5.** The FDLs in Fig. 6 (and their duals) are non-matchable FDLs. Moreover, for positive integers \(m, n \geq 2\), \(A_{2n} \oplus \cdots \oplus A_{2n}\), the vertical sums of \(m\) copies of \(A_{2n}\) are non-matchable.

### 4. Non-matchable FDLs with cut-elements

Recall the definitions of cut-elements from the introduction. Here is a fact about cut-vertices of \(Z(G)\).

**Lemma 4.1** ([31]). Let \(G\) be a plane elementary bipartite graph. If \(Z(G)\) has a cut-vertex \(M\), then \(G\) has both proper and improper \(M\)-alternating cells, and every proper \(M\)-alternating cell intersects every improper \(M\)-alternating cell.

Let \(G\) be a weakly elementary plane bipartite graph such that \(\mathbf{M}(G)\) contains a cut-element. By \(\text{Remark 2.4}\) and the fact that the Cartesian product of two connected graphs other than \(K_1\) is 2-connected, we may assume that \(G\) is elementary. In
the sequel, $G$ always means a plane elementary bipartite graph other than $K_2$ with a given white–black coloring on its vertex set, unless otherwise specified. For a plane elementary bipartite graph $G$ with a perfect matching $M_v$, from the definition of cut-element and Lemma 4.1 it follows that the following three statements are equivalent:

1. $M_v$ is a type $(m, n)$ cut-element of the matchable FDL $M(G)$,
2. $G$ has exactly $n$ improper and $n$ proper $M_v$-alternating cells such that each proper cell intersects each improper cell, and
3. $M_v$ is a cut-vertex of $\tilde{Z}(G)$ with in-degree $m$ and out-degree $n$.

We now give the first main result of this section.

**Lemma 4.2.** If $M(G)$ has a type $(m, n)$ cut-element $M_v$, then $m \leq 2$ or $n \leq 2$.

**Proof.** Suppose to the contrary that $m \geq 3$ and $n \geq 3$. Let $f_1, f_2, f_3$ and $g_1, g_2, g_3$ be three improper and three proper $M_v$-alternating cells of $G$ respectively. By Lemmas 4.1 and 2.11, $f_i \cap g_j \neq \emptyset$ is a proper $M_v$-alternating thread, for $i, j \in \{1, 2, 3\}$. Thus these six cells form a complete bipartite graph $K_{3, 3}$ in the dual graph $G^*$ of $G$. This contradicts the planarity of $G^*$. □

**Theorem 4.3.** Let $L$ be an FDL with a type $(m, n)$ cut-element. If $m \geq 3$ or $n \geq 3$, then $L$ is a non-matchable FDL. □

**Example 4.4.** The FDLs (and their duals) in Fig. 7 are non-matchable FDLs. Moreover, for positive integers $m, n \geq 3$, the FDL $B_m \oplus B_n$ is non-matchable.

Thus, if $M(G)$ has a type $(m, n)$ cut-element $M_v$, then we only need to consider the cases that $m = 2$ and $n = 1$. We consider mainly the former. For a 1-factor $M$ and a subgraph $H$ of $G$, the notation $M|_H$ means the restriction of $M$ to $H$.

Now let us consider matchable FDLs with a type $(2, n)$ cut-element. First, consider two matchable FDLs related to $\Gamma_n$ and $A_{2n}$. Given a grid $L_{2n-1}$, we label the white (resp. black) vertices by $w_i$ (resp. $b_i$), for $0 \leq i \leq n$, from left to right (note that the top-left vertex is white). Similarly, we label the cells by $g_1, g_2, \ldots, g_n$ from left to right.

For $n \geq 2$, let $H_n$ denote the plane graph obtained from grid $L_{2n-1}$ by joining $w_0$ and $b_{2n-1}$ with a line (i.e. an edge) lying above $L_{2n-1}$ and joining $b_0$ and $w_{2n-1}$ with a line lying under $L_{2n-1}$ (see Fig. 8(a)). Note that $H_n$ has two more cells than $L_{2n-1}$; call them $f_1, f_2$. Each of $f_1$ and $f_2$ shares an edge with each square of $L_{2n-1}$.

For $n \geq 2$, let $H'_n$ denote the plane graph obtained from grid $H_{n+1}$ by joining $w_1$ and $b_{2n}$ with a line inside the original $f_2$ and then removing the edges $b_0w_1$ and $b_{2n}w_{2n-1}$ (see Fig. 8(b)). We denote the new cells lying above and under the line $w_1b_{2n}$ by $f_2$ and $g_1$, respectively. Each of $f_1$ and $f_2$ shares an edge with all squares and $g_1$, the latter being bounded by an 8-cycle sharing two edges with $f_1$ and one edge with $f_2$.

For the graphs $H_n$ and $H'_n$, let $M_n$ denote the unique 1-factor of the graph such that both $f_1$ and $f_2$ are improper $M_n$-alternating. Thus each $g_i$ is proper $M_n$-alternating. Let $M_n = M_v \oplus f_1 \oplus f_2$. We can see that $M_v$ and $M_n$ are cut-elements of $M(H_n)$ and $M(H'_n)$. Note that $M(H_2) \cong M(Q_3) \cong A_4$. For any $x, y \in M(G)$, let $I[x, y] = \{z \in M(G) | x \leq z \leq y\}$.
When a poset $\mathbf{L}'$ is isomorphic to a sublattice of a lattice $\mathbf{L}$, we say that $\mathbf{L}'$ is a sublattice of $\mathbf{L}$. Similar to Theorem 3.4(2), we have

**Lemma 4.5.** (1) In $\mathbf{M}(H_n)$, $\mathbf{I}[M_u, M^i_1] \cong \Gamma_{2n-3}$ and $\mathbf{I}[M^0, M_c] \cong \Gamma_{2n-1}$. Thus $\mathbf{M}(H_n) \cong \Gamma_{2n-1} \oplus B_2 \oplus \Gamma_{2n-3}$.
(2) In $\mathbf{M}(H_n')$, $\mathbf{I}[M_u, M^1_1] \cong \Gamma_{2n-1}$ and $\mathbf{I}[M^0, M_c] \cong \Lambda_{2n}$. Thus $\mathbf{M}(H_n') \cong 1 \oplus \Lambda_{2n} \oplus B_2 \oplus \Gamma_{2n-1}$, and
(3) $\mathbf{M}(H_n)$ is a sublattice of $\mathbf{M}(H_n')$.

**Proof.** (1) By Lemma 2.9, the boundary cycle $C$ of $H_n$ is proper $M^i$-alternating. Let $\mathbf{M}_c$ denote the subposet induced by the set of 1-factors of $H_n$ for which $C$ is proper alternating. We have $\mathbf{M}_c \cong \mathbf{M}(G')$, where $G' = H_n - V(C)$. It is clear that $G' \cong L_{2n-3}^i$.

So, by Theorem 3.4(1), $\mathbf{M}_c \cong \mathbf{M}(G') \cong \Gamma_{2n-3}^i$. The maximum element of $\mathbf{M}_c$ is $M^i_1$, and the minimum element of $\mathbf{M}_c$ is $M_u$.

Hence, $\mathbf{I}[M_u, M^1_1] \cong \mathbf{M}_c \cong \Gamma_{2n-3}^i$, and $\mathbf{I}[M_u, M_c] = B_2$.

Now let us prove the rest of the assertion. After both $f_1$ and $f_2$ being twisted, each $g_i$ is proper $M_c$-alternating, and the edges $w_0b_{2n-1}$ and $b_0w_{2n-1}$ never belong to any 1-factor again. Hence, $\mathbf{I}[M^0, M_c] \cong \mathbf{M}(L_{2n-1}) \cong \Gamma_{2n-1}$.

(2) Let $\mathbf{M}_c'$ denote the subposet of $\mathbf{M}(H_n')$ induced by the set of 1-factors $M$ of $H_n'$ such that $w_1b_{2n} \notin M$ and the boundary cycle $C$ of $H'_n$ is proper $M$-alternating. Let $G' = H_n' - V(C) - w_1b_{2n}$. The remaining arguments are similar to (1).

(3) By (1) and (2) and the fact that $\Gamma_{2n-3}$ is a sublattice of $\Gamma_{2n-1}$, it is sufficient to prove that $\Gamma_{2n-1}$ is a sublattice of $\Lambda_{2n}$.

This follows from this fact: A subset $I$ of $\{x_1, \ldots, x_{2n-1}\}$ is an order ideal of $Z_{2n-1}$ if and only if $I \cup \{x_{2n}\}$ is an order ideal of $\Lambda_{2n}$.

Let $\Pi_n = \Gamma_{2n-1} \oplus B_2 \oplus \Gamma_{2n-3}^i$ and $\Pi_n^+ = \Gamma_{2n-1} \oplus B_2 \oplus \Gamma_{2n-3}^i$. So Lemma 4.5(1) implies that $\Pi_n \cong \mathbf{M}(H_n)$.

**Lemma 4.6.** Let $G$ be a plane elementary bipartite graph. If $\mathbf{M}(G)$ has a type $(2, n)$ cut-element for $n \geq 2$, then $\Pi_n$ or $\Pi_n^+$ is a sublattice of $\mathbf{M}(G)$.

**Proof.** By Lemma 4.1, $G$ has exactly two improper and $n$ proper $M$-alternating cells, say $f_1, f_2$ and $g_1, \ldots, g_n$, respectively. Let $G_1 = f_1 \cup f_2 \cup g_1 \cup \cdots \cup g_n$. Note that $G_1$ is elementary and that $M|G_1$ is also a 1-factor of $G_1$. By Lemma 2.11, the maximum degree of $G_1$ is 3, and each component of $f_1 \cap g_i$ is a proper $M$-alternating edge. Also, by Lemma 2.10, $f_1$ and $f_2$ are disjoint, and $g_1, \ldots, g_n$ are pairwise disjoint. Any face $h$ of $G_1$ other than $f_1, f_2$ or $g_1, \ldots, g_n$ is adjacent with at most two cells among $f_1, \ldots, g_n$. Otherwise, $K_3$ would occur as a subgraph of the dual $G_1^*$, that is impossible.

On the other hand, each of $h \cap f_1$ and $h \cap g_i$ is a thread of $G_1$ whenever it is not empty. Otherwise, suppose there exists some $f_i$, say $f_1$, sharing at least two threads of $G_1$ with $h$. Since the threads that encircle $h$ belong alternately to $f_i$'s and $g_i$'s, there must exist a pair of parallel edges (or 2-cycle) between $f_1$ and $h^*$ in $G_1^*$, which separates $f_2$ from some $g_i^*$. This is a contradiction. The case that some $g_i$ shares at least two threads of $G_1$ with $h$ is similar. Thus all such faces $h$ of $G_1$ are encircled by two or four improper $M$-alternating threads. Let $r$ and $s$ denote the numbers of such faces of $G_1$ encircled by two and four threads, respectively. Let $h_1, \ldots, h_r$, denote such faces of $G_1$ encircled by four threads and $c_1, \ldots, c_r$ denote such faces of $G_1$ encircled by two threads. We have $(\bigcup_{i=1}^r h_i) \cup (\bigcup_{j=1}^r c_j) = f_1 \oplus f_2 \oplus g_1 \oplus \cdots \oplus g_n$.

We claim that $s = n$. Let $G_2$ be the subgraph obtained from $G_1$ by removing one thread in some $f_i$ from each $c_i$. In $G_2$, the three faces incident with any vertex of degree 3 are one of cells corresponding to $f_1$ and $f_2$, some cell $g_i$, and some face $h_i$. 

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**Fig. 8.** (a) $H_n$ with 1-factor $M_u$, (b) $H_n'$ with 1-factor $M_u$, (c) $\mathbf{M}(H_n)$ and (d) $\mathbf{M}(H_n')$. 

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Each of $g_i$ and $h_i$ has four vertices of degree 3 in $G_2$. So the number of vertices of degree 3 in $G_2$ is $4s$ and $4n$, respectively, according to counting from $h$-faces and $g$-cells. Hence $4s = 4n$ and the claim holds.

Also, $G_2$ is an even subdivision of $O_{2s}$. Hence we can relabel $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$ as $g_1, h_1, g_2, \ldots, g_n, h_n$ along $f_1$ so that each $h_i$ is adjacent to $g_i$ and $g_{i+1}$, where $g_{n+1} = g_1$. Since $O_{2s}$ is 3-connected, it has a unique embedding in the sphere or plane. Hence the outer face $f_0$ of the plane embedding of $G_1$ is one of $h_1, \ldots, h_n$ and $c_1, \ldots, c_s$. Without loss of generality, let $f_0 = h_n$ or $c_s$. Thus $G_1$ has two plane embeddings as shown in Fig. 9.

Let $L = M \oplus f_1 \oplus f_2$. Each cell $h \in \{h_1, \ldots, h_n\} \cup \{c_1, \ldots, c_s\} \setminus \{f_0\}$ is an improper $L$-alternating cycle, since the threads that encircle cell $h$ are improper $M$-alternating and belong alternately to $f_1$ and $g_i$. Note that $f_0$ is a proper $L$-alternating cycle.

Let $C_0$ be the outer boundary of the subgraph $G_0$ of $G$ defined by $G_0 = (\bigcup_{i=1}^{n-1} I[h_i]) \cup \bigcup_{i=1}^{n} g_i$. Let $L' = M \oplus g_1 \oplus g_n \oplus f_0$ and $L'' = L' \cup C_0$. Note that $f_0$ is an improper $(M \oplus g_1 \oplus g_n)$-alternating cycle and $C_0$ is a proper $L'$-alternating cycle. So both $L'$ and $L''$ are 1-factors of $G$. Also each $g_i$ is a proper $L'$-alternating cell for $2 \leq i \leq n - 1$, and $C_0$ is improper $L''$-alternating. Let $\mathcal{M}_0'$ be the set of 1-factors $F$ of $G_0$ such that for each cycle $h(ih_1, \ldots, h_{n-1}, f_{(h)} = L_{(h)}' = L_{(h)}''$, where $l(h) = l[h] - V(h)$ is the subgraph of $G$ contained in the interior of but not on the cycle $h$. Note that if each $h_i$ for $1 \leq i \leq n - 1$ is a cell of $G$, then $\mathcal{M}_0'$ is just the set of all 1-factors of $G_0$. Thus any two 1-factors in $\mathcal{M}_0'$ differ on $G_0 = (\bigcup_{i=1}^{n-1} h_i) \cup (\bigcup_{i=1}^{n} g_i)$, and $G_0'$ is an even subdivision of $L_n^{2n-3}$. Clearly, both $L'_u$ and $L''_u$ are 1-factors of $G_u$, where $L'_u = L|_{G_u}$ and $L''_u = L'|_{G_u}$. By Proposition 2.5, Lemma 2.7, and Theorem 3.4(c), $M_u' \cong \mathcal{G}_{2n-3}$ and $L'_u$ and $L''_u$ are the maximum element and the minimum element of $M_u'$ respectively, where $M_u'$ is the subposet of $M(G_u)$ induced by the 1-factors in $\mathcal{M}_u'$. For any 1-factor $M_u$ of $G_u$ in $\mathcal{M}_u'$, $M_u \cup (L''_u \setminus L'_u)$ is a 1-factor of $G_u$. Let $\mathcal{M}_u = \{L'_u \setminus L''_u \cup M_u | M_u \in \mathcal{M}_u'\}$. Hence, the 1-factors of $G_u$ in $\mathcal{M}_u$ form a subposet of $M(G_u)$, say $\mathcal{M}_u$, with maximum element $L'_u$ and minimum element $L''_u$, and $M_u \cong M_u \cong \mathcal{M}_u \cong \mathcal{G}_{2n-3}$.

In order to show that $\mathcal{M}_u$ is a sublattice of $M(G)$, it suffices to show that the operations $\wedge$ and $\vee$ of $M(G)$ are closed in $\mathcal{M}_u$. For any two 1-factors $M_1$ and $M_2$ in $\mathcal{M}_u$, $\mathcal{M}_u$ consists of disjoint $M_1$ and $M_2$-alternating cycles and $\mathcal{C} \subset G_u$, where $\mathcal{C} = M_1 \cup M_2$. Also each cycle in $\mathcal{C}$ does not contain any other cycles of $\mathcal{C}$ in its interior. By Corollary 4.3 of [14], $M^* = M_1 \vee M_2 = M_1 \oplus M_2 \cap (\mathcal{C} - (\mathcal{C} - \mathcal{C}))$. Thus $M_u = M_1 \cap \mathcal{C} = M_1 \cap \mathcal{C}' = M_1 \cap \mathcal{C} - (\mathcal{C} - \mathcal{C})$, where $\mathcal{C}' = \mathcal{C} - (\mathcal{C} - \mathcal{C})$ consists of the improper and proper $M_1$-alternating cycles in $\mathcal{C}$, respectively. By the definition of $\mathcal{M}_u$, we have that both $M^*$ and $M_u$ belong to $\mathcal{M}_u$, which completes this case.

In an analogous way, let $G_u = (\bigcup_{i=1}^{n-1} I[h_i]) \cup (\bigcup_{i=1}^{n} g_i)$, and let $C_0$ be the outer boundary of $G_u$, so $C_0$ is a proper $M$-alternating cycle. Letting $M^* = M \oplus C_0$, we conclude that $M^*$ is also a 1-factor of $G$ and that $h_1$ is an improper $M^*$-alternating cell, for $1 \leq i \leq n - 1$. Let $\mathcal{M}_u' = \{L_u' \setminus L_u'' \cup M_u | M_u \in \mathcal{M}_u'\}$. Each element of $\mathcal{M}_u'$ is a 1-factor of $G$. Similarly, if we denote the subposet of $M(G)$ by $\mathcal{M}_u$, then $\mathcal{M}_u$ is a sublattice of $M(G)$, with maximum element $M$ and minimum element $M^*$, and $\mathcal{M}_u \cong M(G_u) \cong \mathcal{G}_{2n-1}$, where $G_u = (\bigcup_{i=1}^{n-1} h_i) \cup (\bigcup_{i=1}^{n} g_i)$.

Now we show that $M \leq L''_u$. It follows that $L \cap L''_u = (M \oplus f_1 \oplus f_2) \cup (M \oplus g_1 \oplus g_n \oplus f_0 \oplus g_0) = f_1 \oplus f_2 \oplus g_1 \oplus g_n \oplus f_0 \oplus g_0 = \bigcup_{i=1}^{n-1} = \neq C_i$. Since each $C_i = \neq C_i$ is improper $L'$-alternating (thus proper $L''$-alternating), $L \leq L''_u$ in $M(G)$.

Remark 4.7. (1) By the proof of Lemma 4.6, if the outer face of $G_1$ is $c_s$, then $\Pi^*_n$ or $\Pi^+_n$ is a sublattice of $M(G)$, where $\Pi^*_n = 1 \oplus A_{2n} \oplus B_2 \oplus \Gamma_{2n-1} \cong \mathcal{M}(H_n)$ and $\Pi^+_n = 1 \oplus A_{2n} \oplus B_2 \oplus \Gamma_{2n-1}'$. (2) Sublattice $\Pi^*_n$ or $\Pi^+_n$ in $M(G)$ may not be convex (cf. [22, p. 98]).
Fig. 10. Non-matchable FDLs with cut-element v.

Fig. 11. Some non-matchable FDLs without cut-elements: (a) $A_4 \times 2$, (b) $(A_4 \uplus 1) \times 2$, (c) $P \times 2$, and (d) $A_4 \times 2 \times 2$.

In $\Pi_n$ or $\Pi_n^+$, the maximum element of $\Gamma_{2n-1}$ is its cut-element. We call it the critical cut-element of $\Pi_n$ or $\Pi_n^+$ (for example, see $M_v$ in Fig. 8(c) and (d)). We now state the contrapositive of Lemma 4.6 as another main result of this section.

**Theorem 4.8.** For $n \geq 2$, if an FDL $L$ with a type $(2, n)$ cut-element $v$ contains neither $\Pi_n$ nor $\Pi_n^+$ as sublattice with the critical cut-element $v$, then $L$ is a non-matchable FDL.

**Example 4.9.** By applying Theorem 4.8, we can show that FDLs in Fig. 10 are non-matchable FDLs. Moreover, for positive integers $n$ with $n \geq 3$, the FDL $B_n \oplus B_2$ is non-matchable.

**Remark 4.10.** If an FDL $L$ has a type $(m, 1)$ cut-element $v$, then the element $v'$ covered by $v$ must be a type $(1, n)$ cut-element unless $v' = 0$. It is easy to check that such a local structure is allowed in matchable FDLs. The simplest example is the following: First, take an even cycle $C$ with length $2l$, where $l \geq \max\{m, n\}$, and let $M$ be its 1-factor such that $C$ is proper $M$-alternating. Next, choose $m$ edges in $M$ and $n$ edges not in $M$ from $C$ and join the end-vertices of each of the edges by a path with odd length at least 3. For such a plane bipartite graph $G$, $M(G)$ has exactly a type $(m, 1)$ cut-element $v$ and a type $(1, n)$ cut-element $v'$ covered by $v$. In fact, $M(G) \cong B_m \uplus B_n$.

By Theorems 2.3, 3.4, 4.3, and 4.8, we can obtain a sequence of non-matchable FDLs without cut-elements.

**Example 4.11.** For any non-matchable FDL $N$ and any nontrivial FDL $L$, the direct product $N \times L$ is a non-matchable FDL without cut-elements (see Theorem 2.3). Only four such examples are presented in Fig. 11 ($P$ denotes the first FDL in Fig. 7).

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