A Branch and Bound approach for truss topology design problems with valid inequalities

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Abstract. One of the classical problems in the structural optimization field is the Truss Topology Design Problem (TTDP) which deals with the selection of optimal configuration for structural systems for applications in mechanical, civil, aerospace engineering, among others. In this paper we consider a TTDP where the goal is to find the stiffest truss, under a given load and with a bound on the total volume. The design variables are the cross-section areas of the truss bars that must be chosen from a given finite set. This results in a large-scale non-convex problem with discrete variables. This problem can be formulated as a Semidefinite Programming Problem (SDP problem) with binary variables. We propose a branch and bound algorithm to solve this problem. In this paper it is considered a binary formulation of the problem, to take advantage of its structure, which admits a Knapsack problem as subproblem. Thus, trying to improve the performance of the Branch and Bound, at each step, some valid inequalities for the Knapsack problem are included.

Keywords: truss topology design, stiffness, semidefinite programming, global optimization, branch and bound, valid inequalities
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INTRODUCTION

Structural optimization problems have received an increasing attention during last years, see [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 15]. We consider the TTDP of finding the stiffest truss under a given load with a bound on the total volume, where the cross-section areas of bars can assume a finite set of possible values. The model is similar to the one given in [11]. There the authors discuss branch and bound methods to the discrete formulation. Here we propose a binary reformulation in order to take advantage of its structure, which admits a Knapsack problem as subproblem. In order to improve the lower bound, valid inequalities obtained from separation of the Knapsack subproblem are added [14].

PROBLEM FORMULATION

This section begins with a brief introduction to the basic engineering concepts and rules typically considered in the design of truss structures. Afterwards, the formulation of the problem is presented.

A truss is a two ($2D$) or three-dimensional ($3D$) structure composed of bars whose ends are connected at joints, called nodes, which may be fixed, free or simply supported. In this work only bars with constant cross section and rectilinear axis are considered. Each node has associated Degrees-of-Freedom (DoF). In this study, only two-dimensional trusses are considered. For $2D$ structures, a fixed node has 0 DoF, a free node has 2 DoF (one associated to each independent direction in the plane, X and Y) and a supported node has just 1 DoF in the direction perpendicular to the support. The total number of DoF of a truss, denoted by $NDoF$, is the sum of the corresponding DoF of each node. In this analysis, it is considered that each bar i has elastic properties, assuming a constant Young's modulus E_i [13].

Initially, it is given a ground structure, that is a grid with a previously chosen set of nodes and connecting bars, called "potential bars". If all possible links between the nodes are considered, then it is called a dense topology; otherwise, if only links between neighbor nodes are considered, it is called a poor topology. In the next is presented the formulation of the problem, considering a ground planar structure with k nodes, $NDoF = n$, m potential bars and an external load vector, $f \in \mathrm{IR}^n$, acting directly on the nodes. The goal is to find the stiffest truss capable of withstand the loading with a total volume not greater than a predefined value, $v > 0$. The volume of the truss has to be distributed among the bars in order to get the stiffest structure. The design variables are the cross-section areas, A_i, of the bars whose values are chosen from a given set, $\mathscr{C} = \{0, c_1, \ldots, c_l\}$. When the external loads are acting on the nodes the structure deforms. That deformation can be described by the vector of nodal displacements, $\delta \in \mathrm{IR}^n$, being the work associated to the external loads given by $f^\top \delta$. The value of $\frac{1}{2} f^\top \delta$ is called compliance. This is a measure of the stiffness of the truss: the smaller the compliance the larger the stiffness of the truss with respect to the loading. The truss should be able to withstand the external loads that are assured by the equilibrium equations system: $K(A)\ \delta = f$, where $K(A) = \sum_{i=1}^m A_i l_i K_i$ is the $n \times n$ stiffness matrix of the structure, l_i is the length of bar i and $K_i = b_i b_i^\top$ is the stiffness matrix of bar i, being $b_i = \frac{\sqrt{E_i}}{l_i} d_i$ and d_i the vector of direction cosines of the bar i. The problem can be formulated as follows [1, 2, 3, 4, 6] [1]:

$$(TTD) \qquad \begin{aligned} & \min\ f^\top \delta \\ & \text{s.t.}\ \sum_{i=1}^m A_i l_i K_i\ \delta = f, \\ & \qquad \sum_{i=1}^m A_i l_i \leq v, \\ & \qquad A_i \in \{0, c_1, \ldots c_l\}. \end{aligned} \tag{1}$$

This is a large-scale non-convex problem which is equivalent to a min-max problem $\min_{A \in \mathscr{A}} \max_{u \in \mathrm{IR}^n} \{2 f^\top u - u^\top K(A) u\}$, where $\mathscr{A} = \{\ A \in \mathrm{IR}^m\ :\ \sum_{i=1}^m A_i l_i \leq v,\ A_i \in \{0, c_1, \ldots c_l\}\ \}$ [4, 12]. This problem can be written as a SDP problem,

$$\begin{aligned} & \min_{\tau, a}\ \tau \\ & \text{s.t.} \begin{bmatrix} \tau & f^\top \\ f & K(A) \end{bmatrix} \succeq 0, \\ & \quad A \in \mathscr{A}, \end{aligned} \tag{2}$$

which can be formulated as a problem with binary variables. In fact,

$$A_i \in \{0, c_1, c_2, \ldots, c_l\} \Leftrightarrow \begin{cases} A_i = \alpha_{i_1} c_1 + \alpha_{i_2} c_2 + \ldots \alpha_{i_l} c_l \\ \alpha_{i_1} + \alpha_{i_2} + \ldots \alpha_{i_l} \leq 1 \\ \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_l} \in \{0, 1\} \end{cases} \tag{3}$$

and so,

$$(TTD01) \qquad \begin{aligned} & \min_{\alpha, \tau}\ \tau \\ & \text{s.t.} \begin{bmatrix} \tau & f^\top \\ f & \sum_{i=1}^m \sum_{j=1}^l \alpha_{j_i} c_j s_i K_i \end{bmatrix} \succeq 0, \\ & \sum_{i=1}^m \sum_{j=1}^l \alpha_{i_j} c_j s_i \leq v, \\ & \sum_{j=1}^l \alpha_{i_j} \leq 1,\ i = 1, \ldots, m, \\ & \alpha_{i_j} \in \{0, 1\},\ i = 1, \ldots, m,\ j = 1, \ldots, l. \end{aligned} \tag{4}$$

BRANCH AND BOUND ALGORITHM

In this section a branch and bound algorithm to solve the problem $(TTD01)$ is described. The main idea of a branch and bound algorithm is to build an enumeration tree search to find an optimal solution and provide its optimality by successive partitioning of the feasible set.

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[1] To simplify, in the objective function, it was considered twice the compliance.

Let $\mathscr{S}$ be the feasible set of the problem $(TTD01)$ and $\mathscr{S}_t$ be the subset of $\mathscr{S}$ corresponding to the feasible set at node t of the tree search. Associated to the enumeration tree we define: a list $\mathscr{L}$ of active subproblems and the global upper bound, $ub = \min_{t \in \mathscr{L}} ub_t$, where ub_t is the value of $(TTD01)$ in the feasible solution found at node t. Next, we will briefly present the details of the branch and bound method:

- For the node selection it was used the depth first search rule.
- At each node t, the **lower bound**, lb_t, is obtained by solving the relaxation of the problem (R_t). In order to improve the lower bound, cover and lifted cover inequalities obtained from separation of the Knapsack subproblem are added [14].
- In order to improve the **upper bound**, ub, at each node of the tree, three heuristics are applied, based on a rounding scheme of the optimal solution of (R_t) to find feasible solutions for problems $(TTD01)$.
- To choose the branching variable, several options are considered, depending on the optimal solution of the relaxation. The feasible set, $\mathscr{S}_t$, is then partitioned in two subsets: $\mathscr{S}_{t1} = \mathscr{S}_t \cap \{\alpha_{i_j} = 0\}$ and $\mathscr{S}_{t2} = \mathscr{S}_t \cap \{\alpha_{i_j} = 1\}$.
- A node is pruned if one of the following conditions holds: (R_t) is infeasible; the optimal value of (R_t), lb_t, is greater or equal than ub; the optimal solution of (R_t) satisfies the discrete constraints.

At each step, before solving the relaxation, a **volume test** is performed in order to prune nodes: If the branching variable is A_i, $A_i = \alpha_{i_1} c_1 + \alpha_{i_2} c_2 + \ldots \alpha_{i_l} c_l$ with $c_1 < \cdots < c_l$, first α_{i_1} is fixed to one. If the remaining volume for the truss does not allow $\alpha_{i_1} = 1$ then the node is pruned and the other variables $\alpha_{i_2}, \ldots \alpha_{i_l}$ are automatically equal to zero.

COMPUTATIONAL RESULTS AND CONCLUSIONS

In this section, some computational experience is reported. The branch and bound method was implemented in C-language. To solve the semidefinite programming problems it was used the CSDP4.9 package [9] and to solve the linear programs it was used GLPK4.8 [10].

For all the considered trusses, it is known the type of the ground structure ($\mathscr{N}_x \times \mathscr{N}_y$-truss): height, width, geometry (regular or irregular), the nodes in the horizontal direction, $\mathscr{N}_x$, and the nodes in the vertical direction, $\mathscr{N}_y$; the maximum volume of the structure, v; the set $\mathscr{C}$ of possible values for the cross-section areas; the nodes restrictions (free, fixed or simple supported); the external loads.

It was considered that, for all bar i, $E_i = 69GPa$, the Young's modulus of the aluminium. For each example it is presented the optimal design, the nodal displacement vector, δ, and the stress at each member i, given by $\sigma_i = \frac{E_i}{l_i} d_i \delta$. For the structure safety verification the stress at each element cannot exceed the aluminium yielding strength which is assumed to be $250MPa$.

Next, results are presented for a 2×6-truss corresponding to a $400cm \times 2000cm$ grid with $v = 140000cm^3$ and $\mathscr{C} = \{0, 3, 6, 9, 12, 15\}$ (in cm^2). The nodes are numbered as shown in the Figure 1. There is one fixed node, node 1, and one supported node, node 11, with a restriction in the vertical displacement. In the ground structure, $m = 26$ and $NDoF = 21$. There are 6 external vertical loads applied at the superior nodes as represented in Figure 1. The values of the loads applied at nodes 2 and 12 are $50kN$ and the values of the loads applied at nodes 4, 6, 8 and 10 are $100kN$. Figure 1 represents the optimal solution obtained with the Branch and Bound method, where the objective function is $o.f. = 2576.7884$ and the volume in the final structure is $v = 139094cm^3$. The design is symmetric, as it was expected, because the topology of the ground structure is symmetric as well as the loading. The thickness of the bars reflects the values of its cross-section, the majority of the bars have a cross-section of $15cm^2$, five bars have inferior cross-section (3, 6 or $9cm^2$), and 3 bars have cross-section zero.

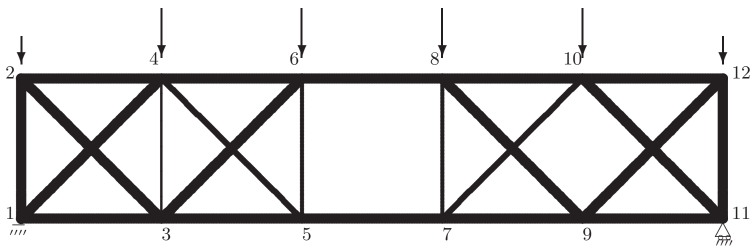

FIGURE 1. Optimal solution by Branch and Bound method ($o.f. = 2576.7884$ $v = 139094.0$)

The stress in the members are between -114MPa and 200MPa. In the design and safety assessment of this kind of structures, as in bridges, the deformations are controlled in order to guarantee the serviceability limit states. The maximum displacement calculated for the example 1 was 15cm, at structure mid-span, was less than 1% of the span, which is a value in the order of the magnitude of the limits imposed for these structures.

If we increase the maximum total volume value to $v = 180000cm^3$, maintaining set $\mathscr{C}$, all the bars in the optimal solution have cross-section area equal to the maximum, i.e. $15cm^2$. For this case, the stress in members are between -161MPa and 205MPa. The largest nodal displacements is 7.33cm for the vertical direction and, for the horizontal direction is 3.45cm. If we consider $v = 180000cm^3$ and set $\mathscr{C} = \{0,4,8,12,14,18\}$, the nodal displacements decrease. The largest nodal displacement is now 7.33cm in the vertical direction and 3.45cm for the horizontal direction. In this case, the stress in members are between -166MPa and 166MPa.

In this paper is proposed a branch and bound algorithm with valid inequalities to solve the discrete truss topology design problem (TTD). This formulation has a rich mathematical structure that enables to find an equivalent formulation with less variables. Nevertheless, from the practical viewpoint, this approach leads to designs which should serve as "reference designs" rather than to readily implementable solutions for construction. There are constraints, such as bounds on the nodal displacements and on stresses, that should be considered. The inclusion of this constraints leads to a large-scale problem hardly tractable. In the examples presented are verified the nodal displacements values as well as the stress in the members.

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