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Euler's Theorem For Homogeneous White Noise Operators

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Abstract In this paper we introduce a new notion of $\lambda-$order homogeneous operators on the nuclear algebra of white noise operators. Then, we give their Fock expansion in terms of quantum white noise (QWN) fields $\{a_t,\ a_t^*\ ;\ t\in\mathbb{R}\}$. The quantum extension of the scaling transform enables us to prove Euler's theorem in quantum white noise setting.

Keywords QWN-Euler operator, Euler's Theorem, QWN-scaling operator, Homogeneous operator, QWN-derivatives.

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1 Introduction and Preliminaries

Let H be the real Hilbert space of square integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$ and $E\equiv\mathcal{S}(\mathbb{R})$ be the Schwartz space consisting of rapidly decreasing C^∞-functions. Then, the nuclear Gel'fand triple

$$\mathcal{S}(\mathbb{R})\subset L^2(\mathbb{R},dx)\subset\mathcal{S}'(\mathbb{R}) \tag{1}$$

can be reconstructed in a standard way (see Ref. [18]) by the harmonic oscillator $A=1+t^2-d^2/dt^2$ and H. The eigenvalues of A are $2n$, $n=1,2,\cdots$, the corresponding eigenfunctions $\{e_n;\ n\geq 1\}$ form an orthonormal basis for $L^2(\mathbb{R})$. In fact (e_n) are the Hermite functions and therefore each e_n is an element of E. The space E is a nuclear space equipped with the Hilbertian norms

$$|\xi|_p=|A^p\xi|_0\,,\qquad \xi\in E,\quad p\in\mathbb{R}$$

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and we have

$$E = \operatorname{proj}\lim_{p\to\infty} E_p\,, \qquad E' = \operatorname{ind}\lim_{p\to\infty} E_{-p},$$

where, for $p \geq 0$, E_p is the completion of E with respect to the norm $|\cdot|_p$ and E_{-p} is the topological dual space of E_p. We denote by $N = E + iE$ and $N_p = E_p + iE_p$, $p \in \mathbb{Z}$, the complexifications of E and E_p, respectively. Throughout, we fix a Young function θ satisfying the condition

$$\limsup_{x\to\infty} \frac{\theta(x)}{x^2} < +\infty\cdot \tag{2}$$

Its polar function θ^* is the Young function defined by

$$\theta^*(x) = \sup_{t\geq 0}(tx - \theta(t)), \quad x \geq 0.$$

For more details , see Refs. [8].

For a complex Banach space $(B, \|\cdot\|)$, $\mathcal{H}(B)$ denotes the space of all entire functions on B and for $m > 0$, $\mathrm{Exp}(B, \theta, m)$ is the Banach space

$$\mathrm{Exp}(B,\theta,m) = \Big\{ f \in \mathcal{H}(B);\ \ \|f\|_{\theta,m} := \sup_{z\in B} |f(z)|e^{-\theta(m\|z\|)} < \infty \Big\}.$$

The projective system $\{\mathrm{Exp}(N_{-p}, \theta, m);\ p \in \mathbb{N},\ m > 0\}$ and the inductive system $\{\mathrm{Exp}(N_p, \theta, m);\ p \in \mathbb{N},\ m > 0\}$ give the two nuclear spaces

$$\mathcal{F}_\theta(N') = \operatorname{proj}\lim_{p\to\infty;m\downarrow 0} \mathrm{Exp}(N_{-p},\theta,m)\,, \qquad \mathcal{G}_\theta(N) = \operatorname{ind}\lim_{p\to\infty;m\to 0} \mathrm{Exp}(N_p,\theta,m). \tag{3}$$

It is noteworthy that, for each $\xi \in N$, the exponential function

$$e_\xi(z) := e^{\langle z,\xi\rangle}, \quad z \in N',$$

belongs to $\mathcal{F}_\theta(N')$ and the set of such test functions spans a dense subspace of $\mathcal{F}_\theta(N')$. In the remainder of this paper we use simply $\mathcal{F}_\theta$ to denote the space $\mathcal{F}_\theta(N')$. The space of continuous linear operators from $\mathcal{F}_\theta$ into its topological dual space $\mathcal{F}_\theta^*$ is denoted by $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and assumed to carry the bounded convergence topology. For $z \in N'$ and $\varphi \in \mathcal{F}_\theta$ with Taylor expansions $\sum_{n=0}^\infty \langle x^{\otimes n}, f_n\rangle$, the holomorphic derivative of φ at $x \in N'$ in the direction z is defined by

$$(a(z)\varphi)(x) := \lim_{\lambda\to 0} \frac{\varphi(x+\lambda z) - \varphi(x)}{\lambda}. \tag{4}$$

We can check that the limit always exists and $a(z) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$. Let $a^*(z) \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta^*)$ be the dual adjoint of $a(z)$, i.e., for $\Phi \in \mathcal{F}_\theta^*$ and $\phi \in \mathcal{F}_\theta$, $\langle\langle a^*(z)\Phi, \phi\rangle\rangle = \langle\langle \Phi, a(z)\phi\rangle\rangle$, where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the standard bilinear form on $\mathcal{F}_\theta^* \times \mathcal{F}_\theta$. Similarly, for $\psi \in \mathcal{G}_{\theta^*}(N)$ with Taylor expansion $\psi(\xi) = \sum_{n=0}^\infty \langle \psi_n, \xi^{\otimes n}\rangle$ we use the common notation $a(z)\psi$ for the derivative (4) with $z \in N$.

The Wick symbol of $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ is by definition [18] a $\mathbb{C}$-valued function on $N \times N$ defined by

$$\sigma(\Xi)(\xi,\eta) = \langle\langle \Xi e_\xi, e_\eta\rangle\rangle e^{-\langle \xi,\eta\rangle}, \quad \xi,\eta \in N. \tag{5}$$

By a density argument, every operator in $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ is uniquely determined by its Wick symbol. In fact, if $\mathcal{G}_{\theta^*}(N \oplus N)$ denotes the nuclear space obtained as in (3) by replacing N_p by $N_p \times N_p$, we have the following characterization theorem for operator Wick symbols.

Theorem 1 *(See Ref. [13]) The Wick symbol map σ yields a topological isomorphism between $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and $\mathcal{G}_{\theta^*}(N \oplus N)$.*

It is a fundamental fact in quantum white noise theory [18] (see, also Ref. [13]) that every white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ admits a unique Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{6}$$

where, for each pairing $l, m \geq 0$, $\kappa_{l,m} \in (N^{\otimes(l+m)})'_{sym(l,m)}$ and $\Xi_{l,m}(\kappa_{l,m})$ is the integral kernel operator uniquely specified via the Wick symbol transform by

$$\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle, \quad \xi, \eta \in N. \tag{7}$$

For any $S_1,\ S_2 \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, there exists a unique $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, denoted $S_1 \diamond S_2$, such that

$$\sigma(S_1 \diamond S_2) = \sigma(S_1)\sigma(S_2). \tag{8}$$

The operator $S_1 \diamond S_2$ will be referred to as *the Wick product* of S_1 and S_2.

Let θ_n be given by $\theta_n = \inf_{r>0} e^{\theta(r)}/r^n$, $n \in \mathbb{N}$. Then, for $p \in \mathbb{N}$ and $\gamma_1, \gamma_2 > 0$, we define the Hilbert space

$$F_{\theta,\gamma_1,\gamma_2}(N_p \oplus N_p) =$$

$$\left\{ \overrightarrow{\varphi} = (\varphi_{l,m})_{l,m=0}^{\infty};\ \varphi_{l,m} \in (N_p^{\otimes l} \otimes N_p^{\otimes m})_{sym(l,m)},\ \sum_{l,m=0}^{\infty} (\theta_l \theta_m)^{-2} \gamma_1^{-l} \gamma_2^{-m} |\varphi_{l,m}|_p^2 < \infty \right\}$$

Put

$$F_\theta(N \oplus N) = \bigcap_{p \in \mathbb{N}, \gamma_1 > 0, \gamma_2 > 0} F_{\theta,\gamma_1,\gamma_2}(N_p \oplus N_p).$$

Theorem 2 *([4]) An operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ if and only if there exists a unique $(\kappa_{l,m})_{l,m} \in F_\theta(N \oplus N)$ such that*

$$\Xi = \Xi_{-\tau} \diamond \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{9}$$

where τ is the usual trace on $N \otimes N$, i.e., $\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle$ and

$$\Xi_{\pm\tau} = \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \Xi_{k,k}(\tau^{\otimes k}).$$

Let $\mathcal{U}_\theta$ be the space of white noise operators given by

$$\mathcal{U}_\theta = \left\{ \Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m});\ (\kappa_{l,m})_{l,m} \in F_\theta(N \oplus N) \right\}.$$

For $x, y \in N$, we put $\kappa_{l,m}(x,y) = \frac{x^{\otimes l}}{l!} \otimes \frac{y^{\otimes m}}{m!}$ and $\Xi^{x,y} := \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}(x,y))$. Then, the set $\{\Xi^{x,y};\ \ x,\ y \in N\}$ spans a dense subspace of $\mathcal{U}_\theta$.

Theorem 3 *([4]) The map f_τ defined by*

$$f_\tau : \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta) \longrightarrow \mathcal{U}_\theta, \quad \Xi \longmapsto \Xi_\tau \diamond \Xi,$$

is a topological isomorphism.

We recall from Ref. [4] the dual pairing: for $T = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{U}_\theta$ and $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, we define

$$\langle\!\langle \Xi, T \rangle\!\rangle := \sum_{l,m=0}^{\infty} l!m! \langle \kappa_{l,m}, \Phi_{l,m} \rangle.$$

For more details see [4], [5], [6],[22], [23] and [24].

In mathematics, a homogeneous function is a function with multiplicative scaling behavior: if the argument is multiplied by a factor, then the result is multiplied by some power of this factor. More precisely, for $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}\backslash\{0\}$, put $S_t f(x) = f(tx)$, $x \in \mathbb{R}^d$. For a given $\lambda \in \mathbb{R}$, an element $f \in L^2(\mathbb{R}^d)$ is said to be $\lambda-$order homogeneous if $S_t f(x) = t^\lambda f(x)$ for each $t \in \mathbb{R}\backslash\{0\}$ and $x \in \mathbb{R}^d$. It is well known that f is $\lambda-$order homogeneous if and only if it satisfies the so-called Euler equation

$$\sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} f = \lambda f \cdot \tag{10}$$

In infinite dimension analysis, an analogue of the Euler operator $\displaystyle\sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i}$ was introduced in [17] as follows

$$\Delta_E = \sum_{i=1}^{\infty} (a^*(e_i) + a(e_i)) a(e_i) = \sum_{i=1}^{\infty} \langle \,\cdot\,, e_i \rangle a(e_i).$$

Moreover, the scaling transformation S_t is defined at $\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, \varphi_n \rangle \in \mathcal{F}_\theta$ by

$$S_t \varphi(x) = \sum_{n=0}^{\infty} \Big\langle x^{\otimes n}, \frac{t^n}{n!} \sum_{l=0}^{\infty} (t^2 - 1)^l \frac{(n+2l)!}{l! 2^l} \tau^{\otimes l} \widehat{\otimes}_l \varphi_{n+2l} \Big\rangle, \quad x \in N' \cdot$$

For $\lambda \in \mathbb{R}$, φ is said to be $\lambda-$order homogeneous if $S_t \varphi = t^\lambda \varphi$ for any $t \in \mathbb{R}\backslash\{0\}$. It is proved in [20] that φ is $\lambda-$order homogeneous if and only if it satisfies the Euler equation

$$\Delta_E \varphi = \lambda \varphi \cdot \tag{11}$$

The main purpose of this paper is the study of the `QWN`-analogue of (11). We start by introducing a `QWN`-Scaling transformation and a `QWN`-second quantization. These transformations will be used to introduce the notion of $\lambda-$order homogeneous operators. Then, as a first main result we give their Fock expansions (see Theorem 5). Our second main result is stated in Theorem 7, where we show that a white noise operator Ξ is $\lambda-$order homogeneous if and only if it satisfies the following `QWN`-Euler equation

$$\Delta_E^Q \Xi = \lambda \Xi.$$

Here Δ_E^Q is the `QWN`-Euler operator defined in [6].

2 Fundamental QWN-Operators

2.1 QWN-Laplacians

From [6], the QWN-Gross Laplacian and QWN-conservation operator can be defined through Theorem 3 on $\mathcal{U}_\theta$, respectively, by

$$\Delta_G^Q = \sum_{j=1}^{\infty} D_{e_j}^+ D_{e_j}^+ + \sum_{j=1}^{\infty} D_{e_j}^- D_{e_j}^-,$$

$$N^Q = \sum_{j=1}^{\infty} (D_{e_j}^+)^* (D_{e_j})^+ + \sum_{j=1}^{\infty} (D_{e_j}^-)^* D_{e_j}^-,$$

where, for $\zeta \in N$,

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \qquad D_\zeta^- \Xi = -[a^*(\zeta), \Xi] \tag{12}$$

are the *creation derivative* and *annihilation derivative* of Ξ, (see [12]).

Lemma 1 *For any $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$, we have*

$$N^Q \Xi = \sum_{l,m=0}^{\infty} (l+m) \Xi_{l,m}(\kappa_{l,m}). \tag{13}$$

Proof From [6], we have, for $x, y \in N$

$$\sigma(N^Q \Xi^{x,y})(\xi, \eta) = (\langle x, \eta \rangle + \langle y, \xi \rangle) \sigma(\Xi^{x,y})(\xi, \eta).$$

On the other hand, denoting the right hand side of (13) by A^Q, we get

$$\sigma(A^Q \Xi^{x,y})(\xi, \eta)$$

$$= \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} \langle x, \eta \rangle \frac{\langle x, \eta \rangle^{l-1}}{(l-1)!} \frac{\langle y, \xi \rangle^m}{m!} + \sum_{l=0}^{\infty} \sum_{m=1}^{\infty} \frac{\langle x, \eta \rangle^l}{l!} \langle y, \xi \rangle \frac{\langle y, \xi \rangle^{m-1}}{(m-1)!}$$
$$= (\langle x, \eta \rangle + \langle y, \xi \rangle) \sigma(\Xi^{x,y})(\xi, \eta).$$

Then, by a density argument we complete the proof.

It is noteworthy that the identity (13) holds true for $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$.

Proposition 1 *Let $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$. Then, we have*

$$(\Delta_G^Q)^* T = \{\Xi_{2,0}(\tau) + \Xi_{0,2}(\tau)\} \diamond T \tag{14}$$

$$(N^Q)^* T = N^Q T. \tag{15}$$

Proof From [1], for $\Xi=\sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})\in\mathcal{U}_\theta$, we have

$$\Delta_G^Q\Xi=\sum_{l,m=0}^{\infty}(l+2)(l+1)\Xi_{l,m}(\tau\otimes^2\kappa_{l+2,m})+\sum_{l,m=0}^{\infty}(m+2)(m+1)\Xi_{l,m}(\kappa_{l,m+2}\otimes_2\tau),\tag{16}$$

where, for $z_p\in(N^{\otimes p})'$, and $\xi_{l+m-p}\in N^{\otimes(l+m-p)}$, $p\le l+m$, the contractions $z_p\otimes_p\kappa_{l,m}$ and $\kappa_{l,m}\otimes^p z_p$ are defined by

$$\langle z_p\otimes^p\kappa_{l,m},\xi_{l-p+m}\rangle=\langle\kappa_{l,m},z_p\otimes\xi_{l-p+m}\rangle,$$

$$\langle\kappa_{l,m}\otimes_p z_p,\xi_{l+m-p}\rangle=\langle\kappa_{l,m},\xi_{l+m-p}\otimes z_p\rangle.$$

Then, for $T=\sum_{l,m=0}^{\infty}\Xi_{l,m}(\Phi_{l,m})\in\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta^*)$, we obtain
$\langle\!\langle T,\Delta_G^Q\Xi\rangle\!\rangle$

$$\begin{aligned}
&=\sum_{l,m=0}^{\infty}l!m!(l+2)(l+1)\langle\Phi_{l,m},\tau\otimes^2\kappa_{l+2,m}\rangle\\
&\quad+\sum_{l,m=0}^{\infty}l!m!(m+2)(m+1)\langle\Phi_{l,m},\kappa_{l,m+2}\otimes_2\tau\rangle\\
&=\sum_{l=2}^{\infty}\sum_{m=0}^{\infty}l!m!\langle\tau\otimes\Phi_{l-2,m},\kappa_{l,m}\rangle+\sum_{l=0}^{\infty}\sum_{m=2}^{\infty}l!m!\langle\Phi_{l,m-2}\otimes\tau,\kappa_{l,m}\rangle.
\end{aligned}$$

Therefore, we get

$$(\Delta_G^Q)^*T=\sum_{l=2}^{\infty}\sum_{m=0}^{\infty}\Xi_{l,m}(\tau\otimes\Phi_{l-2,m})+\sum_{l=0}^{\infty}\sum_{m=2}^{\infty}\Xi_{l,m}(\Phi_{l,m-2}\otimes\tau),$$

which yields

$$\begin{aligned}
\sigma((\Delta_G^Q)^*T)(\xi,\eta)&=\sum_{l=2}^{\infty}\sum_{m=0}^{\infty}\langle\tau\otimes\Phi_{l-2,m},\eta^{\otimes l}\otimes\xi^{\otimes m}\rangle\\
&\quad+\sum_{l=0}^{\infty}\sum_{m=2}^{\infty}\langle\Phi_{l,m-2}\otimes\tau,\eta^{\otimes l}\otimes\xi^{\otimes m}\rangle\\
&=\{\langle\eta,\eta\rangle+\langle\xi,\xi\rangle\}\sigma(T)(\xi,\eta)\\
&=\sigma(\Xi_{2,0}(\tau)+\Xi_{0,2}(\tau))(\xi,\eta)\sigma(T)(\xi,\eta).
\end{aligned}$$

This gives

$$(\Delta_G^Q)^*T=\{\Xi_{2,0}(\tau)+\Xi_{0,2}(\tau)\}\diamond T$$

as desired. (15) follows from (13).

2.2 QWN-Second Quantization

We start by clarifying the topology of the nuclear algebra $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$. From Theorem 1, we have the topological isomorphism:

$$\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*) \simeq \mathcal{G}_{\theta^*}(N \oplus N) = \bigcup_{p\geq 0, \gamma>0} \mathrm{Exp}(N_p \oplus N_p, \theta^*, \gamma).$$

For $p \geq 0$ and $\gamma > 0$, let $\mathcal{L}_{\theta,-p,\gamma}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ denotes the subspace of all $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ which correspond to elements in $\mathrm{Exp}(N_p \oplus N_p, \theta^*, \gamma)$. The topology of $\mathcal{L}_{\theta,-p,\gamma}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ is naturally induced from the norm of the Banach space $\mathrm{Exp}(N_p \oplus N_p, \theta^*, \gamma)$ which will be denoted by $\| \cdot \|_{\theta,-p,\gamma}$, i.e., for $\Xi \in \mathcal{L}_{\theta,-p,\gamma}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$,

$$\|\Xi\|_{\theta,-p,\gamma} = \|\sigma\Xi\|_{\theta^*,-p,\gamma} = \sup_{\xi,\eta\in N_p} |\sigma(\Xi)(\xi,\eta)|\, e^{-\theta^*(\gamma|\xi|_p)-\theta^*(\gamma|\eta|_p)}.$$

For $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and $t \in \mathbb{R}$, we define the operator $\Gamma^Q(t)$ by

$$\Gamma^Q(t)\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(t^{l+m}\Phi_{l,m}). \tag{17}$$

We denote by $GL(\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*))$ the group of all linear homeomorphisms from $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ onto itself.

Proposition 2 *$\{\Gamma^Q(e^t)\}_{t\in\mathbb{R}}$ is a regular one-parameter subgroup of $GL(\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*))$ with infinitesimal generator N^Q.*

Proof The proof of the fact that $\{\Gamma^Q(e^t)\}_{t\in\mathbb{R}}$ is a one-parameter subgroup of $GL(\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*))$ is straightforward. Since we have

$$e^{t(l+m)} = 1 + t(l+m) + t^2 \sum_{k=0}^{\infty} \frac{t^k}{(k+2)!}(l+m)^{(k+2)}$$

then, for $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, one can write

$$\Gamma^Q(e^t)\Xi = \Xi + tN^Q(\Xi) + t^2\Lambda(t)(\Xi) \tag{18}$$

where

$$\Lambda(t)(\Xi) = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Lambda_{l,m}(t)\Phi_{l,m})$$

with

$$\Lambda_{l,m}(t) = \sum_{k=0}^{\infty} \frac{t^k}{(k+2)!}(l+m)^{(k+2)}.$$

Now, for $|t| \leq 1$, using a similar computation as in [7], one can show that, there exist $c, r, r' > 0$ and $p, q \geq 0$ such that

$$\left\| \frac{\sigma(\Gamma^Q(e^t)\Xi) - \sigma(\Xi)}{t} - \sigma(N^Q\Xi) \right\|_{\theta^*,-p,r'} \leq c|t| \|\sigma(\Xi)\|_{\theta^*,-q,r}.$$

It then follows

$$\lim_{t\to 0} \sup_{\|\sigma(\Xi)\|_{\theta^*,-q,r}\leq 1} \left\| \frac{\sigma(\Gamma^Q(e^t)\Xi) - \sigma(\Xi)}{t} - \sigma(N^Q T) \right\|_{\theta^*,-p,\gamma} = 0.$$

This proves the desired statement.

2.3 QWN-Scaling Transformation

Motivated by the classical case studied in [17] and [20], we define the QWN-scaling transformation acting on $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$ by

$$S_t^Q(\Xi) :=$$

$$\sum_{j,k,l,m=0}^{\infty} t^{l+m}(t^2-1)^{j+k} \frac{(l+2j)!(m+2k)!}{2^{j+k}j!k!l!m!} \Xi_{l,m}(\tau^{\otimes j} \otimes^{2j} \kappa_{l+2j,m+2k} \otimes_{2k} \tau^{\otimes k}) \quad (19)$$

We recall from Ref. [6] and Theorem 3 that the QWN-Fourier-Gauss transform $G^Q_{K_1,K_2;B_1,B_2}$ is a continuous linear operator from $\mathcal{U}_\theta$ into itself defined by

$$G^Q_{K_1,K_2;B_1,B_2}\Xi = \sum_{l,m}^{\infty} \Xi_{l,m}(g_{l,m}) \quad (20)$$

where $K_i,\ B_i \in \mathcal{L}(N',N') \cap \mathcal{L}(N,N)$, $i=1,2$ and $g_{l,m}$ is given by

$$g_{l,m} = \sum_{j,k=0}^{\infty} \frac{(l+2j)!(m+2k)!}{l!m!j!k!} \left(B_1^{\otimes l} \otimes B_2^{\otimes m}\right)\left(\tau_{K_1}^{\otimes j} \otimes^{2j} \kappa_{l+2j,m+2k} \otimes_{2k} \tau_{K_2}^{\otimes k}\right).$$

In our setting, we observe that $S_t^Q = G^Q_{\frac{1}{2}(t^2-1)I,\frac{1}{2}(t^2-1)I;tI,tI}$.

Theorem 4 *Let $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$. Then, $(S_t^Q)^*\Xi$ is given by*

$$(S_t^Q)^*\Xi = F(t) \diamond \Gamma^Q(t)(\Xi),$$

where $F(t)$ is given by

$$F(t) = \sum_{j,k=0}^{\infty} \frac{(t^2-1)^{j+k}}{2^{j+k}j!k!} \Xi_{2j,2k}(\tau^{\otimes j} \otimes \tau^{\otimes k}). \quad (21)$$

Proof For $\Xi = \sum_{l,m=0} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and $T = \sum_{l,m=0} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{U}_\theta$, we have

$$\langle\!\langle \Xi, S_t^Q T \rangle\!\rangle$$

$$= \sum_{j,k,l,m=0}^{\infty} l!m! \Big\langle \kappa_{l,m}, (t^2-1)^{j+k} t^{l+m} \frac{(l+2j)!(m+2k)!}{2^{j+k}j!l!} \tau^{\otimes j} \otimes^{2j} \Phi_{l+2j,m+2k} \otimes_{2k} \tau^{\otimes k} \Big\rangle$$

$$= \sum_{p,q=0}^{\infty} p!q! \Big\langle \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{(t^2-1)^{j+k}}{2^{j+k}j!k!} t^{p+q-2j-2k} \tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}, \Phi_{p,q} \Big\rangle.$$

This yields

$$(S_t^Q)^*\Xi = \sum_{p,q=0}^{\infty} \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{t^{p+q-2j-2k}(t^2-1)^{j+k}}{2^{j+k}j!k!} \Xi_{p,q}(\tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}). \quad (22)$$

On the other hand, we have

$$
\begin{aligned}
&\sigma(F(t) \diamond \Gamma^Q(t)(\Xi))(\xi, \eta) \\
&\quad = \sigma(F(t))(\xi, \eta)\sigma(\Gamma^Q(t)(\Xi))(\xi, \eta) \\
&\quad = \sum_{j,k,l,m=0}^{\infty} \frac{t^{l+m}(t^2-1)^{j+k}}{2^{j+k}j!k!} \left\langle \tau^{\otimes j} \otimes \kappa_{l,m} \otimes \tau^{\otimes k}, \eta^{\otimes l+2j} \otimes \xi^{\otimes m+2k} \right\rangle \\
&\quad = \sum_{p,q=0}^{\infty} \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{t^{p+q-2j-2k}(t^2-1)^{j+k}}{2^{j+k}j!k!} \left\langle \tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}, \eta^{\otimes p} \otimes \xi^{\otimes q} \right\rangle
\end{aligned}
$$

or equivalently

$$
F(t) \diamond \Gamma^Q(t)(\Xi)
$$

$$
= \sum_{p,q=0}^{\infty} \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{t^{p+q-2j-2k}(t^2-1)^{j+k}}{2^{j+k}j!k!} \Xi_{p,q}(\tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}).
$$

Comparing with (22), the statement follows.

Remark 1 Using (19), for $x, y \in N$, we have

$$
S_t^Q(\Xi^{x,y}) = \exp\big\{\frac{1}{2}(t^2-1)\langle x, x\rangle + \frac{1}{2}(t^2-1)\langle y, y\rangle\big\}\Xi^{tx,ty}. \tag{23}
$$

Then, for all $s, t \in \mathbb{R}\backslash\{0\}$, by a density argument, one can verify that

$$
S_s^Q S_t^Q = S_{st}^Q. \tag{24}
$$

In particular, for all $s \in \mathbb{R}$ and $t \in \mathbb{R}\backslash\{0\}$, we get

$$
S_{1+\frac{s}{t}}^Q S_t^Q = S_{s+t}^Q. \tag{25}
$$

3 Euler's Theorem For Homogeneous Operator

3.1 Homogeneous Operator

Definition 1 Let $\Xi \in \mathcal{U}_\theta$ and $\lambda \in \mathbb{R}$. We say that Ξ is λ-order homogeneous if for each $t \in \mathbb{R}\backslash\{0\}$ we have

$$
S_t^Q(\Xi) = t^\lambda \Xi. \tag{26}
$$

This definition is motivated by the classical case studied in [17].

Lemma 2 *Let $l,m \geq 0$ and $\kappa_{l,m} \in (N^{\otimes l} \otimes N^{\otimes m})_{sym(l,m)}$. Then, for K_i, $B_i \in \mathcal{L}(N', N') \cap \mathcal{L}(N, N)$, $i = 1, 2$, we have*

$$
G^Q_{K_1,K_2;B_1,B_2}\Xi_{l,m}(\kappa_{l,m}) = \sum_{p=0}^{[l/2]} \sum_{q=0}^{[m/2]} \Xi_{l-2p,m-2q}(g_{l-2p,m-2q}), \tag{27}
$$

where $g_{l-2p,m-2q}$ is given by

$$
g_{l-2p,m-2q} =
$$

$$
\frac{l!m!}{(l-2p)!(m-2q)!p!q!} \left(B_1^{\otimes(l-2p)} \otimes B_2^{\otimes(m-2q)}\right) \left(\tau_{K_1}^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau_{K_2}^{\otimes q}\right) \tag{28}
$$

and τ_{K_i} is the K_i-trace defined by $\langle \tau_{K_i}, z \otimes w\rangle = \langle K_i z, w\rangle$.

Proof The operator $\Xi_{l,m}(\kappa_{l,m})$ can be rewritten as

$$\Xi_{l,m}(\kappa_{l,m}) = \sum_{\alpha,\beta=0}^{\infty} \Xi_{\alpha,\beta}(f_{\alpha,\beta}),$$

where $f_{\alpha,\beta}$ is defined by

$$f_{\alpha,\beta} = \begin{cases} \kappa_{l,m} & if \ \ (l,m) = (\alpha,\beta) \\ 0 & if \ \ (l,m) \neq (\alpha,\beta). \end{cases} \tag{29}$$

Then, by using (20), we get

$$G^{Q}_{K_1,K_2;B_1,B_2} \Xi_{l,m}(\kappa_{l,m}) = \sum_{\alpha,\beta=0}^{\infty} \Xi_{\alpha,\beta}(g_{\alpha,\beta}),$$

with

$$g_{\alpha,\beta} = \sum_{j,k=0}^{\infty} \frac{(\alpha+2j)!(\beta+2k)!}{\alpha!\beta!j!k!} \left(B_1^{\otimes\alpha} \otimes B_2^{\otimes\beta}\right) \left(\tau_{K_1}^{\otimes j} \otimes^{2j} f_{\alpha+2j,\beta+2k} \otimes_{2k} \tau_{K_2}^{\otimes k}\right).$$

From (29), we observe that $g_{\alpha,\beta} = 0$ for $\alpha > l$ or $\beta > m$. Thus, we obtain

$$G^{Q}_{K_1,K_2;B_1,B_2} \Xi = \sum_{0\le\alpha\le l} \sum_{0\le\beta\le m} \Xi_{\alpha,\beta}(g_{\alpha,\beta}),$$

with

$$g_{\alpha,\beta} = \sum_{j,k=0}^{\infty} \sum_{2j=l-\alpha} \sum_{2k=m-\beta} \frac{l!m!}{\alpha!\beta!j!k!} \left(B_1^{\otimes\alpha} \otimes B_2^{\otimes\beta}\right) \left(\tau_{K_1}^{\otimes j} \otimes^{2j} \kappa_{l,m} \otimes_{2k} \tau_{K_2}^{\otimes k}\right).$$

Moreover, when $l-\alpha = 2p+1$ or $m-\beta = 2q+1$, we have $g_{\alpha,\beta} = 0$. The case $l-\alpha = 2p$ and $m-\beta = 2q$ gives

$$g_{\alpha,\beta} = \frac{l!m!}{\alpha!\beta!p!q!} \left(B_1^{\otimes\alpha} \otimes B_2^{\otimes\beta}\right) \left(\tau_{K_1}^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau_{K_2}^{\otimes q}\right).$$

Replacing α by $l-2p$ and β by $m-2q$, we get the desired statement.

The following theorem gives the Fock expansion of the $\lambda-$order homogeneous operator in $\mathcal{U}_\theta$.

Theorem 5 *Let $\lambda \in \mathbb{N}$ and $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$. Then, Ξ is λ-order homogeneous if and only if*

$$\Xi = \sum_{l=0}^{\lambda} \sum_{p=0}^{[l/2]} \sum_{q=0}^{[\frac{\lambda-l}{2}]} \int_{\mathbb{R}^{\lambda-2p-2q}} \Upsilon_{l-2p,\lambda-l-2q}(s_1,\cdots,s_{l-2p},t_1,\cdots,t_{\lambda-l-2q})$$
$$a^*_{s_1}\cdots a^*_{s_{l-2p}} a_{t_1}\cdots a_{t_{\lambda-l-2q}} ds_1\cdots ds_{l-2p} dt_1\cdots dt_{\lambda-l-2q},$$

where $\Upsilon_{l-2p,\lambda-l-2q}$ is given by

$$\Upsilon_{l-2p,\lambda-l-2q} =$$
$$\sum_{j,k=0}^{\infty} \frac{(l+2j)!(\lambda-l+2k)!(-1)^{j+k}}{j!k!p!q!(l-2p)!(\lambda-l-2q)!2^{j+k}2^{p+q}} \left(\tau^{\otimes(j+p)} \otimes^{2(j+p)} \kappa_{l+2j,\lambda-l+2k} \otimes_{2(k+q)} \tau^{\otimes(k+q)}\right).$$

Proof In the following we set

$$\mathcal{G}^Q := G^Q_{-\frac{1}{2}I,-\frac{1}{2}I;-iI,-iI}.$$

Motivated by the classical case (see [16]), we can show that $\mathcal{G}^Q$ is a topological isomorphism from $\mathcal{U}_\theta$ into itself. Moreover,

$$(\mathcal{G}^Q)^{-1}\Xi = G^Q_{-\frac{1}{2}I,-\frac{1}{2}I;iI,iI}\Xi. \tag{30}$$

For any $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$ and $t \in \mathbb{R}$, the technical identity

$$S_t^Q(\Xi) = (\mathcal{G}^Q)^{-1}\Gamma^Q(t)\mathcal{G}^Q(\Xi) \tag{31}$$

holds true. Indeed, by direct computation, we have

$$\mathcal{G}^Q\Xi^{x,y} = \exp\big\{-\frac{1}{2}\langle x,x\rangle - \frac{1}{2}\langle y,y\rangle\big\}\Xi^{-ix,-iy}, \quad x,y \in N.$$

Therefore,

$$\Gamma^Q(t)\mathcal{G}^Q(\Xi^{x,y}) = \exp\big\{-\frac{1}{2}\langle x,x\rangle - \frac{1}{2}\langle y,y\rangle\big\}\Xi^{-itx,-ity}.$$

Then, we obtain

$$(\mathcal{G}^Q)^{-1}\Gamma^Q(t)\mathcal{G}^Q(\Xi^{x,y}) = \exp\big\{\frac{1}{2}(t^2-1)(\langle x,x\rangle + \langle y,y\rangle)\big\}\Xi^{tx,ty}.$$

Hence, by (23) we deduce that

$$S_t^Q(\Xi^{x,y}) = (\mathcal{G}^Q)^{-1}\Gamma^Q(t)\mathcal{G}^Q(\Xi^{x,y}),$$

which proves (31) by density argument.

In view of (31), equation (26) can be rewritten as follows

$$\Gamma^Q(t)\mathcal{G}^Q(\Xi) = t^\lambda \mathcal{G}^Q(\Xi). \tag{32}$$

Let $T = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{U}_\theta$ be the unique Fock expansion of the operator $T = \mathcal{G}^Q(\Xi)$, where $\Phi_{l,m}$ is given by

$$\Phi_{l,m} = \sum_{j,k=0}^{\infty} \frac{(l+2j)!(m+2k)!(-i)^{l+m}(-1)^{j+k}}{l!m!j!k!2^{j+k}} \left(\tau^{\otimes j} \otimes^{2j} \Phi_{l+2j,m+2k} \otimes_{2k} \tau^{\otimes k}\right).$$

Then (32) can be rewritten as

$$\Gamma^Q(t)T = t^\lambda T, \tag{33}$$

or equivalently

$$\sum_{l,m=0}^{\infty} t^{l+m}\Xi_{l,m}(\Phi_{l,m}) = \sum_{l,m=0}^{\infty} t^\lambda \Xi_{l,m}(\Phi_{l,m}).$$

From the uniqueness of the Fock expansion, this last equation is satisfied if and only if $\lambda = l + m$. Then, T satisfies (33) if and only if

$$T = \sum_{l=0}^{\lambda} \varXi_{l,\lambda-l}(\varPhi_{l,\lambda-l}).$$

Therefore, by Eq. (27), we obtain

$$\begin{aligned}\varXi &= (\mathcal{G}^Q)^{-1}T \\ &= \sum_{l=0}^{\lambda} G^Q_{-\frac{1}{2}I,-\frac{1}{2}I;iI,iI}(\varXi_{l,\lambda-l}(\varPhi_{l,\lambda-l})) \\ &= \sum_{l=0}^{\lambda}\sum_{p=0}^{[l/2]}\sum_{q=0}^{[\frac{\lambda-l}{2}]} \varXi_{l-2p,\lambda-l-2q}(\varUpsilon_{l-2p,\lambda-l-2q})\end{aligned}$$

where $\varUpsilon_{l-2p,\lambda-l-2q}$ is given by

$$\varUpsilon_{l-2p,\lambda-l-2q} = \frac{l!(\lambda-l)!i^{\lambda}}{(l-2p)!(\lambda-l-2p)!p!q!2^{p+q}}\left(\tau^{\otimes p}\otimes^{2p}\varPhi_{l,\lambda-l}\otimes_{2q}\tau^{\otimes_q}\right)$$

and

$$\varPhi_{l,\lambda-l} = \sum_{j,k=0}^{\infty}\frac{(l+2j)!(\lambda-l+2k)!(-1)^{\lambda}(-1)^{j+k}}{l!(\lambda-l)!j!k!2^{j+k}}\left(\tau^{\otimes j}\otimes^{2j}\kappa_{l+2j,\lambda-l+2k}\otimes_{2k}\tau^{\otimes k}\right).$$

Then, we get

$$\begin{aligned}\varUpsilon_{l-2p,\lambda-l-2q} = &\sum_{j,k=0}^{\infty}\frac{(l+2j)!(\lambda-l+2k)!(-1)^{j+k}}{j!k!p!q!(l-2p)!(\lambda-l-2q)!2^{j+k}2^{p+q}} \\ &\times\left(\tau^{\otimes(j+p)}\otimes^{2(j+p)}\kappa_{l+2j,\lambda-l+2k}\otimes_{2(k+q)}\tau^{\otimes(k+p)}\right)\end{aligned}$$

as desired.

Example 1 (The 1-order homogeneous operators). For $z, w \in N$, the operator

$$\varXi = a^*(z) + a(w)$$

is a 1-order homogeneous operator. In particular, for $z = w$, the multiplication operator $\varXi = M_{\langle\cdot,z\rangle}$ is 1-order homogeneous.

Example 2 (The 2-order homogeneous operators). For $\kappa_{0,2}$, $\kappa_{2,0}$, $\kappa_{1,1} \in N \otimes N$, the operators

$$\begin{aligned}&\varXi_{2,0}(\kappa_{2,0}) + \varXi_{0,0}(\langle\tau,\kappa_{2,0}\rangle),\\ &\varXi_{0,2}(\kappa_{0,2}) + \varXi_{0,0}(\langle\tau,\kappa_{0,2}\rangle),\\ &\varXi_{1,1}(\kappa_{1,1})\end{aligned}$$

are 2-order homogeneous. Note that if we take $\kappa_{0,2} = \kappa_{2,0} = \tau_K$ for $K \in \mathcal{L}(N', N)$ such that $\langle\tau,\tau_K\rangle = 0$, then the $K-$Gross Laplacian $\varDelta_G(K) = \varXi_{2,0}(\tau_K)$ and its dual $\varDelta_G^*(K) = \varXi_{0,2}(\tau_K)$ are 2-order homogeneous operators. Moreover, for $B \in \mathcal{L}(N', N)$, the conservation operator $N(B) = \varXi_{1,1}(\tau_B)$ is 2-order homogeneous operator.

Remark 2 Let $\lambda \in \mathbb{N}$. Then, using Theorem 3, $\Xi \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ is λ-order homogeneous if and only if

$$\Xi = \Xi_{-\tau} \diamond \Xi_h$$

where Ξ_h is λ-order homogeneous in $\mathcal{U}_\theta$.

3.2 Euler's Theorem For Homogeneous Operator

From Ref. [6], the QWN-Euler operator can be represented, via Theorem 3, as a continuous linear operator on $\mathcal{U}_\theta$ by

$$\Delta_E^Q := \Delta_G^Q + N^Q = \sum_{j=1}^{\infty} M^{Q+}_{\langle \cdot, e_j \rangle} D^+_{e_j} + \sum_{j=1}^{\infty} M^{Q-}_{\langle \cdot, e_j \rangle} D^-_{e_j},$$

where for $z \in N'$,

$$M^{Q-}_{\langle \cdot, z \rangle} = \sigma^{-1}(M_{\langle \cdot, z \rangle} \otimes I)\sigma, \quad M^{Q+}_{\langle \cdot, z \rangle} = \sigma^{-1}(I \otimes M_{\langle \cdot, z \rangle})\sigma,$$

and $M_{\langle \cdot, z \rangle}$ is the multiplication operator by $\langle \cdot, z \rangle$, see [18].

Theorem 6 *Let $\Xi \in \mathcal{U}_\theta$ and $t \in \mathbb{R} \backslash \{0\}$. Then for each $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$*

$$\lim_{s \to 0} \langle\!\langle T, \frac{S_{t+s}^Q \Xi - S_t^Q \Xi}{s} \rangle\!\rangle = \frac{1}{t} \langle\!\langle T, \Delta_E^Q S_t^Q \Xi \rangle\!\rangle.$$

Proof By Theorem 4 and Eq. (25) we have

$$\begin{aligned}
\lim_{s \to 0} \langle\!\langle T, \frac{S_{t+s}^Q \Xi - S_t^Q \Xi}{s} \rangle\!\rangle &= \lim_{s \to 0} \langle\!\langle \frac{1}{s} \left\{ F(1 + \frac{s}{t}) \diamond \Gamma^Q(1 + \frac{s}{t})(T) - T \right\}, S_t^Q \Xi \rangle\!\rangle \\
&= \lim_{s \to 0} \langle\!\langle F(1 + \frac{s}{t}) \diamond \frac{1}{s} (\Gamma^Q(1 + \frac{s}{t})(T) - T) \\
&\quad + \frac{1}{s}(F(1 + \frac{s}{t}) - I) \diamond T, S_t^Q \Xi \rangle\!\rangle.
\end{aligned}$$

Since, we have

$$\lim_{s \to 0} \left\{ \frac{(1 + \frac{s}{t})^{l+m} - 1}{s} \right\} = \frac{d}{ds} \left(1 + \frac{s}{t}\right)^{l+m} \Big|_{s=0} = \frac{1}{t}(l + m),$$

for $T = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m})$, we get

$$\lim_{s \to 0} \frac{\Gamma^Q(1 + \frac{s}{t})(T) - T}{s} = \sum_{l,m=0}^{\infty} \frac{1}{t}(l + m) \Xi_{l,m}(\Phi_{l,m}) = \frac{1}{t} N^Q T. \tag{34}$$

On the other hand using (21), we have

$$\lim_{s \to 0} \sigma(\frac{1}{s}\{F(1 + \frac{s}{t}) - I\})(\xi, \eta)$$

$$
\begin{aligned}
&= \lim_{s\to 0}\frac{1}{s}\left\{\exp\left[\Big(\frac{(1+\frac{s}{t})^2-1}{2}\Big)\big(\langle\xi,\xi\rangle+\langle\eta,\eta\rangle\big)\right]-1\right\}\\
&= \frac{1}{t}\big(\langle\xi,\xi\rangle+\langle\eta,\eta\rangle\big)\\
&= \frac{1}{t}\sigma\big(\Xi_{0,2}(\tau)+\Xi_{2,0}(\tau)\big)(\xi,\eta).
\end{aligned}
$$

Then, from Eqs. (14) and (15) we get

$$
\begin{aligned}
\lim_{s\to 0}\left\langle\!\left\langle T,\frac{S^Q_{t+s}\Xi-S^Q_t\Xi}{s}\right\rangle\!\right\rangle &= \left\langle\!\left\langle \frac{1}{t}N^QT+\frac{1}{t}(\Delta^Q_G)^*T,S^Q_t\Xi\right\rangle\!\right\rangle\\
&= \frac{1}{t}\langle\!\langle T,\Delta^Q_E S^Q_t\Xi\rangle\!\rangle.
\end{aligned}
$$

Which gives the desired statement.

Remark 3 Using Theorem 6, for $\Xi\in\mathcal{U}_\theta$, we have

$$
\lim_{s\to 0}\frac{S^Q_{e^s}\Xi-\Xi}{s}=\lim_{s\to 0}\left(\frac{e^s-1}{s}\right)\frac{S^Q_{(e^s-1+1)}\Xi-\Xi}{e^s-1}=\Delta^Q_E\Xi.
$$

This shows that $\{S^Q_{e^t}\}$ is a semigroup on $\mathcal{U}_\theta$ with infinitesimal generator Δ^Q_E. Hence, we deduce that $S^Q_{e^t}U_0$ is the unique solution of the Cauchy problem

$$
\frac{\partial}{\partial t}U_t=\Delta^Q_E U_t,\quad U_0\in\mathcal{U}_\theta.
$$

Theorem 7 *(Euler's theorem). Let $\Xi\in\mathcal{U}_\theta$. Then Ξ is $\lambda-$order homogeneous if and only if it satisfies the following* `QWN`*-Euler equation*

$$
\Delta^Q_E\Xi=\lambda\Xi\cdot \tag{35}
$$

Proof If Ξ is λ-order homogeneous, then by (26) we have

$$
\Delta^Q_E\Xi=\lim_{t\to 1}\frac{S^Q_t(\Xi)-S^Q_1(\Xi)}{t-1}=\lim_{t\to 1}\frac{t^\lambda-1}{t-1}\Xi=\lambda\Xi.
$$

Conversely, suppose that (35) is satisfied. Let $t\in\mathbb{R}\backslash\{0\}$. Put $G(t)=t^{-\lambda}S^Q_t(\Xi)$. Then, by (35) and Theorem 6 we get

$$
\begin{aligned}
\lim_{s\to 0}\frac{G(t+s)-G(t)}{s} &= \lim_{s\to 0}\frac{1}{s}\Big\{(t+s)^{-\lambda}S^Q_{t+s}(\Xi)-t^{-\lambda}S^Q_t(\Xi)\Big\}\\
&= \lim_{s\to 0}\frac{1}{s}\Big\{(t+s)^{-\lambda}(S^Q_{t+s}(\Xi)-S^Q_t(\Xi))\Big\}\\
&+ \lim_{s\to 0}\frac{1}{s}\Big\{(t+s)^{-\lambda}-t^{-\lambda}\Big\}S^Q_t(\Xi)\\
&= t^{-(\lambda+1)}\Delta^Q_E(S^Q_t(\Xi))-\lambda t^{-(\lambda+1)}S^Q_t(\Xi).
\end{aligned}\tag{36}
$$

Now, let $T\in\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}^*_\theta)$. Then, by Theorem 4 and Theorem 6, we have

$$
\begin{aligned}
\langle\!\langle T,S^Q_t(\Delta^Q_E\Xi)\rangle\!\rangle &= \langle\!\langle F(t)\diamond\Gamma^Q(t)(T),\Delta^Q_E\Xi\rangle\!\rangle\\
&= \lim_{s\to 0}\left\langle\!\left\langle F(t)\diamond\Gamma^Q(t)(T),\frac{S^Q_{1+s}\Xi-\Xi}{s}\right\rangle\!\right\rangle\\
&= \lim_{s\to 0}\left\langle\!\left\langle T,\frac{S^Q_tS^Q_{1+s}\Xi-S^Q_t\Xi}{s}\right\rangle\!\right\rangle.
\end{aligned}
$$

But, by applying the QWN-Scaling transformation on $\Xi^{x,y}$, for $x, y \in N$, we can show that $S_u^Q S_v^Q = S_v^Q S_u^Q$ for all $u, v \in \mathbb{R}$. Then we get

$$\langle\!\langle T, S_t^Q(\Delta_E^Q \Xi)\rangle\!\rangle = \lim_{s\to 0} \langle\!\langle T, \frac{S_{1+s}^Q S_t^Q \Xi - S_t^Q \Xi}{s}\rangle\!\rangle.$$

Hence, using Theorem 6 , we obtain

$$\langle\!\langle T, S_t^Q(\Delta_E^Q \Xi)\rangle\!\rangle = \langle\!\langle T, \Delta_E^Q(S_t^Q \Xi)\rangle\!\rangle,$$

from which we deduce that

$$S_t^Q(\Delta_E^Q \Xi) = \Delta_E^Q(S_t^Q \Xi).$$

Therefore, using (35) we get

$$\Delta_E^Q(S_t^Q(\Xi)) = \lambda S_t^Q(\Xi).$$

Thus, from (36) we deduce that $G'(t) = 0$ for all $t \in \mathbb{R}\backslash\{0\}$. In particular, $G(t) = G(1)$, i.e.,

$$t^{-\lambda} S_t^Q(\Xi) = S_1^Q(\Xi) = \Xi\cdot$$

From which we deduce the desired statement.

Remark 4 Euler's theorem remains valid in $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ where Δ_E^Q is replaced by $\widetilde{\Delta}_E^Q$ acting on $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ as follows

$$\widetilde{\Delta}_E^Q(\Xi) = \Xi_{-\tau} \diamond \Delta_E^Q(\Xi_\tau \diamond \Xi).$$

Corollary 1 *Let $\lambda \in \mathbb{N}$ and $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$ such that $\Delta_G^Q(\Xi) = 0$. Then Ξ is $\lambda-$order homogeneous if and only if $\Xi = \sum_{l=0}^{\lambda} \Xi_{l,\lambda-l}(\kappa_{l,\lambda-l})$.*

Proposition 3 *Let $\Xi \in \mathcal{U}_\theta$ be a $\lambda-$order homogeneous operator such that $\Delta_G^Q(\Xi) = 0$. Then for each $\xi \in N$, $D_\xi^\pm(\Xi)$ are $(\lambda-1)-$order homogeneous and $(D_\xi^\pm)^*(\Xi)$ are $(\lambda+1)-$order homogeneous.*

Proof We recall from [6] that, for any $\xi \in N$, the following identities hold true

$$D_\xi^+ \Xi_{l,m}(\kappa_{l,m}) = l\Xi_{l-1,m}(\xi \otimes^1 \kappa_{l,m})$$

$$D_\xi^- \Xi_{l,m}(\kappa_{l,m}) = m\Xi_{l,m-1}(\kappa_{l,m} \otimes_1 \xi)$$

$$(D_\xi^+)^* \Xi_{l,m}(\kappa_{l,m}) = \Xi_{l+1,m}(\xi \otimes \kappa_{l,m})$$

$$(D_\xi^-)^* \Xi_{l,m}(\kappa_{l,m}) = \Xi_{l,m+1}(\kappa_{l,m} \otimes \xi).$$

Then, if $\Xi = \sum_{l=0}^{\lambda} \Xi_{l,\lambda-l}(\kappa_{l,\lambda-l})$, we have

$$D_\xi^+ \Xi = \sum_{l=0}^{\lambda-1} (l+1)\Xi_{l,\lambda-l-1}(\xi \otimes^1 \kappa_{l+1,\lambda-1-l}). \tag{37}$$

Thus, the fact $\Delta_G^Q(D_\xi^+ \Xi) = D_\xi^+(\Delta_G^Q \Xi) = 0$ and identity (37) proves the statement for $D_\xi^+(\Xi)$ via Corollary 1. The others statements can be verified by slight modification.

Theorem 8 *Let $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$. Then $\Xi_{l,m}(\kappa_{l,m})$ is $(l+m)-$order homogeneous if and only if $\Delta_G^Q(\Xi_{l,m}(\kappa_{l,m})) = 0$.*

Proof From (16) we have

$$\Delta_G^Q \Xi_{l,m}(\kappa_{l,m}) = l(l-1)\Xi_{l-2,m}(\tau \otimes^2 \kappa_{l,m}) + m(m-1)\Xi_{l,m-2}(\kappa_{l,m} \otimes_2 \tau). \quad (38)$$

Then by iterating (38) we get

$$(\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) =$$

$$\sum_{p=0}^{[l/2]} \sum_{p+q=k}^{[m/2]} \frac{l!m!k!}{(l-2p)!(m-2q)!p!q!} \Xi_{l-2p,m-2q}(\tau^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau^{\otimes q}). \quad (39)$$

On the other hand from (27), we obtain

$$S_t^Q \Xi_{l,m}(\kappa_{l,m}) = \sum_{p=0}^{[l/2]} \sum_{q=0}^{[m/2]} \frac{l!m!}{(l-2p)!(m-2q)!p!q!} t^{l+m-2p-2q} \big(\frac{1}{2}(t^2-1)\big)^{p+q} \times \Xi_{l-2p,m-2q}(\tau^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau^{\otimes q}). \quad (40)$$

Then in view of (39), (40) becomes

$$S_t^Q \Xi_{l,m}(\kappa_{l,m})$$

$$= \sum_{k=0}^{[(l+m)/2]} \frac{1}{k!} t^{l+m-2k} \big(\frac{1}{2}(t^2-1)\big)^k (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m})$$
$$= t^{l+m} \Xi_{l,m}(\kappa_{l,m}) + \sum_{k=1}^{[(l+m)/2]} \frac{1}{k!} t^{l+m-2k} \big(\frac{1}{2}(t^2-1)\big)^k (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}).$$

It then follows that $\Xi_{l,m}(\kappa_{l,m})$ is $(l+m)-$homogeneous if and only if

$$\sum_{k=1}^{[(l+m)/2]} \frac{1}{k!} t^{l+m-2k} \big(\frac{1}{2}(t^2-1)\big)^k (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) = 0. \quad (41)$$

Hence, using the fact that $\{P_k(X) = X^{l+m-2k}(X^2-1)^k;\ k = 1, 2, \cdots, [(l+m)/2]\}$ is a linearly independent family of polynomials, one can show that (41) holds if and only if

$$(\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) = 0, \quad \forall k = 1, 2, \cdots, [(l+m)/2].$$

This implies in particular that $\Delta_G^Q \Xi_{l,m}(\kappa_{l,m}) = 0$. The converse is straightforward by Euler's theorem.

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