A Spectral Collocation Method for Eigenvalue Problems of Compact Integral Operators

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ABSTRACT. We propose and analyze a new spectral collocation method to solve eigenvalue problems of compact integral operators, particularly, piecewise smooth operator kernels and weakly singular operator kernels of the form $\frac{1}{|t-s|^{\mu}}$, $0 < \mu < 1$. We prove that the convergence rate of eigenvalue approximation depends upon the smoothness of the corresponding eigenfunctions for piecewise smooth kernels. On the other hand, we can numerically obtain a higher rate of convergence for the above weakly singular kernel for some $\mu$’s even if the eigenfunction is not smooth. Numerical experiments confirm our theoretical results.

Key Words: Eigenvalue problem; spectral collocation method; weakly singular kernel, integral operator, super-geometric convergence.

AMS Subject Classification: 47A10, 47A58, 65J99, 65MR20.

1. Introduction

We consider numerical approximation of the eigenvalue problem for a compact integral operator $T$ on a Banach space. Recent years witness a revitalization of this field and various of methods are applied to solve the problem. The Galerkin, Petrov-Galerkin, collocation, Nyström, and degenerate kernel methods have been extensively studied for the approximation of eigenvalues and eigenvectors of integral operators. The results are well-documented in the literature. Here, we mention a few related to our current work. As early as 1967, Atkinson proved a general theorem showing the convergence of the numerical eigenvalues and eigenvectors to those of compact integral operators [1]. In 1975, he further obtained a convergence rate for the approximation [2], based upon which Osborn established a general spectral approximation theory for compact operators, when a sequence of $\{T_n\}$ approximates $T$ in a collectively compact manner. The analysis of [2, 17] covers many methods and provides a basis for the convergence analysis of our method. In [13], Dellwo and Friedman proposed a new approach by solving a polynomial eigenvalue problem of a higher
degree, base upon which, Alam etc. [4] obtained an accelerated spectral approximation for eigenelements. Kulkarni [16] introduced another method by involving a new approximation operator $T_n$ and obtained a high-order convergence rate. In addition, a multiscale method was discussed in [10]. Comprehensive studies for eigenvalue problem can be found in [6, 9, 21]. For numerical solution of integral equations or integro-differential equations, interested readers are referred to [3, 5].

In this article, we approximate eigenfunctions by some appropriate orthogonal polynomial expansions. Different from previous methods in the literature, we find the exact integration when calculating the convolution of the singular kernel with the orthogonal polynomials. The key ingredients here are some special identities. By doing so, we 1) avoid large numerical quadrature errors accumulated with the singular kernels and thereby obtain higher accuracy for eigenvalue approximations, and 2) avoid product integration method and therefore reduce computational cost. Furthermore, if the kernel is positive definite and piecewisely smooth, a refined result can be obtained.

To fix the idea, we consider problems of the form

$$\int_0^1 k(t, s)u(s)ds = \lambda u(t), \quad t \in [0, 1],$$

where $k(t, s) = |t - s|^{-\mu}$ for $0 < \mu < 1$, $k(t, s)$ is piecewisely smooth or smooth. We will develop algorithms for all three types of problems separately.

This paper is organized as follows: In section 2, some preliminary knowledge is given. In section 3, algorithms for all types of equations are listed. Section 4 is devoted to convergence analysis of algorithms. Finally, we illustrate our theories with numerical examples in section 5. Throughout the paper, $C$ stands for a generic constant that is independent of collocation points $p$ but which may depend on the index $\mu$ and the number of pieces a piecewise kernel has.

2. Preliminary

Let $T : X \to X$ be a compact linear operator on a Banach space $X$ and $\sigma(T)$ and $\rho(T)$ be the spectrum and resolvent of $T$ respectively. Let $\lambda$ be a nonzero eigenvalue of $T$ with multiplicity $m$ and let $\Gamma$ be a circle centered at $\lambda$ which lies in $\rho(T)$ and which encloses no other points in $\sigma(T)$. Then, the spectral projection associated with $T$ and $\lambda$ is defined by

$$E = -\frac{1}{2\pi i} \int_{\Gamma} (T - zI)^{-1}dz$$

and $\max_{z \in \Gamma} \|(T - zI)^{-1}\| \leq C$.

Let $\{T_n\}$ be a sequence of operator in $B(X)$ that converges to $T$ in a collectively way, i.e., the set $\{T_n x : \|x\| \leq 1, n = 1, 2, \cdots\}$ is sequentially compact. For $n$ large enough, $\Gamma \in \rho(T_n)$ and the associated projection,

$$E_n = -\frac{1}{2\pi i} \int_{\Gamma} (T_n - zI)^{-1}dz$$
exists and \( \max \|(T_n - zI)^{-1}\| \leq C \). Clearly, \( \dim(E) = \dim(E_n) = m \) and \( T_n E_n = E_n T_n \). Furthermore, the spectrum of \( T_n \) inside \( \Gamma \), contains \( m \) approximations of \( \lambda \), i.e. \( \lambda_{n,1}, \lambda_{n,2}, \cdots, \lambda_{n,m} \), counted according to their algebraic multiplicities \([9, 17]\). Let

\[
\hat{\lambda}_n = \frac{\lambda_{n,1} + \lambda_{n,2} + \cdots + \lambda_{n,m}}{m}.
\]

Then we have the following theorem.

**Theorem 2.1** [17] For all \( n \) sufficiently large,

\[
|\lambda - \hat{\lambda}_n| \leq C \|\frac{(T_n - T_n})R(E)\|,
\]

where \( R(E) \) is the range of the projection \( E \).

This is a rather general result. We may refine the result if the kernel is positive definite. Let

\[
a(u, v) = \int_0^1 \int_0^1 k(t, s)u(s)v(t)dsdt, \quad b(u, v) = \int_0^1 u(t)v(t)dt,
\]

where \( v \) is a test function in the \( L^2 \) space \( V \). If the bilinear operator \( a(u, v) \) is coercive, then we can list eigenvalues of \( T \) by

\[
\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0
\]

with zero the only possible cluster point.

Let us consider a numerical approximation of the first eigen-pair \((\lambda, u)\). Let \((\lambda_p, v_p)\) be their Galerkin approximation, and let \( u_p \) be the Legendre expansion of \( u \). We have

\[
\lambda = \frac{a(u, u)}{b(u, u)} = \sup_{v \in V} \frac{a(v, v)}{b(v, v)}, \quad \lambda_p = \frac{a(v_p, v_p)}{b(v_p, v_p)} = \max_{v \in \mathcal{P}_p} \frac{a(v, v)}{b(v, v)}.
\]

Here \( \mathcal{P}_p \) is the polynomial space with degree no more than \( p \). Denote \( \tilde{\lambda}_p = \frac{a(u_p, u_p)}{b(u_p, u_p)} \), then we have the following lemma.

**Lemma 2.2** Let \( \lambda, \lambda_p \) and \( \tilde{\lambda}_p \) be defined as above and \( a(u, v) \) be coercive, then

\[
0 \leq \lambda - \lambda_p \leq \lambda - \tilde{\lambda}_p = \lambda \frac{\|u - u_p\|_b^2}{\|u\|_b^2} - \frac{\|u - u_p\|_a^2}{\|u\|_a^2}.
\]

**Proof:** From Lemma 9.1 of [6] on page 701, we have

\[
0 \leq \nu_p - \nu \leq \tilde{\nu}_p - \nu \leq \frac{\|u - u_p\|_b^2}{\|u\|_a^2} - \nu \frac{\|u - u_p\|_a^2}{\|u\|_a^2},
\]

where \( \nu = \frac{1}{\lambda}, \nu_p = \frac{1}{\lambda_p} \) and \( \tilde{\nu}_p = \frac{1}{\tilde{\lambda}_p} \). Hence,

\[
0 \leq \frac{\lambda - \lambda_p}{\lambda} \leq \lambda_p \frac{\|u - u_p\|_b^2}{\|u\|_a^2} - \frac{\lambda}{\lambda} \frac{\|u - u_p\|_a^2}{\|u\|_a^2}.
\]
Using the fact that
\[ a(u_p, u_p) = \lambda_p b(u_p, u_p), \]
we derive (2.3) from (2.5).

Next, we introduce some identities, which will be essential in this paper. Towards this end, we define the class of Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \). Under the normalization \( P_k^{(\alpha,\beta)}(1) = (k+\alpha\beta) \), one has the expression, namely,
\[
P_k^{(\alpha,\beta)}(x) = \frac{1}{2^k} \sum_{l=0}^k \left( \binom{k + \alpha}{k - l} \binom{k + \beta}{l} (x-1)^l (x+1)^{k-l} \right). \tag{2.6}
\]
Jacobi polynomials satisfy the three-term recursive relations:
\[
P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{1}{2}[(\alpha - \beta) + (\alpha + \beta + 2)x],
\]
\[
a_{1,k}P_{k+1}^{(\alpha,\beta)}(x) = a_{2,k}P_k^{(\alpha,\beta)}(x) - a_{3,k}P_{k-1}^{(\alpha,\beta)}(x), \tag{2.7}
\]
where
\[
a_{1,k} = 2(k+1)(k + \alpha + \beta + 1)(2k + \alpha + \beta),
\]
\[
a_{2,k} = (2k + \alpha + \beta + 1)(\alpha^2 - \beta^2) + x\Gamma(2k + \alpha + \beta + 3)/\Gamma(2k + \alpha + \beta),
\]
\[
a_{3,k} = 2(k + \alpha)(k + \beta)(2k + \alpha + \beta + 2). \tag{2.8}
\]
Especially, if \( \alpha = 0, \beta = 0 \), Jacobi polynomials become Legendre polynomials.

**Lemma 2.3** ([19]) Let \( a, b \) be positive constants and \( L_n(x) \) be the Legendre polynomials with degree \( n \) on \([-1, 1]\), then
\[
\int_a^b (s - a)^{\alpha-1}L_n(s) ds = \frac{n!}{(\alpha)_{n+1}}(b - a)^\alpha P_n^{(\alpha,-\alpha)}(\frac{a}{b}), \quad -b < a < b; \alpha > 0, \tag{2.9}
\]
\[
\int_a^b (b - s)^{\beta-1}L_n(s) ds = \frac{n!}{(\beta)_{n+1}}(b + a)^\beta P_n^{(-\beta,\beta)}(\frac{b}{a}), \quad -a < b < a; \beta > 0, \tag{2.10}
\]
where \((k)_{n+1} = k(k+1) \cdots (k+n)\).

Specifically, if we choose \( a = 1, b = x, \beta = 1 - \mu \) in (2.10), then we obtain
\[
\int_{-1}^{x} \frac{L_n(t)}{(x-t)^\mu} dt = \frac{n!}{(1-\mu)_{n+1}}(1 + x)^{1-\mu} P_n^{(\mu-1,1-\mu)}(x), \tag{2.11}
\]
and \( a = x, b = 1, \alpha = 1 - \mu \) in (2.9), we arrive at
\[
\int_{x}^{1} \frac{L_n(t)}{(t-x)^\mu} dt = \frac{n!}{(1-\mu)_{n+1}}(1 - x)^{1-\mu} P_n^{(1-\mu, \mu-1)}(x). \tag{2.12}
\]

**Remark 1:** We use identities (2.11) and (2.12) in our algorithm for weakly singular kernels after we expand eigenvectors by Legendre polynomials.
Lemma 2.4 ([18]) Let $\alpha > -1, \beta > -1$ and $0 < \nu < 1$, then for $-1 < x < 1$

$$
\int_{-1}^{1} \frac{(1-t)^\alpha (1+t)^\beta F_m^{(\alpha, \beta)}(t)}{|x-t|^{\nu}}\, dt = \frac{\cos \frac{\pi \nu}{2} \Phi_1(x) + \cos \pi \left(\frac{\nu}{2} - \beta\right) \Phi_2(x)}{\Gamma(\nu) \cos \frac{\pi \nu}{2}}, \; m = 0, 1, 2, \ldots,
$$

(2.13)

where

$$
\Phi_1(x) = \frac{\Gamma(m + \alpha + 1)\Gamma(m + \nu)\Gamma(\beta - \nu + 1)(-1)^m}{2^{-\alpha - \beta + \nu - 1}\Gamma(m + \alpha + \beta - \nu + 2)m!}
\times _2 F_1(m + \nu, \nu - m - \alpha - \beta - 1; -\beta + \nu; \frac{1+x}{2}),
$$

(2.14)

$$
\Phi_2(x) = \frac{\Gamma(m + \beta + 1)\Gamma(\nu - \beta - 1)(-1)^{m+1}}{2^{-\alpha}(1 + x)^{\nu - \beta - 1}m!}
\times _2 F_1(m + \beta + 1, -m - \alpha; \beta - \nu + 2; \frac{1+x}{2}).
$$

(2.15)

Here, $_2 F_1(a, b; c; z)$ is known as Gauss’ hypergeometric functions.

For the sake of convergence analysis, we need to introduce the error estimate of Gauss quadrature.

Lemma 2.5 ([12]) Let $f \in C^{2n}$, $x_i$ and $w_i$ be the Gauss points and their corresponding quadrature weights on the interval $[a,b]$. Then

$$
\int_a^b f(x)\, dx - \sum_{i=0}^{n} w_i f(x_i) = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)]^3} f^{(2n)}(\xi), \; \xi \in (a,b).
$$

(2.16)

3. Algorithms

In this section, we develop algorithms for eigen-problem with all three kinds of kernels mentioned before. Models that we consider in this article are

1. Weakly singular kernels:

$$
\lambda y(t) = \int_0^1 \frac{y(s)}{|t-s|^\mu} ds, \; \quad 0 < \mu < 1, \; t \in [0,1];
$$

(3.1)

2. Piecewise smooth kernels:

$$
\lambda y(t) = \int_0^1 k(t, s)y(s) ds, \; \quad t \in [0,1],
$$

(3.2)

where $k(t, s) = \begin{cases} 
  t-s/2, & \text{if } 0 \leq t \leq s \leq 1 \\
  s/2, & \text{if } 0 \leq s < t \leq 1;
\end{cases}$

3. Smooth kernels:

$$
\lambda y(t) = \int_0^1 e^{st} y(s) ds, \; \quad t \in [0,1].
$$

(3.3)
3.1 The first algorithm for (3.1)

It is clear that (3.1) is equivalent to

\[
\lambda y(t) = \int_0^t \frac{y(s)}{(t-s)^\mu} ds + \int_t^1 \frac{y(s)}{(s-t)^\mu} ds.
\]  

(3.4)

We make a change of variable \( t = \frac{1+x}{2} \) and obtain

\[
\int_0^{(1+x)/2} \left( \frac{1+x}{2} - s \right)^{-\mu} y(s) ds + \int_{(1+x)/2}^1 \left( s - \frac{1+x}{2} \right)^{-\mu} y(s) ds = \lambda u(x),
\]  

where \( x \in [-1,1] \) and \( u(x) = y(\frac{1+x}{2}) \). Next, we make another change of variable, \( s = \frac{1+x}{2} \), and reach

\[
\left( \frac{1}{2} \right)^{1-\mu} \int_{-1}^{x} (x-\tau)^{-\mu} u(\tau) d\tau + \left( \frac{1}{2} \right)^{1-\mu} \int_{x}^{1} (\tau-x)^{-\mu} u(\tau) d\tau = \lambda u(x), \; x \in [-1,1]
\]  

(3.6)

Let \( u_p(x) = \sum_{j=0}^{p} c_j L_j(x) \) be the approximation of \( u(x) \). Obviously, \( c_j \)'s satisfy the equation

\[
\left( \frac{1}{2} \right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{-1}^{x_i} \frac{L_j(\tau)}{(x_i-\tau)^\mu} d\tau + \left( \frac{1}{2} \right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{x_i}^{1} \frac{L_j(\tau)}{(\tau-x_i)^\mu} d\tau = \lambda_p \sum_{j=0}^{p} c_j L_j(x_i).
\]  

(3.7)

Substituting (2.11) and (2.12) into (3.7), we obtain

\[
\sum_{j=0}^{p} c_j \left[ \left( \frac{1}{2} \right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \right] + \left( \frac{1}{2} \right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i) = \lambda_p \sum_{j=0}^{p} c_j L_j(x_i), \; i = 0, \ldots, p.
\]  

(3.8)

If we write

\[
a_{ij} = \left( \frac{1}{2} \right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i)
\]

and

\[
b_{ij} = \frac{1}{2} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i)
\]

then we have \( AC_p = \lambda_p BC_p \), where \( A = (a_{ij}), B = b_{ij}, C_p = (c_0, c_1, \ldots, c_p)^T \).

3.2 The second algorithm for (3.1)

From Theorem 1 of [20], we assume that the first true eigenvector is of the form

\[
y(t) = \hat{d}_1 t^{-\mu} + \hat{d}_2 (1-t)^{-\mu} + a \text{ smoother function } \phi(t).
\]  

(3.9)
Hence, we approximate the eigenvector by \( u_p(t) = d_1 t^{1-\mu} + d_2 (1-t)^{1-\mu} + \sum_{j=0}^{p} c_j P_j(t) \), where \( P_j(t) \) is the shifted Legendre polynomial on \([0, 1], j = 0, 1, \cdots, p\). Substituting it into (3.1) and taking the same change of variable as the previous algorithm, we obtain

\[
\left( \frac{1}{2} \right)^{2-2\mu} \left( d_1 \int_{-1}^{1} \frac{(1+\tau)^{1-\mu}}{|x-\tau|^\mu} d\tau + d_2 \int_{-1}^{1} \frac{(1-\tau)^{1-\mu}}{|x-\tau|^\mu} d\tau \right) + \left( \frac{1}{2} \right)^{1-\mu} \sum_{j=0}^{p} c_j \left( \int_{-1}^{x} (x-\tau)^{-\mu} L_j(\tau) d\tau + \int_{x}^{1} (\tau-x)^{-\mu} L_j(\tau) d\tau \right) = d_1 \left( \frac{1+x}{2} \right)^{1-\mu} + d_2 \left( \frac{1-x}{2} \right)^{1-\mu} + \sum_{j=0}^{p} c_j L_j(x), \quad x \in [-1, 1]. \tag{3.10}
\]

From Lemma 2.3 and Lemma 2.4, and (3.10), we obtain

\[
\left( \frac{1}{2} \right)^{2-2\mu} \frac{\Gamma(2-2\mu)}{2^{2\mu-2}\Gamma(3-2\mu)} 2F_1(\mu, 2\mu - 2; 2\mu - 1; \frac{1+x_i}{2}) d_1 + \left( \frac{1}{2} \right)^{2-2\mu} \frac{1}{\Gamma(\mu)} \\
\left( \frac{\Gamma(2-\mu)\Gamma(1-\mu)}{2^{2\mu-2}\Gamma(3-2\mu)} 2F_1(\mu, 2\mu - 2; 1 + \frac{x_i}{2}) - \frac{\Gamma(\mu - 1)}{2^{\mu-1}(1+x_i)^{\mu-1}} \right) d_2 + \sum_{j=0}^{p} c_j \left[ \left( \frac{1}{2} \right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(\mu-1, 1-\mu)}(x_i) \right] \\
+ \left( \frac{1}{2} \right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(\mu-1, 1-\mu)}(x_i) \right] = \lambda_p \left( \sum_{j=0}^{p} c_j L_j(x_i) + d_1 \left( \frac{1+x_i}{2} \right)^{1-\mu} + d_2 \left( \frac{1-x_i}{2} \right)^{1-\mu} \right), \quad i = 0, \cdots, p + 2. \tag{3.11}
\]

Note that the first hypergeometric function is not well-defined when \( \mu = 1/2 \). However, the integration of the two singular terms with the kernel are simpler, in which case, the linear system is

\[
\left( \frac{\pi(1+x_i)}{4} + \sqrt{1-x_i} + \frac{1+x_i}{2} \log(1 + \sqrt{\frac{1-x_i}{2}}) - \frac{1+x_i}{4} \log(1 + \frac{1+x_i}{2}) \right) d_1 + \left( \sqrt{\frac{1+x_i}{2}} - \frac{x_i-1}{2} \tanh^{-1}(\sqrt{\frac{1+x_i}{2}}) - \frac{\pi(x_i-1)}{4} \right) d_2 + \sum_{j=0}^{p} c_j \left[ \left( \frac{1}{2} \right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(\mu-1, 1-\mu)}(x_i) \right] \\
(1+x_i)^{1-\mu} P_j^{(\mu-1, 1-\mu)}(x_i) + \left( \frac{1}{2} \right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(\mu-1, 1-\mu)}(x_i) \right] = \lambda_p \left( d_1 \sqrt{\frac{1+x_i}{2}} + \frac{1-x_i}{2} + \sum_{j=0}^{p} c_j L_j(x_i) \right), \quad i = 0, \cdots, p + 2. \tag{3.12}
\]

3.3 Algorithm for (3.2)
We make change of variables as before and let \( u(x) = y \left( \frac{1+x}{2} \right) \), we obtain

\[
\lambda u(x) = \int_{-1}^{1} \frac{1 + \tau}{8} u(\tau) d\tau + \frac{1}{4} \int_{x}^{1} (x - \tau) u(\tau) d\tau.
\]  

Let \( u_p(x) = \sum_{j=0}^{p} c_j L_j(x) \) be the approximation of \( u(x) \). Then \( c_j \)'s satisfy

\[
\lambda_p \sum_{j=0}^{p} c_j L_j(x_i) = \sum_{j=0}^{p} c_j \int_{-1}^{1} \frac{1 + \tau}{8} L_j(\tau) d\tau + \sum_{j=0}^{p} c_j \int_{x_i}^{1} (x_i - \tau) L_j(\tau) d\tau,
\]  
i.e.

\[
\lambda_p \sum_{j=0}^{p} c_j L_j(x_i) = \left( \frac{c_0}{4} + \frac{c_1}{12} \right) + \frac{1 - x_i}{8} \sum_{j=0}^{p} c_j \sum_{k=0}^{p} w_k (x_i - x_k) L_j \left( \frac{1 + x_i}{2} + \frac{1 - x_i}{2} x_k \right),
\]  
i = 0, \ldots, p. \hspace{0.5cm} (3.14)

Here, the numerical integration is exact. The scheme is of the form

\[
AC_p = \lambda_p BC_p,
\]

where

\[
b_{ij} = L_j(x_i)
\]  
\[
a_{ij} = \begin{cases} 
\frac{1 - x_i}{8} \sum_{k=0}^{p} w_k (x_i - x_k) L_j \left( \frac{1 + x_i}{2} + \frac{1 - x_i}{2} x_k \right), & \text{if } j \neq 0, 1; \\
\frac{1 - x_i}{8} \sum_{k=0}^{p} w_k (x_i - x_k) L_j \left( \frac{1 + x_i}{2} + \frac{1 - x_i}{2} x_k \right) + \frac{1}{4}, & \text{if } j = 0; \\
\frac{1 - x_i}{8} \sum_{k=0}^{p} w_k (x_i - x_k) L_j \left( \frac{1 + x_i}{2} + \frac{1 - x_i}{2} x_k \right) + \frac{1}{12}, & \text{if } j = 1.
\end{cases}
\]

### 3.4 Algorithm for (3.3)

Substitute the Legendre expansion \( y(t) = \sum_{j=0}^{p} y_j L_j(t) \) into (3.3) and collocating at \( n \) Gaussian points, we have

\[
\sum_{j=0}^{p} y_j \int_{0}^{1} e^s t_i L_j(s) ds = \lambda \sum_{j=0}^{p} y_j L_j(t_i), \quad i = 1, 2, \cdots, n. \hspace{0.5cm} (3.16)
\]

The matrix form of (3.16) is

\[
K y = \lambda L y,
\]

where

\[
K_{ij} = \int_{0}^{1} e^s t_i L_j(s) ds, \quad L_{ij} = L_j(t_i), \quad y = (y_1, y_2, \cdots, y_n)^T.
\]  

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\( K_{ij} \) can be calculated by the \( n \)-point Gaussian quadrature
\[
\int_0^1 e^{s_i L_j(S)} ds \approx \sum_{l=0}^{p} e^{s_l t_l} L_j(s_l) w_l, \quad s_k = t_k. \tag{3.19}
\]

4. Convergence Analysis

Let \( L_k \) be the standard Legendre polynomial of degree \( k \) and \( \pi_p f \in P_p[-1, 1] \) interpolate a smooth function \( f \) at \((p + 1)\)-Gauss points: \(-1 < x_0 < \cdots < x_p < 1\). Let \( T_k \) be the first kind Chebyshev polynomial of degree \( k \). Then the remainder of the interpolation is
\[
f(x) - \pi_p f(x) = f[x_0, x_1, \cdots, x_p, x] \nu(x), \tag{4.1}
\]
where \( \nu(x) = (x - x_0)(x - x_1) \cdots (x - x_p) \).

Note that
\[
L_n(x) = \frac{1}{2^n n!} \frac{d^n}{d x^n}(x^2 - 1)^n = \frac{1}{2^n n!} \frac{d^n}{d x^n} \sum_{j=0}^{n} \binom{n}{j} x^{2(n-j)} (-1)^j
= \frac{1}{2^n n!} \sum_{j=0}^{n} \binom{n}{j} (2n - 2j)(2n - 2j - 1) \cdots (2n - 2j - n + 1) x^{n-2j} (-1)^j \tag{4.2}
\]
From the term with \( j = 0 \) we get the leading coefficient
\[
\frac{1}{2^n n!} \binom{n}{0} (2n)(2n - 1) \cdots (2n - n + 1)(-1)^0 = \frac{(2n)!}{2^n (n!)^2} \tag{4.3}
\]
By the Stirling formula,
\[
\frac{(2n)!}{2^n (n!)^2} \approx \frac{(2n)^{2n} \sqrt{4\pi n}}{2^n [(n/e)^n \sqrt{2\pi n}]^2} = 2^n. \tag{4.4}
\]
Hence,
\[
f(x) - \pi_p f(x) \approx f[x_0, x_1, \cdots, x_p, x] \frac{L_{p+1}(x)}{2^{p+1}}, \tag{4.5}
\]
If \( f \in C^{p+1}[-1, 1] \), the divided difference
\[
f[x_0, x_1, \cdots, x_p, x] = \frac{f^{(p+1)}(\xi_x)}{(p+1)!}, \quad \xi_x \in (-1, 1). \tag{4.6}
\]
The result can be concluded as the following theory.

**Theorem 4.1**

(1) If \( y(t) \) satisfies condition (K): \( \|y^{(k)}\|_{L^\infty[0,1]} \leq C k! R^{-k} \), then
\[
\|y - \pi_p y\|_{L^\infty[0,1]} \leq \frac{C}{(4R)^{p+1}}. \tag{4.7}
\]
(2) If \( y(t) \) satisfies condition (M): \( \|y^{(k)}\|_{L^{\infty}[0,1]} \leq CM^k \), we have

\[
\|y - \pi_p y\|_{L^{\infty}[0,1]} \leq \frac{C}{\sqrt{p+1}} \left( \frac{eM}{4(p+1)} \right)^{p+1} .
\]  

(4.8)

**Proof:** We make change of variables

\[
t = \frac{1 + x}{2}, \quad s = \frac{1 + \tau}{2}, \quad x, \tau \in [-1,1],
\]

and let \( u(x) = y\left( \frac{1+x}{2} \right) \), then the result for \( y \) under condition (K) follows directly from (4.6) and the fact that \( dt = \frac{1}{2} dx \).

If \( y \) satisfies condition (M), by applying the Stirling’s formula,

\[
\|y - \pi_p y\|_{L^{\infty}[0,1]} \leq CM^p\frac{1}{(4R)^{p+1}(p+1)!} \leq CM^p\frac{1}{\sqrt{2\pi}(p+1)(4(p+1))^{p+1}} \approx cM^p\frac{1}{\sqrt{2\pi}(p+1)(4(p+1))^{p+1}} \leq \frac{C}{\sqrt{p+1}} \left( \frac{eM}{4(p+1)} \right)^{p+1} .
\]  

(4.9)

For non-smooth functions, we need some other estimates.

**Theorem 4.2** ([8]) (1) For any \( f \in H^k(-1,1) \),

\[
\|f - \pi_p f\|_{L^2(-1,1)} \leq Cp^{-k} |f|_{H^k_p(-1,1)} .
\]  

(4.10)

(2) For any \( f \in H^k_w(-1,1) \),

\[
\|f - \pi_p^c f\|_{L^2(-1,1)} \leq Cp^{-k} |f|_{H^k_w(-1,1)} ,
\]  

(4.11)

where two seminorms are defined by

\[
|f|_{H^k_p(-1,1)} = \left( \sum_{s=\min(k,p+1)}^k \|f^{(s)}\|_{L^2(-1,1)}^2 \right)^{1/2},
\]

\[
|f|_{H^k_w(-1,1)} = \left( \sum_{s=\min(k,p+1)}^k \|f^{(s)}\|_{L^2_w(-1,1)}^2 \right)^{1/2},
\]

and the weight \( w(x) = (1 - x)^{-1/2}(1 + x)^{-1/2} \) and \( \pi_p^c \) is the interpolatory operator on Chebyshev points.

10
Let \( R(E) \) and \( R(E_p) \) be the range of \( E \) and \( E_p \), respectively. Define \( \pi_p : R(E) \to R(E_p) \) as an interpolatory projection by 
\[
\pi_p(t) = \sum_{j=0}^{p} \xi_j L_j(t) \quad \text{and} \quad \xi_j \text{ is determined by}
\]
\[
\sum_{j=0}^{p} \xi_j L_j(t_i) = x(t_i), \quad i = 0, \cdots, p,
\]
then, our algorithms can be written as
\[
T_p u_p = \lambda_p u_p, \quad \text{where} \quad T_p = \pi_p T. \quad (4.12)
\]

**Theorem 4.3.** Let \( y \) be the exact first eigenvector and \( T \) be a compact operator in (3.1), (3.2) or (3.3) and \( T_p \) be defined as above, then

1. If \( y \in H^k(0, 1) \),
\[
|\lambda - \hat{\lambda}_p| \leq \frac{C}{(2p)^k}; \quad (4.13)
\]
2. Furthermore, if \( y \) satisfies condition (K),
\[
|\lambda - \hat{\lambda}_p| \leq \frac{C}{(4R)^{p+1}}; \quad (4.14)
\]
3. Furthermore, if \( y \) satisfies condition (M),
\[
|\lambda - \hat{\lambda}_p| \leq \frac{C}{\sqrt{p+1}} \left( \frac{eM}{4(p+1)} \right)^{p+1}. \quad (4.15)
\]

**Proof:** The result follows directly from Theorem 2.1, 4.1 ,4.2, and Theorem 2.2 of [16]. To make the paper self-contained, we put the proof here.

Let \( \hat{E}_p = E_{p|R(E)} : R(E) \to R(E_p) \). Then for large \( p \), \( \hat{E}_p \) is bijective and \( \|\hat{E}_p^{-1}\| \leq 2 \) [17]. Define \( \hat{T} = T_{R(E)} \), and \( \hat{T}_p := \hat{E}_p^{-1}T_p \hat{E}_p \). Then
\[
|\lambda - \hat{\lambda}_p| = \frac{1}{m} |\text{trace}(\hat{T} - \hat{T}_p)| \leq \|\hat{T} - \hat{T}_p\|
\]
\[
= \|\hat{E}_p^{-1}(\hat{E}_p T - \hat{E} T_p)\|
\]
\[
\leq C\| (T - T_p)|_{R(E)}\|
\]
\[
= C\| (I - \pi_p) T |_{R(E)}\|. \quad (4.16)
\]

Since \( Tu \) is smoother than \( u \), see [11, 15], then the result follows.

**Theorem 4.4** Let \( \lambda \) and \( \lambda_p \) be the exact eigenvalue and its numerical approximation of a positive definite operator \( T \) whose kernel is a piecewise smooth function, respectively. Then

1. If \( u \) satisfies condition (K),
\[
|\lambda - \lambda_p| \leq C \left( \frac{1}{(4R)^{2p+2}} + \frac{e^{2p}}{p^{2p-3/2} 2^{6p}} \right), \quad (4.17)
\]
(2) If \( u \) satisfies condition (M),
\[
|\lambda - \lambda_p| \leq C \left( \frac{1}{p+1} \left( \frac{eM}{4(p+1)} \right)^{2p+2} + \frac{e^{2p}}{p^{2p-3/2}2^{2p}} \right),
\]
(4.18)

(3) If \( u \in H^k[0,1] \),
\[
|\lambda - \lambda_p| \leq C \left( \frac{1}{(2p)^2} + \frac{e^{2p}}{p^{2p-3/2}2^{2p}} \right),
\]
(4.19)

**Proof:** By our algorithms, we have
\[
\int_0^1 k(t_i,s)u_p(s)ds = \lambda_p u_p(t_i),
\]
(4.20)
where \( t_i \) are \((p+1)\)-Gauss points on \([0,1] \).

Multiplying both sides by \( L_j(t_i)w_i \) and summing up from 0 to \( p \), we obtain
\[
\sum_{j=0}^p \int_0^1 k(t_i,s)u_p(s)L_j(t_i)w_i ds = \lambda_p \sum_{j=0}^p u_p(t_i)L_j(t_i)w_i.
\]
(4.21)
Here, \( w_i \) are weights of the Gauss quadrature.

If we write \( \tilde{A} = (\int_0^1 \int_0^1 k(t,s)L_j(s)L_i(t)dsdt)_{ij} \), \( \tilde{B} = (\int_0^1 L_j(t)L_i(t)dt)_{ij} \) and recall that \( u_p(x) = \sum_{i=0}^p \tilde{u}_i L_i(x) \), we obtain
\[
\tilde{A}\tilde{u} = \lambda_p \tilde{B}\tilde{u},
\]
where \( \tilde{u} = [\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_p]^T \).

However, for most cases, we can only apply numerical quadrature to find elements of \( \tilde{A} \) and \( \tilde{B} \). If the kernel is piecewise smooth, we apply the Gauss quadrature piece by piece. Therefore, the system that we actually solve is
\[
Au = \lambda_p Bu.
\]
(4.22)

Now we are ready to analyze errors of eigenvalue approximations.

First, we analyze the case when the kernel is a linear piecewise polynomial. Noting the fact that \((p+1)\)-Gauss quadrature is exact for all polynomials of degree less than or equal \( 2p+1 \), we have
\[
A = \tilde{A}, \quad \text{and} \quad B = \tilde{B}.
\]
(4.23)
Here, the integration is piecewise, so is the numerical integration.

Denote the arithmetic mean of the approximation of \( \lambda \) by \( \lambda_p \) again, if it is a multiple eigenvalue. We derive from Lemma 2.2 that
\[
|\lambda - \lambda_p| = |\lambda - \tilde{\lambda}_p|
\leq C \begin{cases} \frac{1}{4^{2p+2}}, & \text{if } u \text{ satisfies condition (K)}; \\ \frac{1}{p+1} \left( \frac{eM}{4(p+1)} \right)^{2p+2}, & \text{if } u \text{ satisfies condition (M)}; \\ \frac{1}{(2p)^2}, & \text{if } u \in H^k[0,1]. \end{cases}
\]
(4.24)

If the kernel is piecewise smooth or smooth, from the analysis of previous case, we only need to estimate $A - \tilde{A}$ since $B = \tilde{B}$. If we write the remainder of Gaussian quadrature as $\epsilon$, then

$$A - \tilde{A} \leq C\epsilon \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$  \hspace{1cm} (4.25)

Here, we define $E < F$ if and only if $|E|_{ij} < |F|_{ij}$.

By the error estimate of the Gauss quadrature and the Stirling’ formula, we have

$$\epsilon \leq C \left( \frac{[p]!^4}{(2p+1)(2p)!^3} \right) \approx C \left( \frac{\sqrt{2\pi p} (\frac{p}{e})^p}{(2p+1)\sqrt{2\pi (2p)} (\frac{2p}{e})^p} \right)^4 \leq C \left( \frac{e^{2p}}{p^{2p+1/2}6p} \right).$$  \hspace{1cm} (4.26)

Hence,

$$\|A - \tilde{A}\|_n \leq C \left( \frac{pe^{2p}}{p^{2p+1/2}26p} \right), \quad n = 1, \infty.$$  \hspace{1cm} (4.27)

Clearly, (4.22) is equivalent to

$$B^{-1}Au = \lambda_p u.$$  \hspace{1cm} (4.28)

Thus,

$$\|\tilde{B}^{-1}A - B^{-1}A\|_n \leq \|B^{-1}\|_n \|\tilde{A} - A\|_n \leq C \left( \frac{p^2 e^{2p}}{p^{2p+1/2}26p} \right), \quad n = 1, \infty,$$  \hspace{1cm} (4.29)

by noting that $B = \tilde{B} = \text{diag}(1, 1/3, \ldots, 1/(2p + 1))$. Therefore, by a perturbation theory, see [9] (Page 30), we have

$$|\lambda_p - \tilde{\lambda}_p| \leq C \left( \frac{e^{2p}}{p^{2p-3/2}26p} \right).$$  \hspace{1cm} (4.30)

Denote the arithmetic mean of the approximation of $\lambda$ by $\lambda_p$ again, if it is a multiple eigenvalue. We derive from Lemma 2.2 and (4.30) that

$$|\lambda - \lambda_p| \leq |\lambda - \tilde{\lambda}_p| + |\tilde{\lambda}_p - \lambda_p| \leq C \begin{cases} \left( \frac{1}{2p+2} + \frac{e^{2p}}{p^{2p+1/2}26p} \right), & \text{if } u \text{ satisfies condition (K)}; \\ \left( \frac{1}{p+1} (\frac{eM}{4(p+1)})^{2p+2} + \frac{e^{2p}}{p^{2p-1/2}26p} \right), & \text{if } u \text{ satisfies condition (M)}; \\ \left( \frac{1}{(2p)^{2p}} + \frac{e^{2p}}{p^{2p+1/2}26p} \right), & \text{if } u \in H^k[0, 1]. \end{cases}$$
Table 1: Example 5.1: $\lambda - \lambda_p$

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Remark 2: Theorem 4.4 shows that though numerical integration contributes the error of eigenvalue approximation, it is trivial compared with truncation error for our method. Hence, in our numerical experiments, we ignore it for reference curves.

5. Numerical Examples

In this section, we will find numerical approximations to solutions of some examples to demonstrate our theory.

Example 5.1 ([4]) We consider a problem with form (3.2). Then each $\lambda_j = \frac{1}{(2j-1)^2\pi^2}$, $j = 1, 2, \cdots$ is an eigenvalue of $T$ of algebraic multiplicity $m = 2$. Let $\hat{\lambda}$ denote the arithmetic mean of the two eigenvalues of $T_p$ to the largest two eigenvalues $\lambda = 1/\pi^2$. Numerical errors are presented in Table 1 and the left part of Fig.1, from which, we see that the error decays super-geometrically. Here Reference Curve is the graph of $f(p) = \frac{1}{100(p+1)}\left(\frac{e\pi}{4(p+1)}\right)^{2p+2}$.

Example 5.2 Now let us consider an eigen-problem of the form (3.1) with $\mu = 1/3$. From [20], eigenfunctions belong to $H^{\frac{5}{2}-\epsilon}(0, 1)$, where $\epsilon$ is a sufficiently small positive number and we expect to obtain a convergence rate of $O(p^{-7/3})$ based on Theorem 4.3 for the first algorithm. Here, we apply both our spectral collocation methods and the three-point
Gaussian collocation on equally spaced intervals with discontinuous piecewise quadratic elements method. Unfortunately, we do not know the exact eigenvalues for such type of kernels. However, we list some of our numerical approximations in Table 2 and we use the numerical approximation of the second algorithm for $p = 70$ as our “exact” value to obtain Fig. 2. It is easy to see that we can only obtain a seven-digit of accuracy for the first algorithm; we obtain an eleven-digit of accuracy and a convergence rate of $O(p^{-14/3})$ for the second algorithm, see Table 3. However, the convergence rate for the three-point Gaussian collocation with discontinuous piecewise quadratic element is only $O(h^{7/6})$. This fact also confirms results in [7], which says that the convergence rate for $p$-version methods doubles the convergence rate for $h$-version method if the true solution is singular.

**Example 5.3** We consider an eigen-problem of the form (3.1) with $\mu = 1/2$. In this

![Fig. 2](image_url)

Fig. 2. Kernel: $|t - s|^{1/3}$: the first algorithm (Left), the second algorithm (Middle), and the discontinuous linear elements method (Right).
Table 4: Example 5.3: $\lambda_p$ (The first algorithm)

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Table 5: Example 5.3: $\lambda_p$ (The second algorithm)

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</table>

case, eigenfunctions belong to $H^1(0,1)$. Again, we use both algorithms to solve it and consider the numerical approximation of the second algorithm for $p = 70$ as the “exact” first eigenvalue. Numerical results are shown in Table 4, Table 5 and Figure 3.

Example 5.4 Consider the eigenvalue problem of the form (3.3). We apply the algorithm in the section 3. Since the kernel is smooth, the first eigenvalue converges very fast, see Table 6 and the right part of Fig.1. In this case, Reference Curve is the graph of $f(p) = \frac{1}{10(p+1)} \left( \frac{e^p}{2(p+1)} \right)^{2p+2}$.

Acknowledgments The work of the third author is supported in part by the US National Science Foundation through grant DMS-1115530.

References

Fig. 3. Kernel: $|t-s|^{1/2}$ (The first algorithm)  

Kernel: $|t-s|^{1/2}$ (The second algorithm)


