Brief paper

$\mathcal{L}_2$ gain analysis for a class of switched systems

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\begin{abstract}
This paper considers the problem of disturbance tolerance/rejection for a family of linear systems subject to actuator saturation and $\mathcal{L}_2$ disturbances. For a given set of linear feedback gains, a given switching scheme and a given bound on the $\mathcal{L}_2$ norm of the disturbances, conditions are established in terms of linear or bilinear matrix inequalities under which the resulting switched system is bounded state stable, that is, trajectories starting from a bounded set will remain inside the set or a larger bounded set. With these conditions, both the problem of assessing the disturbance tolerance/rejection capability of the closed-loop system and the design of feedback gain and switching scheme can be formulated and solved as constrained optimization problems. Disturbance tolerance is measured by the largest bound on the disturbances for which the trajectories from a given set remain bounded. Disturbance rejection is measured by the restricted $\mathcal{L}_2$ gain over the set of tolerable disturbances. In the event that all systems in the family are identical, the switched system reduces to a single system under a switching feedback law. It will be shown that such a single system under a switching feedback law has stronger disturbance tolerance/rejection capability than a single linear feedback law can achieve.
\end{abstract}

\section{Introduction}

The literature on analysis and design of switched systems has been growing rapidly in recent years (see, for example, Branicky (1994), Cheng (2005), DeCarlo, Branicky, Pettersson, and Lennartson (2000), Liberzon and Morse (1999), Pettersson and Lennartson (2001), Sun and Ge (2005), Wicks, Peleties, and DeCarlo (1998) and Xi, Feng, Jiang, and Cheng (2003) and the references therein). Motivated by the results reported in this literature, we consider in this paper the following family of linear systems subject to input saturation and disturbances,

\begin{equation}
\dot{x} = A_i x + B_i \text{sat}(u) + E_i w, \\
\dot{z} = C_i x, \quad i \in \mathcal{I}_N := \{1, 2, \ldots, N\}
\end{equation}

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $z \in \mathbb{R}^p$ are respectively the state, input and output of the system, $w \in \mathbb{R}^q$ represents the disturbances, and $\text{sat} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the vector valued standard saturation function $\text{sat}(u) = [\text{sat}(u_1) \; \text{sat}(u_2) \; \cdots \; \text{sat}(u_m)]^T$, $\text{sat}(u_i) = \text{sign}(u_i) \min(|u_i|, 1)$. A switched system then results by defining a controller/supervisor which chooses one of the systems at each time instant based on the measurement of the state and according to an index function, say, $i = \sigma(x)$. A typical form of the index function is $\sigma(x) = i$ for $x \in \Omega_i$ with $\bigcup_{i=1}^N \Omega_i = \mathbb{R}^n$. Thus, the control design involves the construction of both feedback gains for individual systems and the index function so that the resulting switched system possesses certain desired performances.

In the absence of the disturbances $w$, a basic design objective is the local asymptotic stability of the resulting switched system with as large a domain of attraction as possible. By utilizing some techniques in dealing with actuator saturation (Hu & Lin, 2001) and the form of the largest region index function proposed by Pettersson (2003, 2004, 2005), we recently proposed a method for the design of the individual feedback gains and the index function that result in a locally asymptotically stable switched system (Lu and Lin, 2008). The design is formulated and solved as a constrained optimization problem with the objective of enlarging the domain of attraction of the resulting stable equilibrium at the origin. It was shown by numerical examples that such a design may result in a domain of attraction larger than that of a switched system, designed without taking actuator saturation into account.

In this paper, we will carry out an analysis of, and design for, the disturbance tolerance/rejection capability of the switched system resulting from the family of systems (1). We will restrict ourselves...
to a class of disturbances whose energies are bounded by a given value, i.e.,

\[ \mathcal{W}_a^r := \left\{ w : \mathbb{R}_+ \to \mathbb{R}^n : \int_0^\infty w^T(t)w(t)dt \leq \alpha \right\}, \]

for some positive number \( \alpha \). For a given set of linear feedback gains, a given index function and a given value of \( \alpha \), conditions will be established in terms of linear or bilinear matrix inequalities under which the resulting switched system is bounded state stable.

A system is said to be bounded state stable if its trajectories starting from a bounded set will remain inside the set or a larger bounded set. With these conditions, both the problem of assessing the disturbance tolerance/rejection capability of the closed-loop system and the design of feedback gain and switching scheme can be formulated and solved as constrained optimization problems.

Disturbance tolerance is measured by the largest bound on the energy of the disturbance, \( \alpha^r \), for which the trajectories from a given set remain bounded. Disturbance rejection is measured by the restricted \( \mathcal{L}_2 \) gain over \( \mathcal{W}_a^r \).

An interesting special class of the systems we consider in this paper is the case when all the systems in (1) are identical. In this case, the switched system reduces to a single system under a switching linear feedback law. It will be shown that for a single linear system of the form (1), a switching feedback law will result in stronger disturbance tolerance/rejection capability than a single linear feedback law of Fang, Lin, and Hu (2004) and Fang, Lin, and Shamash (2006). The \( \mathcal{L}_2 \) gain analysis and design for linear systems under actuator saturation has been studied by several authors. A small sample of their works include Chitour, Liu, and Sontag (1995), Fang et al. (2004, 2006), Hindi and Boyd (1998), Hu and Lin (2001), Lin (1997), Nguyen and Jabbari (1999) and Xie, Wang, Hao, and Xie (2004). In particular, in our recent work (Fang et al., 2004, 2006), we considered the \( \mathcal{L}_2 \) gain analysis and design for a linear system under actuator saturation. The disturbance tolerance capability of the closed-loop system under a given feedback law was assessed, and the linear feedback law that results in a minimized restricted \( \mathcal{L}_2 \) gain was designed.

The remainder of this paper is organized as follows. In Section 2, we state our problem and recall some preliminary materials that will be needed in the development of the results of this paper. Section 3 establishes bounded state stability conditions. Disturbance tolerance and disturbance rejection are addressed in Sections 4 and 5, respectively. Simulation results are presented in Section 6. Section 7 concludes the paper.

2. Problem statement and preliminaries

For the family of systems (1), we would like to design a linear feedback law for each individual system in the family and an index function such that the resulting switched system possesses a high degree of disturbance tolerance and a high level of disturbance rejection capabilities. We will adopt the switching strategy of Pettersson (2003, 2004).

Such a switching strategy is defined based on some appropriately chosen symmetric matrices \( Q_i \in \mathbb{R}^{n \times n}, i \in I_N \). More specifically, at a given state \( x \), the subsystem \( i \) will be activated if the quadratic function \( x^T Q_i x \) is greater or equal to any other \( x^T Q_j x, j \neq i \). More specifically, this switching scheme is defined by the following index function Pettersson (2003, 2004), referred to as the largest region function,

\[ i(x) = \arg \max_{i \in I_N} x^T Q_i x. \]

Based on the matrices \( Q_i \)’s, we define the following sets

\[ \Omega_i = \{ x \in \mathbb{R}^n | x^T Q_i x \geq 0 \}, \quad i \in I_N, \]

\[ \Omega_{ij} = \{ x \in \mathbb{R}^n | x^T Q_i x = x^T Q_j x \geq 0 \}, \quad i, j \in I_N. \]

Then, a well-defined switched system must satisfy the following properties:

- Covering property: \( \Omega_i \cup \Omega_2 \cup \cdots \cup \Omega_{N} = \mathbb{R}^n \);
- Switching property: \( \Omega_{ij} \subseteq \Omega_i \cap \Omega_j, i \in I_N, j \in I_N \).

The first condition says that there are no regions in the state space where none of the subsystem is activated. The second condition, which is automatically satisfied by this choice of \( \Omega_i \) and \( \Omega_{ij} \), means that a switch from subsystem \( i \) to \( j \) occurs only for states where the regions \( \Omega_i \) and \( \Omega_j \) are adjacent. Consequently, switching occurs on the switching surface \( x^TQ_i x = x^TQ_j x \). The following regarding the covering property was established in Pettersson (2003, 2004).

**Lemma 1** (Covering property). If for every \( x \in \mathbb{R}^n \), \n\[ \theta_1 x^T Q_i x + \theta_2 x^T Q_j x + \cdots + \theta_N x^T Q_N x \geq 0, \]

where \( \theta_i > 0, i \in I_N \), then \( \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N = \mathbb{R}^n \).

3. Bounded state stability

We recall a tool from Hu and Lin (2001) for expressing a saturated linear feedback \( u = \text{sat}(Fx) \) on the convex hull of a mixture of the unsaturated control inputs and the auxiliary inputs. For an \( F \in \mathbb{R}^{n \times n} \), let \( \mathcal{L}(F) = \{ x \in \mathbb{R}^n : f(x) \leq 1, i \in I_N \} \), where \( f \) represents the ith row of matrix \( F \). We note that \( \mathcal{L}(F) \) represents the region in \( \mathbb{R}^n \) where \( F \) does not saturate. Also, let \( \mathcal{V} \) be the set of \( m \times n \) diagonal matrices whose diagonal elements are either 1 or 0. There are \( 2^m \) elements in \( \mathcal{V} \). Suppose these elements of \( \mathcal{V} \) are labeled as \( D_+ \), \( s \in I_{2m} \). Denote \( D_{-} = I - D_+ \). Clearly, \( D_{-} \in \mathcal{V} \) if \( D_{-} \in \mathcal{V} \). The following lemma is adopted from Hu and Lin (2001).

**Lemma 2.** Let \( F, H \in \mathbb{R}^{n \times n} \). Then, for any \( x \in \mathcal{L}(H) \), \n\[ \text{sat}(Fx) \in co \{ D_+Fx + D_{-}Hx : s \in I_{2m} \}, \]

where co stands for the convex hull.

For a positive definite matrix \( P \in \mathbb{R}^{n \times n} \) and a scalar \( \rho > 0 \), we define \( \mathcal{E}(P, \rho) := \{ x \in \mathbb{R}^n : x^T P x \leq \rho \} \). The following theorem characterizes the bounded state stability of the switched system that results from the family of systems (1) and the switching scheme (3).

**Theorem 1.** Consider system (1). If there exist \( P_i > 0, \xi > 0, Q_i = Q_i^T, F_i \in \mathbb{R}^{n \times n}, H_i \in \mathbb{R}^{n \times n}, \theta_i \geq 0, \theta_i > 0 \) and \( n_{ij} \) such that

1. \( (A_i + B_i(D_iF_i + D_{-i}H_i))^T P_i + P_i (A_i + B_i(D_iF_i + D_{-i}H_i)) + \frac{1}{\xi} P_i E_i^T E_i P_i + \theta_i Q_i \leq 0, \quad s \in I_{2m}, i \in I_N \);
2. \( P_i = P_i + n_{ij} (Q_i - Q_j), i \in I_N, j \in I_N \);
3. \( \theta_i Q_i + \theta_j Q_j + \cdots + \theta_{n_{ij}} Q_{N} \geq 0 \);

and \( \mathcal{E}(P_i, 1 + \alpha \xi) \cap \Omega_i \subseteq \mathcal{L}(H_i), i \in I_N \), then every trajectory of the closed-loop system that starts from inside of \( \cap_{i=1}^{N} (\mathcal{E}(P_i, 1) \cap \Omega_i) \) will remain inside of \( \cap_{i=1}^{N} (\mathcal{E}(P_i, 1 + \alpha \xi) \cap \Omega_i) \) for every \( w \in \mathbb{R}^2_+ \) as long as no sliding motion occurs or sliding motions only occur along switching surfaces with the corresponding \( n_{ij} \geq 0 \). If the condition \( \mathcal{E}(P_i, 1 + \alpha \xi) \cap \Omega_i \subseteq \mathcal{L}(H_i) \) is replaced with \( \mathcal{E}(P_i, \alpha \xi) \cap \Omega_i \subseteq \mathcal{L}(H_i) \), then any trajectory starting from the origin will remain inside the region \( \cap_{i=1}^{N} (\mathcal{E}(P_i, \alpha \xi) \cap \Omega_i) \) for every \( w \in \mathbb{R}^2_+ \) as long as no sliding motion occurs or sliding motions only occur along switching surfaces with the corresponding \( n_{ij} \geq 0 \).
Proof. By Lemma 1 and 2, Condition 3 of the theorem and the largest region function strategy (3) guarantee that the resulting switched system is well defined.

We will define the energy of the system as its trajectory evolves in the state space. In each region \( \Omega_i \), we will measure its energy as \( V_i(x) = \frac{1}{2} x^T P_i x \). The derivative of \( V_i(x) \) along the trajectory of the subsystem \( i \) is then given by

\[
\dot{V}_i = 2x^T P_i (A_i x + B_i \text{sat}(F_i x) + E_i w).
\]

By Lemma 2, for every \( x \in \mathcal{E}(P, 1 + \alpha \xi) \cap \Omega_i \subset \mathcal{L}(H_i) \) (or \( x \in \mathcal{E}(P, \alpha \xi) \cap \Omega_i \subset \mathcal{L}(H_i) \)),

\[
\text{sat}(F_i x) \in \{ D_i F_i x + D_i^T H_i x, \ s \in I_{2m} \}.
\]

It follows that

\[
A_i x + B_i \text{sat}(F_i x) \in \{ A_i x + B_i (D_i F_i + D_i^T H_i) x, \ s \in I_{2m} \}.
\]

Noting that,

\[
2x^T P_i E_i w \leq \frac{1}{\xi} x^T P_i E_i^T P_i x + \xi w^T w, \quad \forall \xi > 0,
\]

we have

\[
\dot{V}_i = 2x^T P_i (A_i x + B_i \text{sat}(F_i x) + E_i w) \leq \max_{s \in I_{2m}} 2x^T P_i (A_i x + B_i (D_i F_i + D_i^T H_i) x)
\]

\[
+ \frac{1}{\xi} x^T P_i E_i^T P_i x + \xi w^T w.
\]

It then follows from Condition 1 in Item 1 of the theorem that

\[
\dot{V}_i \leq -\bar{\theta} x^T Q_i x + \xi w^T w, \quad \forall x \in \mathcal{E}(P, 1 + \alpha \xi) \cap \Omega_i
\]

(0 \in \mathcal{E}(P, \alpha \xi) \cap \Omega_i).

Recall that system \( i \) is activated in the region \( \mathcal{E}(P, 1 + \alpha \xi) \cap \Omega_i \) (or \( \mathcal{E}(P, \alpha \xi) \cap \Omega_i \)). In this region, \( x^T Q_i x \geq 0 \).

Thus,

\[
\dot{V}_i \leq \xi w^T w, \quad \forall x \in \mathcal{E}(P, 1 + \alpha \xi) \cap \Omega_i
\]

(0 \in \mathcal{E}(P, \alpha \xi) \cap \Omega_i).

Let \( V(x) = V_{\alpha}(x) \) be the overall energy of the switched system for use as a Lyapunov function candidate. Such a function is, in general, not differentiable along the boundaries between \( \Omega_i \)'s, i.e., \( \Omega_{ij}, i \in I_N, j \in I_N \), where the switch from subsystem \( i \) to subsystem \( j \) occurs and where \( V_i(x) = V_j(x) \) due to Condition 2 of the theorem.

But, for any state not on the boundaries, we have

\[
\dot{V}_i \leq \xi w^T w, \quad \forall x \in \bigcap_{i=1}^{N} \mathcal{E}(P, 1 + \alpha \xi) \cap \Omega_i
\]

(0 \in \bigcup_{i=1}^{N} \mathcal{E}(P, \alpha \xi) \cap \Omega_i).

Consequently, if sliding mode does not occur on any of the boundaries, then let \( t_j, j = 1, 2, \ldots, N_i \), be the times when the trajectory crosses the boundaries of \( \Omega_i \)'s before time \( t \). Then, integrating both sides of the above inequality from 0 to \( t \) results in

\[
V(x(t)) \leq V(x(0)) + \xi \sum_{j=1}^{N_i-1} \int_{t_j}^{t_{j+1}} w^T(w) \text{d}t
\]

\[
+ \xi \int_{t_N}^{t} w^T(w) \text{d}t
\]

\[
= V(x(0)) + \xi \int_{t_0}^{t} w^T(w) \text{d}t
\]

\[
\leq \begin{cases} 1 + \alpha \xi, & \text{if } x(0) \in \bigcup_{i=1}^{N} \mathcal{E}(P, 1) \cap \Omega_i, \\ \alpha \xi, & \text{if } x(0) = 0. \end{cases}
\]

where we have set \( t_0 = 0 \). This shows that any trajectory that starts from \( \bigcap_{i=1}^{N} \mathcal{E}(P, 1) \cap \Omega_i \) will remain inside \( \bigcap_{i=1}^{N} \mathcal{E}(P, 1 + \alpha \xi) \cap \Omega_i \), and any trajectory that starts from \( x(0) = 0 \) will remain inside \( \bigcap_{i=1}^{N} \mathcal{E}(P, \alpha \xi) \cap \Omega_i \).

We next consider the situation when sliding motion does occur. Denote \( \bar{A}_i = A_i + B_i (E_i F_i + H_i E_i) \), \( i \in I_N, s \in I_{2m} \), and \( \bar{A}_{ij} = A_j + B_j (E_j F_j + H_j E_j) \), \( j \in I_N, s \in I_{2m} \). A sliding motion occurs along the hyper surface \( x^T Q_i x = x^T Q_i x \geq 0 \), between two neighboring regions \( x^T Q_i x \geq 0 \) and \( x^T Q_i x \geq 0 \), if vector fields \( \bar{A}_i x, s \in I_{2m} \), all point into region \( x^T Q_i x \geq 0 \) and vector fields \( \bar{A}_{ij} x, s \in I_{2m} \), all point into region \( x^T Q_i x \geq 0 \). (Pettersson, 2005). A sliding motion may lead to either stable or unstable dynamics along the switching surface according to Filippov’s convex combination (Filippov, 1988).

\[
\dot{x} = \lambda \left[ \sum_{s=1}^{m} \rho_{s,i}(x) \bar{A}_{s,i} x + E_i w \right]
\]

\[
+ (1 - \lambda) \left[ \sum_{s=1}^{m} \rho_{s,j}(x) \bar{A}_{s,j} x + E_j w \right], \quad \forall 0 \leq \lambda \leq 1,
\]

where \( \rho_{s,i}(x) > 0, \sum_{s=1}^{m} \rho_{s,i}(x) = 1 \), and \( \rho_{s,j}(x) > 0, \sum_{s=1}^{m} \rho_{s,j}(x) = 1 \), are such that (see Lemma 2)

\[
A_i x + B_i \text{sat}(F_i x) = \sum_{s=1}^{m} \rho_{s,i}(x) \bar{A}_{s,i} x,
\]

\[
A_j x + B_j \text{sat}(F_j x) = \sum_{s=1}^{m} \rho_{s,j}(x) \bar{A}_{s,j} x.
\]

We will show that a sliding motion will not destroy the closed-loop property established above, if it occurs on a switching surface with corresponding \( \eta_{ij} \geq 0 \). Indeed, if \( \eta_{ij} = 0 \), Condition 2 of the theorem implies that \( P_j = P_i \), which result in a common Lyapunov function for dynamics in both \( \Omega_i \) and \( \Omega_j \). That is, the Lyapunov function \( V(x) \) is differentiable along \( \Omega_{ij} \) and (5) is also valid for any \( x \in \Omega_{ij} \).

We next consider the case of \( \eta_{ij} > 0 \). According to the analysis of sliding motions in (Pettersson, 2005), the sliding motion occurring along the surface \( x^T Q_i x = x^T Q_i x \geq 0 \) implies that

\[
x^T \sum_{s=1}^{m} \rho_{s,i}(x) \bar{A}_{s,i}^T (Q_j - Q_i)x + x^T (Q_j - Q_i)E_i w < 0,
\]

\[
x^T \sum_{s=1}^{m} \rho_{s,j}(x) \bar{A}_{s,j}^T (Q_i - Q_j)x + x^T (Q_i - Q_j)E_j w > 0.
\]

Using Condition 2 and noting that \( \eta_{ij} > 0 \), we have

\[
x^T \sum_{s=1}^{m} \rho_{s,i}(x) \bar{A}_{s,i}^T (P_j - P_i)x + x^T (P_j - P_i)E_i w < 0,
\]

\[
x^T \sum_{s=1}^{m} \rho_{s,j}(x) \bar{A}_{s,j}^T (P_i - P_j)x + x^T (P_i - P_j)E_j w > 0.
\]
which is equivalent to

\[
\begin{align*}
&\sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w \\
&< \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w, \\
&\sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w, \\
&< \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w.
\end{align*}
\]

Hence, by Condition 1 of the theorem, we have

\[
\begin{align*}
&\sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w \\
&< -\partial x^T Q x + \xi w^T w, \\
&\sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w \\
&< -\partial x^T Q x + \xi w^T w,
\end{align*}
\]

from which and Condition 1 of the theorem, we have that, for all \(0 \leq \lambda \leq 1,

\[
\begin{align*}
&\sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w \\
&+ (1 - \lambda) \left( \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w \right) \\
&+ 2\lambda x^T P_1 E_1 w + 2(1 - \lambda)x^T P_1 E_1 w < \xi w^T w,
\end{align*}
\]

This, in turn, implies that

\[
\begin{align*}
&\sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w \\
&+ P_1 \left( \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1}^T P_s x + \sum_{s=1}^{2^m} \rho_{s,1}(x)A_{s,1} x + 2x^T P_1 E_1 w \right) \\
&+ 2\lambda x^T P_1 E_1 w + 2(1 - \lambda)x^T P_1 E_1 w < \xi w^T w.
\end{align*}
\]

Therefore, the inequality (5), and hence (6), is still valid for any \(x \in \Omega_{i,j} \text{.} \) This completes the proof. \( \square \)

### 4. Disturbance tolerance

A fundamental problem to be addressed before determining the restricted \( L_2 \) gain is the assessment of the disturbance tolerance capability of the closed-loop system. The disturbance tolerance capability is measured by the largest bound on the energy of the disturbances, say \( \alpha^* \), under which the closed-loop trajectories starting from the origin or a given set of initial conditions remain bounded. As the restricted \( L_2 \) gain will be defined with zero initial conditions, we will only assess disturbance tolerance with zero initial conditions in this section. Disturbance tolerance with a given set of initial conditions can be dealt with similarly by resorting to the first part of Theorem 1.

As established in Theorem 1, under the three itemized conditions and the condition that \( \varepsilon(P, \alpha \xi) \subset \mathcal{L}(H_i), \) \( i \in I_n, \) the trajectories of the closed-loop system that start from origin will remain inside the region \( \mathcal{C} \varepsilon_{\alpha \xi}(\varepsilon(P, \alpha \xi) \cap \Omega_i) \) for every \( w \in W, \) if no sliding motion occurs or sliding motions only occur along switching surfaces with the corresponding \( n_{1j} \geq 0. \)

Without loss of generality, we can assume that \( \xi = 1 \) in the above mentioned result. If \( \xi \neq 1, \) we can multiply both sides of Condition 1 of Theorem 1 with \( 1/\xi \) to obtain

\[
(A_i + B_i(D_iF_i + D_i^T H_i)) P_i + \frac{P_i}{\xi} (A_i + B_i(D_iF_i + D_i^T H_i)) \\
\]

\[
+ \frac{P_i}{\xi} E_i^T E_i + \frac{1}{\xi} \geq 0, \quad s \in I_{2m}, \quad i \in I_n.
\]

Let \( \tilde{P}_i = P_i/\xi \) and \( \tilde{Q}_i = Q_i/\xi. \) Then, \( \tilde{P}_i \text{ and } \tilde{Q}_i \) satisfy all conditions of Theorem 1 with \( \xi = 1 \) and \( \varepsilon(\tilde{P}_i, \alpha \xi) = \varepsilon(P_i, \alpha \xi). \) As a result, the disturbance tolerance capability of the closed-loop system under zero initial conditions can be assessed through solving the following optimization problem,

\[
\begin{align*}
&\sup_{P_i > 0, Q_i > 0, \alpha > 0, n_{1j} \geq 0} \alpha, \\
&s.t. \begin{cases}
(A_i + B_i(D_i F_i + D_i^T H_i))^T P_i + P_i (A_i + B_i(D_i F_i + D_i^T H_i)) \\
+ P_i E_i^T E_i + \theta_i Q_i \leq 0, \quad s \in I_{2m}, \quad i \in I_n, \\
(P_i + n_{1j} Q_i) \geq 0, \quad i \in I_{n-1}, \quad j \in I_n, \\
H_i Q_i + \eta_i H_i \geq 0, \quad i \in I_{n-1}, \quad j \in I_n, \\
E_i P_i + \theta_i H_i \geq 0, \quad i \in I_{n-1}, \quad j \in I_n.
\end{cases}
\end{align*}
\]

Let \( \nu = 1/\alpha. \) Then, Constraint (d) is implied by

\[
\begin{align*}
&n_{1j}(P_i - \delta_i Q_i)^{-1} \nu_1 \leq \nu, \quad k \in I_m, \\
&n_{1j}(P_i - \delta_i Q_i)^{-1} \nu_1 \leq \nu, \quad k \in I_m,
\end{align*}
\]

which, by Schur complements, is equivalent to,

\[
\begin{align*}
&\nu \nu_1 (P_i - \delta_i Q_i)^{-1} \nu_1 \geq \nu, \quad k \in I_m, \\
&\nu \nu_1 (P_i - \delta_i Q_i)^{-1} \nu_1 \geq \nu, \quad k \in I_m, \\
&\nu \nu_1 (P_i - \delta_i Q_i)^{-1} \nu_1 \geq \nu, \quad k \in I_m, \quad i \in I_n.
\end{align*}
\]

where \( \delta_i > 0 \) and \( \nu_k \) denotes the kth row of \( H_i. \)

Consequently, the optimization problem (7) can be written as the following BMI problem,

\[
\begin{align*}
&\inf_{P_i > 0, Q_i > 0, \alpha > 0, \eta_i, \nu > 0} \nu, \\
&s.t. \begin{cases}
(A_i + B_i(D_i F_i + D_i^T H_i))^T P_i + P_i (A_i + B_i(D_i F_i + D_i^T H_i)) \\
+ P_i E_i^T E_i + \theta_i Q_i \leq 0, \quad s \in I_{2m}, \quad i \in I_n, \\
(P_i + n_{1j} Q_i) \geq 0, \quad i \in I_{n-1}, \quad j \in I_n, \\
H_i Q_i + \eta_i H_i \geq 0, \quad i \in I_{n-1}, \quad j \in I_n, \\
E_i P_i + \theta_i H_i \geq 0, \quad i \in I_{n-1}, \quad j \in I_n.
\end{cases}
\end{align*}
\]

If we define \( P_n = P \) and \( \eta_i = n_{1j}, \) \( i \in I_{n-1}, \) then, Constraint (b) simplifies to

\[
\begin{align*}
P_n = P + n_{1j} Q_n \geq 0, \quad i \in I_{n-1},
\end{align*}
\]

and consequently, Constraint (a) simplifies to

\[
\begin{align*}
(A_i + B_i(D_i F_i + D_i^T H_i))^T P_n + n_{1j} Q_n \geq 0, \quad i \in I_{n-1}, \\
(A_i + B_i(D_i F_i + D_i^T H_i))^T P_n + n_{1j} Q_n \geq 0, \quad i \in I_{n-1}, \\
(A_i + B_i(D_i F_i + D_i^T H_i))^T P_n + n_{1j} Q_n \geq 0, \quad i \in I_{n-1}, \\
(A_i + B_i(D_i F_i + D_i^T H_i))^T P_n + n_{1j} Q_n \geq 0, \quad i \in I_{n-1},
\end{align*}
\]
In case of switching between only two subsystems, we can set $Q_1 = Q$ and $Q_2 = -Q$, where $Q$ is a symmetric matrix. Furthermore, we can, without loss of generality, scale $\theta_1 = \theta_2 = 1$. This implies that Constraint (c) in (9) is automatically satisfied. The optimization problem (9) then simplifies to

$$\inf_{P > 0, P - 2Q > 0, \eta, j_1 > 0, j_2 > 0, \eta H_1, H_2} V,$$

s.t. (a) \((A_1 + B_1(D_1 F_1 + D_1^T H_1))T (P - 2\eta Q) + (P - 2\eta Q)(A_1 + B_1(D_1 F_1 + D_1^T H_1)) + (P - 2\eta Q)(P - 2\eta Q) + \eta \delta_1 Q \leq 0, \quad s \in I_{2^m}, \)

\((A_2 + B_2(D_2 F_2 + D_2^T H_2))T P + (P + B_2(D_2 F_2 + D_2^T H_2))(P - 2\eta Q) + \eta \delta_2 Q \leq 0, \quad s \in I_{2^m}, \)

(b) \(v_i T (h_{1,k} P - 2\eta Q - \delta_1 Q) \geq 0, \quad k \in I_m, \)

\(v_i T (h_{2,k} P + \delta_2 Q) \geq 0, \quad k \in I_m. \)

The problem of verifying the existence of the unknown variables solving the optimization problem (10), is a bilinear matrix inequality (BMI) problem, which is NP-hard and difficult to solve. However, many algorithms for BMI problems have been proposed on the basis of approximations, heuristics, branch & bound, or local search. For example, PENOPT offers a commercial solver PENBMI for solving the optimization problems with bilinear matrix inequality constraints. We will use PENBMI to obtain all our numerical results in Section 6. While the conservativeness of these numerical results is not clear, we do demonstrate the effectiveness of the proposed method.

The above optimization problem can be adapted for the design of feedback gains $F_i$’s. This can be readily done by viewing $F_i$’s as additional optimization parameters.

5. $\mathcal{L}_2$ gain analysis

The restricted $\mathcal{L}_2$ gain of the closed-loop system is defined over a set of tolerable disturbances, say, $W_2^c$, as

$$\gamma^* = \sup_{x(0) = 0, u \in W_2^c} \frac{\|z(t)\|_{L_2}}{\|w(t)\|_{L_2}},$$

where $\| \cdot \|_{L_2}$ is the $\mathcal{L}_2$ norm of a signal. Thus, the closed-loop system has a restricted $\mathcal{L}_2$ gain over $W_2^c$ less than or equal to $\gamma$ if, for $x(0) = 0$,

$$\int_0^T z^T(t)z(t)dt \leq \gamma^2 \int_0^T w^T(t)w(t)dt, \quad \forall t \geq 0, \forall w \in W_2^c.$$

The following theorem characterizes the conditions under which the switched linear system has a restricted $\mathcal{L}_2$ gain less than or equal to $\gamma$.

**Theorem 2.** Consider system (1) and an $\alpha \in (0, \alpha^*)$. If there exist $P_i > 0$, $Q_i = Q_i^T$, $F_i \in \mathbb{R}^{n \times m}$, $H_i \in \mathbb{R}^{m \times n}$, $\delta_i \geq 0$, $j_{i,i}$ and $\theta_i > 0$ such that

1. \((A_i + B_i(D_i F_i + D_i^T H_i))T P_i + (P_i + B_i(D_i F_i + D_i^T H_i))T (A_i + B_i(D_i F_i + D_i^T H_i)) + (P_i + B_i(D_i F_i + D_i^T H_i))(P_i + B_i(D_i F_i + D_i^T H_i)) \leq \gamma^2 \|z(t)\|_{L_2}, \quad \forall s \in I_{2^m}, i \in I_n, \)

2. $P_i = P_i^T$, $Q_i = Q_i^T$, $i \in I_{2^m}, j \in I_{2^n}$,

3. $\delta_i Q_i + \theta_i Q_i + \cdots + \theta_i Q_i \leq 0,$

and $\mathcal{E}(P, \alpha) \cap \Omega_i \subset \mathcal{E}(H_i), i \in I_{2^m}$, then the restricted $\mathcal{L}_2$ gain from $w$ to $z$ over $W_2^c$ is less than or equal to $\gamma$, if no sliding motion occurs or sliding motions only occur along switching surfaces with the corresponding $j_{i,i} \geq 0$.

**Proof.** By Lemmas 1 and 2, Condition 3 of the theorem and the largest region function strategy (3) guarantee that the resulting switched system is well defined. To continue with the proof, we define the energy of the system as its trajectory evolves in the state space. In each region $\Omega_j$, we will measure its energy as $V_j(x) = x^T P_j x$. The derivative of $V_j(x)$ along the trajectory of the subsystem $i$ is then given by

$$\dot{V}_i = 2x^T P_i(A_i x + B_i sat(F_i x) + E_i w).$$

By Lemma 2, for every $x \in \mathcal{E}(P, \alpha) \cap \Omega_i \subset \mathcal{E}(H_i)$, $sat(F_i x) \in \{D_i F_i x + D_i^T H_i x, s \in I_{2^n}\}$.

It follows that

$$A_i x + B_i sat(F_i x) \in \{A_i x + B_i(D_i F_i + D_i^T H_i) x, s \in I_{2^n}\}.$$

Recall that system $i$ is activated in the region $\mathcal{E}(P, \alpha)$, where $x^T P_i x \geq 0$. Thus, in view of Condition 1 of the theorem, and noting that,

$$2x^T P_i E_i w \leq x^T P_i E_i P_j x + w^T w,$$

we have

$$\dot{V}_i = 2x^T P_i(A_i x + B_i sat(F_i x) + E_i w) \leq \max_{x \in I_{2^n}} 2x^T P_i(A_i x + B_i(D_i F_i + D_i^T H_i) x) + x^T P_i E_i P_j x + w^T w,$$

$$\leq -\frac{1}{\gamma^2} x^T C_i^T C_i x + w^T w,$$

$$= -\frac{1}{\gamma^2} z^T z + w^T w, \forall x \in \mathcal{E}(P, \alpha) \cap \Omega_i.$$

Now, let $V(x) = V_{|\mathcal{L}_2}(x)$ be the overall energy of the switched system as for a Lyapunov function candidate. Such a function is in general not differentiable along the boundaries between $\Omega_j$’s, i.e., $\Omega_j$’s, where the switch from subsystem i to subsystem j occurs and where $V_j(x) = V(x)$ due to Condition 2 of the theorem. However, for any state not on these boundaries, we have

$$\dot{V} \leq -\frac{1}{\gamma^2} z^T z + w^T w, \forall x \in \cap_{i=1}^n (\mathcal{E}(P, \alpha) \cap \Omega_i).$$

It was shown in the proof of Theorem 1, the set $\cap_{i=1}^n (\mathcal{E}(P, \alpha) \cap \Omega_i)$ is an invariant set. Consequently, if sliding mode does not occur on any of the boundaries, then let $t_j, j = 1, 2, \ldots, N$, be the times when the trajectory crosses the boundaries of $\Omega_j$’s before time $t$. Then, integrating both sides of the above inequality from 0 to $t$ results in

$$V(x(t)) \leq V(x(0)) + \sum_{j=1}^{N} \int_{t_{j-1}}^{t_{j}} \left(-\frac{1}{\gamma^2} z^T z(t) + w^T w(t)\right) dt + \int_{t_0}^{t} \left(-\frac{1}{\gamma^2} z^T z(t) + \frac{w^T w(t)}{2}\right) dt,$$

$$= V(x(0)) + \int_{t_0}^{t} \left(-\frac{1}{\gamma^2} z^T z(t) + \frac{w^T w(t)}{2}\right) dt,$$

$$\forall x \in \cap_{i=1}^n (\mathcal{E}(P, \alpha) \cap \Omega_i),$$

where we have set $t_0 = 0$. Noting that $V(x(t)) \geq 0$ and $V(0) = 0$, we have that, for $x(0) = 0$,

$$\int_{t_0}^{t} z^T z(t) dt \leq \gamma^2 \int_{t_0}^{t} w^T w(t) dt,$$

which indicates that, if no sliding motion occurs, the restricted $\mathcal{L}_2$ gain from $w$ to $z$ over $W_2^c$ is less than or equal to $\gamma$. We next examine the situation when a sliding motion occurs. We will show that the restricted $\mathcal{L}_2$ gain from $w$ to $z$ over $W_2^c$ is still less than or equal to $\gamma$, if a sliding motion occurs on a switching
surface with corresponding \( n_{ij} \geq 0 \). Indeed, if \( n_{ij} = 0 \), Condition 2 of the theorem implies that \( P_i = P_j \), resulting in a common Lyapunov function for the dynamics in both \( \Omega_1 \) and \( \Omega_2 \). That is, the Lyapunov function \( V(x) \) is differentiable along \( \Omega_{ij} \) and (12) is also valid for \( x \in \Omega_{ij} \).

For the case when \( n_{ij} > 0 \), as has been shown in the proof of Theorem 1, the sliding motion occurring along the surface \( x^TQx \geq 0 \) implies (3), from which and Condition 1 of this theorem, we have

\[
\begin{align*}
\mathbf{x}(t) &= \mathbf{x}(0) e^{A_T} t + \int_0^t e^{A_T (t-s)} \sum_{i=1}^m \left( \rho_i(x) \bar{A}_{i,s} P_i + P_i \sum_{j=1}^m \rho_j(x) \bar{A}_{j,s} \right) x(s) ds + \int_0^t e^{A_T (t-s)} \sum_{i=1}^m \left( \rho_i(x) \bar{A}_{i,s} P_i + P_i \sum_{j=1}^m \rho_j(x) \bar{A}_{j,s} \right) w(s) ds,
\end{align*}
\]

Thus, based on the proof of Theorem 1, the restricted \( L_z \) gain can also be adapted for the design of feedback gains \( F_i \)'s by simply viewing \( F_i \)'s as additional optimization parameters.

6. Numerical examples

Example 1. Let us consider system (1) with \( w \in W^2_a \) and

\[
A_1 = \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},
\]

\[
C_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.
\]

To design a switching law that maximizes the disturbance tolerance capacity of the resulting switched system, we solve the optimization problem (10) and obtain

\[
\nu^* = 0.3409, \quad \alpha^* = 2.9336,
\]

\[
H_1 = \begin{bmatrix} -0.3094 \\ 0.1259 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 3.3443 \\ -1.3757 \end{bmatrix},
\]

\[
Q_1 = -Q_2 = Q = \begin{bmatrix} -0.0422 \\ 0.0466 \end{bmatrix}, \quad 0.0466, \quad 0.0422,
\]

\[
P_1 = \begin{bmatrix} 11.9804 \\ 9.1899 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 28.8187 \\ -13.1624 \end{bmatrix}, \quad 7.1183.
\]

Plotted in Fig. 1 are the ellipsoids \( \mathcal{E}(P_1, \alpha^*) \) and \( \mathcal{E}(P_2, \alpha^*) \), along with a trajectory starting from the origin and under a pulse disturbance of duration 0.2s and with a maximum energy \( \alpha^* \). A zoom in plot of this trajectory is shown in Fig. 2.

We next estimate the restricted \( L_z \) gain of the resulting switched system over \( W^2_a, \alpha \in (0, \alpha^*) \). This can be done by solving the optimization problem (17). Plotted in Fig. 3 is the obtained \( \gamma^* \) for different values of \( \alpha \).

Example 2. Consider system (1) with \( w \in W^2_a \) and

\[
A_1 = A_2 = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix},
\]

\[
E_1 = E_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]

\[
F_1 = \begin{bmatrix} 1.2231 \\ -2.2486 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.8396 \\ -1.7221 \end{bmatrix}.
\]

We note that two subsystems result from the two different stabilizing feedback gains. For each of these two subsystems, we
which results in a switched system with an estimated disturbance tolerance capability of $\alpha^* = 632.8927$.

We next proceed to design a switching law that minimizes the restricted $L_2$ gain of the resulting switched system. To this end, let us set $\alpha = 500$. Solving the optimization problem (17), we obtain a switching law characterized by

$$Q_1 = -Q_2 = Q = \begin{bmatrix} -433.9681 & -10.2274 \\ -10.2274 & 433.9681 \end{bmatrix},$$

with

$$P_1 = \begin{bmatrix} 822.9329 & -688.0880 \\ -688.0880 & 677.0671 \end{bmatrix},$$

$$P_2 = 10^3 \times \begin{bmatrix} 1.0000 & -0.6839 \\ -0.6839 & 0.5000 \end{bmatrix},$$

$$H_1 = 10^{-51} \times \begin{bmatrix} -0.9690 & -0.1154 \\ -0.1154 & -0.4108 \end{bmatrix},$$

$$H_2 = 10^{-43} \times \begin{bmatrix} -0.0459 \\ -0.4108 \end{bmatrix}.$$

This switching law results in a switched system with an estimated restricted $L_2$ gain of $\gamma^* = 0.1187$. It is clear that $\gamma^* < \min(\gamma^*_1, \gamma^*_2)$.

On the other hand, when the actuator saturation does not occur, the closed-loop system resulting from the two individual feedback gains, each behaves as a linear system for which the restricted $L_2$ gain can be exactly determined as its $H_\infty$ norm. They are $\gamma^*_1 = 0.1253$ and $\gamma^*_2 = 0.1183$. However, in the absence of actuator saturation, the $L_2$ gain of the switched system can be estimated as $\gamma^* = 0.1072$. This demonstrates the effectiveness of the proposed switching scheme in reducing the $L_2$ gain from the disturbance to the system output, as it results in an $L_2$ gain that is smaller than the $H_\infty$ norm of each of the closed-loop system under the two individual feedback gains.

7. Conclusions

This paper considered the problem of disturbance tolerance/rejection of a switched system resulting from a family of linear systems subject to actuator saturation and $L_2$ disturbances. Design algorithms for both feedback gains for individual systems and the switching scheme were developed. Several examples were worked out to illustrate the effectiveness of the proposed design method and indeed the power of switching control itself.

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References


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