Scheduling with an outsourcing option on both manufacturer and subcontractors

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A R T I C L E   I N F O

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Abstract

This paper addresses a single-stage scheduling problem with outsourcing allowed where each job can be either scheduled for in-house production or outsourced to one of the outside subcontractors available. The manufacturer has an unrelated parallel machine system, and each subcontractor has its own single machine. Subcontractors are capable to process all the jobs. Unlike most of past research, our study considers the joint scheduling of both in-house and outsourced jobs simultaneously. The objective is to minimize sum of the total weighted completion time and total outsourcing cost. An integer programming formulation is presented and then improved through an optimality property on subproblems and solve them to optimality.

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1. Introduction

Outsourcing is commonly required as a way to improve overall scheduling performance in various companies, especially, including electronics industries and motor industries [15]. The outsourcing of non-critical activities to subcontractors allows firms to focus more on high value operations. Nowadays, many companies outsource their jobs to a third party instead of managing them directly. Manufacturers often face with situations where demands requested from customers are too much to be taken care by internal capacity. In such situations, they transfer all or part of their activities to an external service provider so as to handle demand fluctuations without the need to store more inventory or design for high production capacity. A proper plan for outsourcing can improve lead times, reduce total costs, and make a company more competitive. As a result, recent interest has been developed towards finding an appropriate outsourcing policy that enables companies to compete with others, and help them emerge as a market winner [6]. While a manufacturer can benefit from outsourcing, the potential maximum benefit cannot be achieved unless there is an efficient production plan that can cope with the complexity of outsourcing [24]. To achieve this benefit, management needs to decide what quantities of each product to be manufactured and what quantities to be outsourced to external subcontractors. For this purpose, a joint scheme between production and outsourcing plans is necessary in an efficient scheduling scheme. The complexity of this problem is that we have to decide on scheduling plans for manufacturer and subcontractors, as well as their coordination. In the last three decades, the production planning problem known as aggregate production planning (APP) has been widely analyzed jointly with outsourcing scenarios. However, there are few studies on production scheduling with outsourcing decisions. The APP involves the problem of determining production quantity and outsourcing quantity over multiple time periods where the objective is often to minimize total costs including production cost, inventory cost, and outsourcing cost [2,10,12]. In APP, the quantity of outsourcing is determined in each time period and the scheduling of individual jobs is not decided, whereas in joint scheduling and outsourcing models it should be determined which jobs to be outsourced and how to schedule the jobs.

In this paper, we investigate the problem of single-stage scheduling with an option of outsourcing with the objective of minimizing sum of total weighted completion time and total outsourcing cost. The manufacturer receives a set of jobs at the beginning of planning horizon; each of which needs one stage processing. A set of decisions should be made simultaneously: which set of jobs should be outsourced, which subcontractor is appropriate for outsourcing and which sequence of jobs is more beneficial. Mathematically, this problem can be stated as follows. There are a set of n independent jobs \( N = 1, 2, \ldots, n \), \( M_1 \) unrelated in-house machines \( M_1 = 1, 2, \ldots, m_1 \), and \( M_2 \) outside subcontractors \( M_2 = 1, 2, \ldots, m_2 \) and each subcontractor has a single machine. We show the set of total machines by \( M = M_1 \cup M_2 \). All machines in \( M \) are available in the whole planning horizon without breakdown. A job can be processed on only one machine, and a machine can process at most one job at a time.
ease of presentation, we denote this class of scheduling problems by $P$. In the problem $P$, each job $j \in N$ has $|M|$ processing times $p_{jk}(k \in M)$, a weight factor $w_j$, and an outsourcing cost $\beta_j^h$ for machine $k \in M_h$. All the jobs in $N$ are available at time zero. Similar to the classic scheduling problem, pre-emption of jobs is prohibited and cancelation is not allowed. Since machines are unrelated, the processing times $p_{jk}$, for all $j \in N$ and $k \in M$, are arbitrary and dissimilar on different machines. Let $C_j$ indicates the completion time of job $j$ in a feasible schedule $S$. We define $S_R$ as outsourced jobs set in schedule $S$, and $O_j$ as the outsourcing cost of job $j \in S_R$ after its processing machine is determined. Then the total weighted completion time is $\sum_{j \in N} w_j C_j$, and total outsourcing cost $OC$ is $\sum_{j \in S_R} \beta_j^h$. Without the loss of generality, it is assumed that parameters $p_{jk}$ and $\beta_j^h$ are integer values for all $j \in N$, $k \in M$ ($p_{jk}, \beta_j^h \in Z^+$). Using the conventional three-field classification notation for scheduling problems presented by Lawler et al. [13], we denote problem $P$ by $R\mid \sum w_j C_j + OC$. Since the problem $R\mid \sum w_j C_j + OC$ is a generalization of classical unrelated parallel machine scheduling, it can be concluded that this problem belongs to the class of NP-hard.

The remainder of this paper is organized as follows. Section 2 discusses the related literature. The integer programming formulation of the problem is given in Section 3. A Lagrangian relaxation scheme is presented in Section 4, and then Section 5 describes a dynamic programming algorithm for solving subproblems. Section 6 gives details of the proposed Lagrangian heuristic for solving dual problem, and Section 7 analyzes the performance of solution procedure via computational experiments. In Section 8, we conclude the paper.

2. Literature review

In general, the problem of scheduling with outsourcing option is a new research direction and a few researchers have studied this problem in the literature. The main part of these researches focused on single-stage environment. For example, Ruiz-Torres et al. [25] presented a bi-objective model to deal with the problem of finding outsourcing strategies and considered trade-offs between the following measures: (i) average tardiness $T = \sum_{h \in M} T_h$, in which $T_h$ is the tardiness of jobs on machine $h$ and $M$ is the machines set, and (ii) total outsourcing cost $OC = \sum_{k \in K} \phi(MC_k)$ in which $K$ is a subset of machines set $M$, $MC_k$ is completion of last job on machine $h$ and $\phi(x)$ represents the cost function of $x$. A single machine with $n$ one-operation jobs are available; each one can be completed by either in-house machine or $m$ identical outsourcing machines. A heuristic algorithm is introduced in their work and evaluated by a lower bounding procedure. Moreover, Lee and Sung [15] suggested a single-machine scheduling problem with outsourcing allowed. They considered minimizing weighted sum of the outsourcing cost $\sum_{j \in O_0} o_j$, in which $O_0$ is the set of outsourced jobs in schedule $\pi$, and sum of completion times $\sum C_j$, subject to an outsourcing budget $OC(\pi) \leq K$. In the analysis of the problem which is denoted by $B|\sum(1-\delta)C_j+\delta \cdot OC$, some problem properties were characterized and also two associated heuristics and a branch and bound were derived. A similar research on single-machine scheduling has been presented by Lee and Sung [16] in which the scheduling measure is one of the following: maximum lateness, total tardiness, and total completion time. The authors showed that the problem is NP-hard in the ordinary sense. As another example of single-stage problem, Qi [22] dealt with a general outsourcing model with transportation considerations. It was assumed that all outsourced jobs have to be transported back to the company in batches. The authors discussed four commonly used objective functions, i.e., (i) total completion time $\sum C_j$, (ii) makespan $C_m = \max \{\sum_{j \in S} C_j\}$. (iii) maximum lateness $L_m = \max \{\sum_{j \in S} C_j - dj\}$, and (iv) total number of tardy jobs $\sum U_j$ where $C_j$ and $d_j$ represent the completion time and due date of job $j$, and $U_j$ is a binary. It indicates 1 if job $j$ is late ($C_j > d_j$) and 0 otherwise ($C_j \leq d_j$). Optimal algorithms for all variants of the problem were also developed. Besides, Chen and Li [7] dealt with a model with make-to-order and identical parallel machines subject to a constraint on delivery performance. Each job $j$ is processed by manufacturer’s own plant with processing time $p_0$ and processing cost $q_0$ dollars. In addition to in-house production, outsourcing of job $j$ to subcontractor $h$ is possible with delivery time $p_d$ and processing cost of $q_d$ dollars. The objective is to minimize total sum of production cost and subcontracting cost $\sum \sum q_{0j} x_{ij} + \sum \sum q_{hj} y_{ij}$ in which $x_{ij}$ is a 0–1 variable indicating whether job $j$ is assigned to machine $i$. Also $y_{ij}$ is a 0–1 variable indicating whether job $j$ is assigned to subcontractor $s$, and $V$ is the set of jobs that can be subcontracted. The problem is shown to be NP-hard even when there is a single machine at the manufacturer plant and there is a single-remote subcontractor. The computational results revealed that a significant reduction in the production costs is achieved when the subcontracting option is used.

Additionally, some of the researchers were dealt with the two-stage environment for the problem of production scheduling with outsourcing option. Cai et al. [5] investigated a manufacturing model that involves two machines, with one owned by manufacturer and the other by third party, and all the outsourced jobs have to be transported back from the third-party after completion. In order to minimize the total cost, including lateness cost, cost to use third-party machine, and overtime cost, the authors developed an integer nonlinear program, obtained some optimality properties and designed an algorithm to find optimal solution. In another study, Lee and Choi [18] considered a two-stage production problem in which each job requires two operations denoted by $(1, j)$ and $(2, j)$ to be processed on machines 1 and 2, and there is one option of outsourcing. The objective is to minimize sum of makespan $C_m (\pi)$ associated with sequence $\sigma$ and total outsourcing cost $\alpha p_0 \sum x_{ij} + \sum x_{ij} y_{ij}$ where $x_{ij}$ represents unit outsourcing cost for operations at machine $i$ and $p_0$ is the processing time of operation $(i, j)$. In their work, the computational complexity was discussed for all cases and an approximation algorithm was developed for NP-hard cases. Besides, a study on two-stage outsourcing model of production scheduling was presented by Qi [23] in which each job needs two sequential operations. In their model, there exist two in-house production facilities where each one can process both operations, and also another option of outsourcing stage-one operations to a remote outside supplier. Outsourcing cost is denoted by $x x \delta$ where $x$ indicates the difference between in-house and outsourcing unit costs, and $x$ represents total outsourcing time. The batch transportation and transportation delay $\tau$ is also considered and makespan criterion was minimized. Recently, Qi [24] presented an outsourcing model for scheduling problem in a two-stage flow shop and analyzed different scenarios for outsourcing. He assumed that the outsourcing cost is proportional to the job processing time with a constant coefficient.

Research on this problem in multi-stage production environment is very limited. In this class, Lee et al. [17] studied an advanced planning and scheduling problem in a supply chain which can be modeled as a flow shop and devised a genetic algorithm to solve the problem. In another study, Chung et al. [9] studied a multi-stage job shop scheduling where an operation of jobs can be performed either on an in-house machine or on an external machine with an extra cost. They presented a decomposition-based algorithm that converts the original problem into smaller subproblems, and also improved the efficiency of their heuristic by executing operations sequencing and picking
out steps, repeatedly. Besides, Mokhtari et al. [19] investigated the problem of flow shop scheduling under four outsourcing scenarios and due date constraints, and then developed a heuristic for solving the problem in different scenarios. More recently, Tavares Neto and Godinho Filho [27] dealt with the permutation flow shop problem with the objective of minimizing weighted sum of makespan and outsourcing costs under budget constraint \( F_{prmu} \), Budget \((1 - \delta)M + \delta \cdot OC\), and designed two sequential ant colony optimization algorithms to solve the problem.

One major drawback in almost all previous studies is that they limit the number of subcontractors available to just one company. However, in practice, there exist often several suppliers to choose for each job. Using several suppliers makes the problem more realistic and, at the same time, more complicated to formulate and solve, because doing so, a supplier selection decision is embedded into the traditional problem. The authors could find very few papers like [7,19] that considered several available subcontractors to be selected for subcontracting. Another drawback in some previous studies like [15,16] is that they assume unlimited capacity for subcontractors and hence ignore scheduling of outsourced orders which should be considered in parallel with in-house scheduling. It is also recognized that the problem of scheduling with outsourcing has not been treated in unrelated parallel machine environment yet. As a remedy, the current paper supports several available suppliers to be selected for subcontracting and also considers scheduling of outsourced jobs as well as in-house in an unrelated parallel machine system. This problem has not been considered in the literature yet. In sequel, details of formulation and solving approach will be presented.

\[ x_{0j} \] a binary variable indicating whether job \( j \) is processed first on machine \( k \)

\[ x_{j,n+1} \] a binary variable indicating whether job \( j \) is processed last on machine \( k \)

In terms of the above notations, the problem can be formulated as follows:

\[
(\text{Problem } \mathbf{P}) \quad \min \sum_{j \in M} \sum_{i \in N} \left( w_j C_{jk} + p_{jk} \right) x_{ij} \sum_{i \in N \cup \{0\}} x_{ij}
\]

subject to

\[
\sum_{k \in M} \sum_{i \in N \cup \{0\}} x_{0j} = 1, \quad \forall j \in N
\]

\[
\sum_{j \in N} x_{ij} \leq 1, \quad \forall k \in M
\]

\[
\sum_{i \in N \cup \{0\}} x_{ij} = \sum_{i \in N \cup \{n+1\}} x_{ij}, \quad \forall j \in N, \quad k \in M
\]

\[
C_{jk} = p_{jk} x_{0j} + \sum_{i \in N} \left( C_{ik} + p_{jk} \right) x_{ij}, \quad \forall j \in N, \quad k \in M
\]

\[
x_{0j} \in \{0,1\}, \quad \forall i, j \in N, \quad k \in M
\]

The objective (1) seeks to minimize sum of total weighted completion time and total processing cost. Constraint (2) guarantees that each job is processed exactly once, and constraint (3) ensures that at most one job has the starting position on each machine. Constraint (4) which has the same role as flow conservation condition in network flow problems ensures that the jobs are correctly sequenced within a partial schedule on machine \( k \in M \) and also assures that at most one job has the finishing position on each machine. Constraint (5) defines the completion time \( C_{ik} \) considering the relationship between two adjacent jobs. The last constraint (6) ensures binary nature of decision variables.

In the sequel, we are going to improve the above formulation by taking into account a job ordering restriction on machines. The given formulation can be improved, if we consider those schedules where the partial schedule on each machine must follow any ordering pattern within jobs. In our case, the well-known Smith’s rule [26] is applicable. This rule states that in any optimal schedule of problem \( R | \sum w_j C_j \), jobs within each partial schedule follow the weighted shortest processing time (WSPT) order. So it is needed only to consider those schedules where the jobs on each machine follow the WSPT rule. That means if jobs \( i \) and \( j \) are both processed on a certain machine \( k \), and \( p_{ik}/w_i \leq p_{jk}/w_j \), then we only need to consider partial schedules on machine \( k \) where job \( i \) precedes job \( j \). Let \( A_i^k \) is defined as a set of jobs \( i \) in the sequence of WSPT on machine \( k \) where job \( j \) precedes job \( i \), and \( B_j^k \) as a set of jobs \( j \) in the sequence of WSPT on machine \( k \) where job \( j \) succeeds job \( i \). By use of WSPT as a problem-specific job-ordering restriction, we have the following new formulation for problem \( \mathbf{P} \)

3. Problem formulation

In this section, we aim at formulating the problem considered in current paper as an integer program. To this end, a general kind of problem is considered at first step where we have just one set of machines such that any job \( j \) can be processed by any machine \( k \) for a processing time \( p_{jk} \) and a processing cost \( p_{jk} \). A special case of this problem would be when some of the machines are in-house and others are out of house as in our problem. In such a case, the processing costs for in-house jobs are set to be zero and those of outsourced jobs can be interpreted as outsourcing cost charged by subcontractors. The following notations are used in the formulation of the general problem. Sets, indices and parameters

\( n \) number of jobs

\( m \) total number of machines

\( m_1 \) number of in-house machines

\( m_2 \) number of subcontractors

\( i, j, l \) job indices

\( k \) machine indices

\( w_j \) weight of job \( j \)

\( p_{jk} \) processing time of job \( j \) on machine \( k \)

\( q_{jk} \) processing cost of job \( j \) on machine \( k \)

\( N \) set of jobs, \( N = 1, 2, \ldots, n \)

\( M \) set of total machines, \( M = 1, 2, \ldots, m_1 + m_2 \)

\( M_1 \) set of in-house machines, \( M_1 = 1, 2, \ldots, m_1 \)

\( M_2 \) set of outsourcing machines, \( M_2 = 1, 2, \ldots, m_2 \)

Variables

\( C_{jk} \) completion time of job \( j \) on machine \( k \)

\( x_{ij} \) a binary variable indicating whether job \( i \) is processed immediately before job \( j \) on machine \( k \)
denoted by \( P \)

\[
\text{(Problem } P \text{)} \quad \min \sum_{j \in N} \sum_{k \in M} \left( w_{jk} C_{jk} + \gamma_{jk} \sum_{i \in B^k_j, j} x_{ij} \right)
\]

subject to

\[
\sum_{k \in M} \sum_{i \in B^k_j, j} x_{ij} = 1, \quad \forall j \in N
\]

\[
\sum_{i \in N} \sum_{k \in M} \sum_{j \in B^k_j} x_{ij} \leq 1, \quad \forall k \in M
\]

\[
C_{jk} = p_{jk} x_{jk} + \sum_{i \in N} \left( C_{ik} + p_{ik} \right) x_{ij}, \quad \forall j \in N, \quad k \in M
\]

\[
x_{ij} \in \{0, 1\}, \quad \forall i, j \in N, \quad k \in M
\]

As can be seen, the integer programming formulation (7)–(12) is a 
ika larger size unrelated parallel machine scheduling problem with \( n \) one-operation jobs and \( m_1 + m_2 \) machines. However, the structure of objective function is a special case we cannot find in any classical problem of parallel machine scheduling. In the sequel, we exploit a special structure of problem \( P \) for designing a decomposition scheme to solve it.

4. A decomposition scheme for problem \( P \)

As described before, the problem \( P \) includes a special kind of 
ika machine scheduling problem with unrelated parallel machines. Hence we can conclude that this problem belongs to the class of NP-hard. In addition, it involves a nonlinear term in Eq. (11) which makes it hard to be solved by traditional methods. We can also observe that this problem is composed of similar subsystems, represented by \( m_1 + m_2 \) machines. As an interesting fact, there are some independent blocks linked by a coupling constraint in the technological coefficient matrix. We can exploit this observation, and apply a Lagrangian relaxation (LR) approach to decompose the problem into smaller and easier subproblems and solve the problem via standard LR technique. In fact, such an idea has been employed in several past publications for designing efficient solution procedure for different scheduling problems [28,8,14]. The idea of LR is within this fact that many hard problems can be viewed as smaller and easier problems complicated by a set of coupling constraints. Dualizing the coupling constraints generates a Lagrangian problem which is easier to solve and whose optimal solution gives a lower bound of the original problem. An upper bound is also obtainable from a Lagrangian heuristic where the solution of relaxed problem is modified. The interested reader is referred to Geoffrion [11] and Bazaraa and Goode [3] for detailed discussion on the LR technique.

We consider Lagrangian relaxation of the problem \( P \) by dualizing constraint (8) that couples different machines. Using multiplier \( \lambda_j \) for all \( j \in N \), the LR problem is constructed as follows:

\[
\text{LR:} \quad L(\lambda) = \min Z_{LR}
\]

subject to

\[
Z_{LR} = \sum_{j \in N} \sum_{k \in M} \left( w_{jk} C_{jk} + \gamma_{jk} \sum_{i \in B^k_j, j} x_{ij} \right) + \sum_{j \in N} \lambda_j \left( \sum_{k \in M} \sum_{j \in B^k_j} x_{ij} - 1 \right)
\]

\[
= \sum_{j \in N} \lambda_j + \sum_{j \in N} \sum_{k \in M} \left( w_{jk} C_{jk} + \left( \gamma_{jk} + \lambda_j \right) \sum_{i \in B^k_j, j} x_{ij} \right)
\]

Eqs. (9)–(12) and \( \lambda_j \) is unrestricted in \( \text{sign} \lambda_j \in N \) (14)

Problem LR can be decomposed into \(|M| \) single machine subproblems (one for each machine) as follows:

\[
\text{LR}_k: \quad L^k(\lambda) = \min Z^k_{LR}
\]

subject to

\[
Z^k_{LR} = \sum_{j \in N} \left( w_{jk} C_{jk} + \left( \gamma_{jk} + \lambda_j \right) \sum_{i \in B^k_j, j} x_{ij} \right),
\]

Eqs. (9)–(12), and (14).

These subproblems can be solved independently and then their solutions are utilized to obtain the solution of Lagrangian problem LR whose objective function value is calculated by \( Z_{LR} = \sum_{k \in M} Z^k_{LR} - \sum_{j \in N} \lambda_j \). In subsequent section, we will describe a dynamic programming algorithm for solving the constructed subproblems.

5. Dynamic programming for solving subproblems

Dynamic programming (DP) is a useful general optimization technique for making a sequence of dependent decisions. In general, when performance measure of a single machine problem has an additive form, as in our subproblems, we can find an optimal solution through a dynamic programming approach [1]. In first stage of DP, it starts with a small part of the whole problem and finds the optimal solution for that small part. It then slightly enlarges this small part and finds the optimal solution of the new problem using the preceding one. This procedure is repeated until the current problem encompasses the original problem, and the problem solves optimally at the last stage of algorithm. Then, tracking back from original problem to the smaller parts, DP determines the optimal variables.

Here, we aim at designing a forward DP algorithm for solving single-machine subproblems presented in Eq. (15). Let us first give a brief description of these subproblems. We have a set of independent jobs \( N = 1, 2, \ldots, n \) with a processing time \( p_{jk} \) and a weight factor \( w_{jk} \) both of which are positive integers numbers. The only decision is to find a subset of jobs \( A \subseteq N \) in such a way that the total cost value \( \sum_{j \in A} \left( w_{jk} C_{jk} + \lambda_j \gamma_{jk} \right) \) is minimized. We construct the DP algorithm incorporating with WSPT rule. By using this rule, as a job-ordering restriction in our problem, only jobs in \( B^k_j \) can be sequenced before job \( j \) on machine \( k \). So we first re-number the jobs in such a way that \( p_{jk} / w_{jk} \leq p_{jk} / w_{jk} \leq \ldots \leq p_{jk} / w_{jk} \). Let \( R_k = 1, 2, \ldots, N \) denotes the new order of jobs on machine \( k \), and \( H = 1, 2, \ldots, l \) denote a subset of jobs \( (H \subseteq R_k) \) which follows the WSPT order. At each stage \( l \in \{1, 2, \ldots, n\} \), it is decided whether job \( l \) is included in partial schedule or not. It is also assumed that \( G(L_t) \) represents the minimum cost of all partial schedules that consist of jobs from the set \( H \) and their last jobs are completed at time \( t \). Algorithm starts with the first job where \( G(L_t) \) is initially set to \( 0 \) for \( t \geq 0, l = 0 \), and \( \infty \) for \( t < 0, l = 0, \ldots, n \). In next stages, the minimum objective value of partial schedules is calculated via the recursive relation \( G(L_t) = \min \left( G(L_{t-1} - p_{lk}) + w_{lt} + \lambda_j \gamma_{jk} \right) \), \( G(L_{t-1}) \) for all \( l = 1, 2, \ldots, n \) and \( t = 0, \ldots, p \). Note that \( p \) represents the planning horizon on machine \( k \), and is equal to \( \sum_{j \in N} p_{jk} \). If the partial schedule consists of job \( l \), we add it to the best solution for the first \( l-1 \) jobs that complete at time \( t - p_{lk} \), the value of this solution is calculated by \( G(L_{t-1} - p_{lk}) + w_{lt} + \lambda_j \gamma_{jk} \). If job \( l \) is not included in the partial schedule, we then select the best schedule with respect to the
first \(l-1\) jobs that complete at time \(t\); the value of this solution is calculated by \(G(l-1,t)\). After executing all stages, current subproblem can be solved by computing minimum value of \(G(l,t)\) for all \(t=0, \ldots, P_k\) in last stage.

**Example.** Assume that a manufacturer receives three jobs (1,2,3) with weight factor \(w=(2,2,1)\), processing time \(p=(2,1,1)\) and processing cost \(\gamma=(5,10,5)\) on a certain in-house machine \(k\). The Lagrangian multiplier is given as \(\lambda=(-7.5,-9)\). The aim of single machine subproblem \(LR(k)\) is to find the best set of jobs from (1,2,3) with the minimum objective value \(Z_{LR}^k\). According to the WSPT rule, the new order of jobs is \(R_k=(2,1,3)\). After calculating initial values of DP algorithm, the recursive relation is used to obtain the objective value \(G(l,t)\) for all \(l=1,2,3,4\) as follows:

First stage on job 2:

\[
\begin{align*}
G(1,0) &= \min G(0,0-1)+2 \times 0+(5)+10, \quad G(0,0) = 0 \\
G(1,1) &= \min G(0,1-1)+2 \times 1+(5)+10, \quad G(0,1) = 0 \\
G(1,2) &= \min G(0,2-1)+2 \times 2+(5)+10, \quad G(0,2) = 0 \\
G(1,3) &= \min G(0,3-1)+2 \times 3+(5)+10, \quad G(0,3) = 0 \\
G(1,4) &= \min G(0,4-1)+2 \times 4+(5)+10, \quad G(0,4) = 0 
\end{align*}
\]

Second stage on job 1:

\[
\begin{align*}
G(2,0) &= \min G(1,0-2)+2 \times 0+(7)+5, \quad G(1,0) = 0 \\
G(2,1) &= \min G(1,1-2)+2 \times 1+(7)+5, \quad G(1,1) = 0 \\
G(2,2) &= \min G(1,2-2)+2 \times 2+(7)+5, \quad G(1,2) = 0 \\
G(2,3) &= \min G(1,3-2)+2 \times 3+(7)+5, \quad G(1,3) = 0 \\
G(2,4) &= \min G(1,4-2)+2 \times 4+(7)+5, \quad G(1,4) = 0 
\end{align*}
\]

Third stage on job 3:

\[
\begin{align*}
G(3,0) &= \min G(2,0-1)+1 \times 0+(9)+5, \quad G(2,0) = 0 \\
G(3,1) &= \min G(2,1-1)+1 \times 1+(9)+5, \quad G(2,1) = -3 \\
G(3,2) &= \min G(2,2-1)+1 \times 2+(9)+5, \quad G(2,2) = -2 \\
G(3,3) &= \min G(2,3-1)+1 \times 3+(9)+5, \quad G(2,3) = -1 \\
G(3,4) &= \min G(2,4-1)+1 \times 4+(9)+5, \quad G(2,4) = 0 
\end{align*}
\]

According to above calculations, the best set of jobs should be scheduled on machine \(k\) is \(H=\{3\}\). The value of this solution is \(-3\). Similar approach can be employed for all other in-house and outsource machines.

6. A Lagrangian heuristic procedure

In this section, we are going to present a Lagrangian heuristic to solve the original problem. The Lagrangian dual (LD) is formulated as follows:

\[
\text{(LD) } \max_{x} L(\lambda) = \min Z_{LR}
\]

subject to

\[
Z_{LR} = \sum_{k \in M} Z_{LR}^k - \sum_{j \in H} \lambda_j \tag{19}
\]

\([9)-(12), \text{ and } (14)\]

The solution of dual problem (19) provides a lower bound for the original problem. The general steps of the Lagrangian heuristic for scheduling problems are introduced by Nishi et al. [20] as follows.

**Step 0:** Initialization. Set the number of iterations, and Lagrange multipliers.

**Step 1:** Solving relaxed problem. Each subproblem is solved to optimality, and the lower bound of original problem is calculated.

**Step 2:** Construction of a feasible solution. Generate a feasible solution from the solution of relaxed problem by using a heuristic, and calculate the upper bound.

**Step 3:** Evaluation of convergence. If the GAP = \(UB - LB(\lambda)\) is equal to zero (or less than a specific threshold), or lower bound has not been updated for a pre-known number of iterations, the algorithm terminates.

**Step 4:** Solving dual problem. Update the Lagrangian multipliers, and then return to Step 1.

A crucial step of LR approach is to optimize the dual function, and the success of it depends greatly on the ability to generate better Lagrangian multipliers. The subgradient method is commonly utilized to achieve a good value for multipliers. In this method, multiplier is updated according to

\[
x_{h+1} = x_h + s_h \frac{g(x_h)}{\|g(x_h)\|^2}, \quad 0 < \alpha < 2
\]

in which \(L^*\) represents the optimal value estimated by the best feasible solution achieved so far and \(L^h\) is the value of \(L\) at the \(h\)th iteration, and \(g(x)\) is the subgradient of Lagrangian function at \(x^h\) which is equal to \(\sum_{k \in M} \sum_{i \in \phi_k} \sigma_k \phi_i x_{ji} \). The variables \(x_j\) are calculated by

\[
x_j = \arg \min \left( \sum_{k \in M} Z_{LR}^k \right) \text{ subject to } (9)-(12) \text{ and } (14). \tag{22}
\]

The initial value of \(x\) is usually set to be 0.5. When the duality gap GAP is less than a predefined small number or a fixed CPU time is reached, the algorithm terminates. According to (21) and (22), the subgradient method needs to solve all subproblems at each iteration. However, the solution of Lagrangian dual problem obtained in each iteration of subgradient method is not necessarily identical to the solution of the original problem which is due to the existence of duality gaps in scheduling problems [20]. Thus the dual solution is generally associated with an infeasible schedule and some of relaxed constraints may be violated. In this regard, there are two set of jobs that may make our problem infeasible. The first set \(UF_1\) includes jobs that may be repeated in more than one machine, and the second set \(UF_2\) is related to jobs which are not chosen in any partial schedule at all. To modify these solutions, we present a Lagrangian heuristic approach to construct a feasible solution from the solution of relaxed problem.

In this algorithm, the jobs which have been scheduled only once remain fixed form both assigned machine and starting time. By the end of this stage, we may have a schedule that involves machine idle time. Let \(V_j\) represents the set of all jobs assigned to the machine \(k\) at this moment, and \(\mu_j^k\) be the value of \(p_j^k/w_j\) for all unselected jobs \(j \in UF_2\) and \(k \in M\). Under Smith’s rule, all jobs should be scheduled in WSPT order, and therefore, the potential position of unselected job \(j\) on machine \(k\) can be obtained by comparing \(\mu_j^k\) and \(p_j^k/w_j\) for all
jobs \(i\) in \(Y_k\). Doing so, the potential position of all unselected jobs on every machine is determined, under Smith's rule. It is clear that assigning an unselected job to a machine under WSPT rule may need shifting the subsequent jobs to the right, and this will incur additional cost in terms of total weighted completion time. Hence, the unselected job should be assigned to the machine \(k\) where the total additional cost incurred by shifted jobs plus the weighted completion time of inserted job is minimized. This procedure is implemented for all jobs in \(U_F\) in ascending order of their processing times hoping that the gaps of idle time are filled as much as possible without further shifting.

At the end of this stage, all the jobs are contained in the schedule once and thus the feasibility of solution is assured. However, we may still have a schedule with machine idle time, at this moment.

**Definition 1.** A performance measure \(Z\) is regular if (a) the scheduling objective is to minimize \(Z\), and (b) \(Z\) can increase only if at least one of the completion times in the schedule increases \([1]\).

It can be shown that in an optimal solution of a scheduling problem with regular measure, as in our problem, there is no machine idle time. By using this rule, we simply shift all the jobs in the current schedule to the left in such a way that the machine idle time is equal to zero, and there is no overlap between successive jobs, to obtain a perfect schedule. The value of this schedule gives an upper bound for the original problem.

### 7. Computational experiments

This section describes the computational experiments which were designed to evaluate the performance of the proposed solution procedure. First of all, the processing cost \(\frac{y_{ji}}{x_{ji}}\) is set to be zero for in-house jobs, and be outsourcing cost for outsourced jobs in order to more concentrate on original problem (scheduling with outsourcing option). In order to capture a wide range of problem structures, 20 problem sets were considered and 10 problems were randomly generated in each set, resulting in a total of 200 test problems used in experiments. The statistical values of results are computed in order to enhance the confidence level of the performance of the solution procedure. Data required for our problem consists of number of jobs, number of in-house machines, number of subcontractors, processing times, and outsourcing cost. The number of jobs, number of in-house machines, and number of subcontractors vary from 20 to 100, 4 to 10, and 2 to 5, respectively. The processing times and job weights were generated from a discrete uniform distribution over \((1,10)\) and \((1,10)\), respectively. It is also considered that the outsourcing cost is drawn from a uniform distribution between \((1,100)\) and \((1,110)\), respectively. It is also considered that the outsourcing cost is drawn from a uniform distribution between \((1,10)\) and \((1,100)\), respectively. It is also considered that the outsourcing cost is drawn from a uniform distribution between \((1,110)\), \((1,120)\), and \((1,110)\), respectively.

The results of the test experiments are shown in Table 1. These results are described by providing number of jobs \(N\), number of in-house machines \(M_1\), number of outsourcing machines \(M_2\), and average percentage gap (gap %) between the value of obtained feasible solution and the lower bound. The gap % which is used to evaluate the quality of the solutions is defined as \(\text{gap} \% = 100 \times (\text{value of obtained feasible solution} – \text{lower bound})/\text{lower bound}\). The average CPU time is also reported in seconds. In all experiments, the algorithm terminates when a predefined number of iterations \((300)\) or a value of average percentage gap \((0.5\%)\) is met. The results reported in Table 1 show that the solution procedure does well for a wide range of problem sizes, with an average gap of 2.85% over all 200 test problems. The number of jobs has a significant effect on the computational time of the solution procedure. It also seems that, for problems with a given number of machines, the gaps approximately increase when the number of jobs increases.

<table>
<thead>
<tr>
<th>ID</th>
<th>(N)</th>
<th>(M_1)</th>
<th>(M_2)</th>
<th>L-gap</th>
<th>CPU (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>6</td>
<td>5</td>
<td>5.23</td>
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</tr>
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<td>30</td>
<td>6</td>
<td>5</td>
<td>8.32</td>
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<td>60</td>
<td>8</td>
<td>5</td>
<td>10.95</td>
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</tr>
<tr>
<td>4</td>
<td>80</td>
<td>8</td>
<td>5</td>
<td>11.21</td>
<td>13.20</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>10</td>
<td>5</td>
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<td>14.63</td>
</tr>
<tr>
<td>Average</td>
<td>10.034</td>
<td>20.104</td>
<td>279.69</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Additionally, we compared the tightness of the lower bound obtained by the Lagrangian relaxation (LR) presented in current paper with the lower bound obtained by the commonly used linear programming relaxation (LP-relaxation). To this end, we designed additional computational experiments on some set of problems with \(N = \{20,30,60,80,100\}\). The LP-relaxation lower bounds are generated using the well-known commercial solver LINGO 8.0. The comparisons were carried out through lower bound gap (L-gap), which is defined \(L\)-gap \(100 \times (LR \text{ bound} – LP \text{ bound})/LP \text{ bound}\). The average values of results are shown in Table 2. As obvious, the lower bound gap is on average 10.034% which shows the higher performance of Lagrangian approach with respect to the LP-relaxation bound. It is also revealed that the computational time required by LR procedure is averagely less than the CPU time required to obtain the LP bound. Since the resulting LP-relaxation problem is still a nonlinear program, the LP-relaxation procedure is more time consuming than our LR approach as shown in Table 2.

Furthermore, additional experiments were carried out to investigate the quality of the feasible solution (upper bound) obtained by the Lagrangian heuristic proposed in our paper. For this purpose, the optimal/best solution of original problem obtained by LINGO is recorded. The average value of comparison results is reported in Table 3 in terms of quality of solutions and CPU time. The quality of solutions is measured by \(Q\)-gap = \(00 \times (LR \text{ solution value} – LINGO \text{ value}/LINGO \text{ value})\). The results show that our Lagrangian procedure is capable of finding good
feasible solutions compared to the optimal/best solutions generated by LINGO solver. As shown in Table 3, LINGO failed to generate the optimal solution for three of the six problem sets after running for 12 h. It is also shown that the average CPU time required by LINGO is significantly more than that of our LR heuristic.

In addition to above comparisons, a new set of experiments were conducted to analyze the sensitivity of results to variability in \( \gamma^j_k, p_{jk} \) and \( w_j \). In order to create each class of problems, the value of some of the input data (\( \gamma^j_k, p_{jk}, w_j \)) should be changed. The pattern of changes should be associated with a valid range of parameters \( \gamma^j_k, p_{jk}, w_j \). For this purpose, we are going to discuss an industrial interpretation of parameters \( w_j \) and \( \gamma^j_k \) which are appeared in the objective function \( \sum_{j=1}^{N} \sum_{k=1}^{2} w_{ij} C_{jk} + \sum_{j=1}^{N} \sum_{k=2}^{2} \frac{\gamma^j_k}{2} \sum_{i=1}^{2} x_{ij} \) which the weight \( w_j \) may be interpreted as a holding cost of job \( j \) per unit time [21]. In such a case, the objective function is the sum of holding and processing cost. It is assumed that the processing cost of jobs charged by the subcontractors is drawn from the discrete uniform distribution over interval \([20,100]\), and holding cost \( w_j \) is calculated by \( \gamma_j \), where \( \gamma = [0.2,1,2] \) is a transformation factor and \( \gamma_j = \sum_{i=1}^{2} \frac{\gamma_i}{\gamma} \) is mean outsourcing cost. These values of \( \gamma \) represent three scenarios described below.

**Scenario I.** The holding cost of a job is two times the processing cost of the job charged by subcontractors \((\gamma = 2)\). This scenario can be occurred when we have hazardous goods such as explosives, flammable, poison, corrosive, radioactive substances. These materials must be packaged, labeled, handled and transported according to strict regulations from several agencies. Mitigating the risks associated with hazardous materials may require the application of safety precautions during their transport and storage. All these circumstances lead to a greater holding cost than processing cost.

**Scenario II.** The holding cost of a job is same as the processing cost of the job charged by subcontractors \((\gamma = 1)\). This scenario can be occurred when we have sensitive goods which can be categorized into three classes: (i) temperature sensitive goods, (ii) stress sensitive goods, and (iii) light sensitive goods. The pharmaceuticals, foodstuffs and chemicals are some examples of sensitive goods. Holding costs of sensitive goods are generally less than those of hazardous goods and hence holding cost of a sensitive good can be assumed to be same as the production cost of that good.

**Scenario III.** The holding cost of a job is 0.2 times the production cost of the job charged by subcontractors \((\gamma = 0.2)\). This scenario can be occurred when we have regular goods. The good that is not included in hazardous and sensitive categories, called regular good.
We carried out analysis of sensitivity on a sample problem with $n=20$, $m_1=4$ and $m_2=2$ for three aforementioned scenarios separately. Tables 4–6 show the results for scenarios I, II and III. As the results show, the optimal objective value of all the three scenarios is approximately increased when processing time $p_{jk}$ or processing cost $g_{jk}$ increase. As another observation, all the Gap indices, i.e., duality gap, L-gap and Q-gap, increase when the processing times increase. At a given level of processing time, duality gap, L-gap and Q-gap decrease from scenarios I to III. These observations are also depicted in Figs. 1–3. Note that the average duality gap, L-gap and Q-gap at a given processing time and scenario over all processing cost $g_{jk}$.

Fig. 1. The effect of processing time on average duality gap in scenarios I–III.

Fig. 2. The effect of processing time on average L-gap in scenarios I–III.

Fig. 3. The effect of processing time on average Q-gap in scenarios I–III.

8. Conclusion

Because of increasing importance of on-time delivery of jobs to the customers, this paper suggests a single-stage production scheduling model with an option of outsourcing. Unlike most of past research, our model allows for multiple available subcontractors and also supports joint decision on scheduling of in-house and outsourced jobs on manufacturer and subcontractors simultaneously. We designed an integer programming model and also presented a dynamic programming based Lagrangian relaxation procedure for solving the problem. Results of extensive computational experiments revealed high performance of the proposed approach for a wide range of problem sizes.

References


