Iterative methods for simultaneous computing arbitrary number of multiple zeros of nonlinear equations

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Abstract

In this work we present two modifications of an earlier presented iterative method for solving nonlinear equations in the case of multiple zeros. The new methods are two-point, derivative free iterative methods for the simultaneous extraction of one or several of the multiple zeros. It is proved that the proposed methods possess quadratic convergence locally. Numerical examples are given to illustrate the efficiency and performance of the methods presented.

Key words: Nonlinear equations, Root-finding methods, Simultaneous methods, Order of convergence

1 Introduction

We consider the problem of solving the nonlinear equation

\[ f(x) = 0, \]

where \( f : \mathcal{D} \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a nonlinear, continuous function on \( \mathcal{D} \). The problem of solving such nonlinear equations is one of the oldest problems of applied mathematics. It still remains an important research topic, arising in many areas of engineering sciences, physics, computer science, finance, and so on. There are many numerical methods which have been developed for solving this problem. Basically, the methods concerning this topic may be classified in two groups: methods for finding only one root of nonlinear equations at a time and methods for finding all roots of algebraic polynomials simultaneously. We should note that the methods for the simultaneous finding of all roots of polynomial have some advantages, e.g., they have a wider region of convergence, they are more stable, and they can be implemented for parallel computing. The list of publications concerning this topic is very extensive. We want to refer the readers’ attention to some interesting publications on the two groups, respectively, published in the last few years: [7], [8], [9], [10], [11], [12], and [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26]. For some classical books, monographs and summary papers covering many of the valuable results, see [27], [28], [29], [30], [31], [32].

But sometimes in practise it necessary to find more than one or only few zeros of a given nonlinear equation. In recent years, the problem of simultaneous extraction of only a part of all roots of polynomials became very popular. One of the first works on this topic...
dates from 1966, by S. Presic [2]. Some recent results are presented by N. Kyurkchiev and A. Iliev in [3, 4, 5]. See also [6]. In our recent publication [1] we have introduced a new iterative method for simultaneous extraction of several simple zeros of a nonlinear equation. In this study we extend the suggested method in the case of multiple zeros.

The paper is organized as follows. Some preliminary results are included in Section 2. In Section 3 we generalize the theorems from Section 2 in the case of random simultaneous iterative method. The new modified iterative methods are presented in Section 4, where we prove that the new modifications have second order of convergence. Numerical examples are given in Section 5 to demonstrate the convergence behavior of the methods considered.

2 Preliminary results

Let $I_m = \{1, \ldots, m\}$ be the index set. In [2] we have presented the following two-point iterative method for simultaneous extraction of $m$ simple zeros of (1)

\[
\begin{align*}
&\begin{cases}
y_i^{(k)} = x_i^{(k)} - G_i^{(k)}, & i \in I_m, \\
x_i^{(k+1)} = x_i^{(k)} - \frac{f(x_i^{(k)})}{f[x_i^{(k)}, y_i^{(k)}]}, & k = 0, 1, 2, \ldots,
\end{cases}
\end{align*}
\]

(2)

where $G_i^{(k)}$ is the quantity

\[
G_i^{(k)} = G_i(x_1^{(k)}, \ldots, x_m^{(k)}) = \frac{f(x_i^{(k)})}{\prod_{j \neq i}^m (x_i^{(k)} - x_j^{(k)})} \quad \text{for } i \in I_m.
\]

The notation $f[., .]$ represents divided difference of order one. For simplicity, further we will omit the iteration index. The following two theorems give the convergence results of the presented method in the both cases: polynomials and random nonlinear functions.

2.1 Application to polynomials

If we restrict our consideration to $f$ algebraic (monic) polynomial of degree $n$ with simple zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$. The following theorem concerned with the convergence order of the proposed iterative method (2).

**Theorem 1** If $x_1^{(0)}, x_2^{(0)}, \ldots, x_m^{(0)}$ ($m \geq 1$) are sufficiently close approximations to the simple zeros $\alpha_1, \alpha_2, \ldots, \alpha_m$ of a polynomial $f$ of degree $n$ (and $m \leq n$), then the order of convergence of the iterative method (2) is two.

**Remark 1.** In the case of extracting all of the zeros of a polynomial $f$ of order $n$, i.e. $m = n$, from (2) we get the known third order iterative method

\[
\begin{align*}
&\begin{cases}
y_i = x_i - \frac{f(x_i)}{\prod_{j \neq i}^n (x_i - x_j)}, \\
\hat{x}_i = x_i - \frac{f(x_i)}{f[x_i, y_i]}, & i \in I_n,
\end{cases}
\end{align*}
\]

(3)
suggested and analyzed in [14, 15].

2.2 Application to nonlinear equations

The following theorem is concerned with the order of convergence of method (2) for nonlinear, continuous and sufficiently differentiable function \( f : D \subset \mathbb{R} \rightarrow \mathbb{R} \).

**Theorem 2** Let \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are simple zeros of the nonlinear function \( f \). If \( x_1^{(0)}, x_2^{(0)}, \ldots, x_m^{(0)} \) are sufficiently close approximations to \( \alpha_1, \alpha_2, \ldots, \alpha_m \), then the order of convergence of the iterative method (2) is two.

**Remark 2.** In the case of approximating only one root of nonlinear equation (1), i.e. \( m = 1 \), from (2) we get the well known second order iterative method

\[
\hat{x}_i = x_i - \frac{f(x_i)}{f[x_i, y_i]} = x_i - \frac{f(x_i)^2}{f(x_i) - f(y_i)},
\]

where \( y_i = x_i - f(x_i) \), see [27].

3 Extended modifications

In this section we suggest the following generalized version of method (2)

\[
\begin{cases}
    y_i = x_i - f(x_i) \varphi_i(x), \\
    \hat{x}_i = x_i - \frac{f(x_i)}{f[x_i, y_i]}, \quad i = 1, \ldots, m,
\end{cases}
\]

where \( \varphi_i(x) \) be continuous functions with \( \varphi_i(\alpha) \neq 0 \) for \( i = 1, \ldots, m \).

3.1 Application to polynomials

We restrict our consideration to algebraic (monic) polynomial of degree \( n \)

\[
f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0,
\]

with simple zeros \( \alpha_1, \alpha_2, \ldots, \alpha_n \).

In this case the following theorem implies that the convergence order of the proposed iterative method (5) becomes quadratic as the number of the iterations is increasing.

**Theorem 3** If \( x_1, x_2, \ldots, x_m \) (\( m \geq 1 \)) are sufficiently close approximations to the simple zeros \( \alpha_1, \alpha_2, \ldots, \alpha_m \) of a polynomial \( f \) of degree \( n \) (where \( m \leq n \) defined by (6), and there exist functions \( \varphi_i(x) = \varphi_i(x_1, \ldots, x_m) \) such that \( \varphi_i(\alpha_1, \ldots, \alpha_m) \neq 0 \), then the iterative function (5) generates convergent process with second order of convergence.

**Proof.** Let us denote \( \varepsilon_i = x_i - \alpha_i, \quad \hat{\varepsilon}_i = \hat{x}_i - \alpha_i \) and \( v_i = v_i(x) = f(x_i) \varphi_i(x) \). According to the assumption of the theorem, the errors \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \) are sufficiently small in moduli. Let us assume that the errors \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \) are of the same order in moduli and let \( |\varepsilon_i| = O(|\varepsilon|) \), where \( \varepsilon \in \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\} \) is the error such that \( |\varepsilon| = \max_{1 \leq k \leq m} |\varepsilon_k| \).
Similarly $|\hat{\varepsilon}| = \max_{1 \leq k \leq m} |\hat{\varepsilon}_k|$. Let use the notation $\beta = O_m(\gamma)$ for two complex numbers $\beta$ and $\gamma$ if their moduli are of the same order, i.e. $|\beta| = O(|\gamma|)$.

Using definition of the divided difference

$$f \left[ x_i, y_i \right] = \frac{f(x_i) - f(y_i)}{x_i - y_i} = \frac{f(x_i) - f(y_i)}{v_i(x)},$$

we can represent (5) in the following way

$$\bar{x}_i = x_i - v_i(x) \frac{f(x_i)}{f(x_i) - f(y_i)} , \ i \in I_m ,$$

where $y_i = x_i - v_i(x)$. Observe that

$$f(x_i) = \varepsilon_i \prod_{j \neq i} (x_i - \alpha_j) = O_m(\varepsilon) \quad \text{and} \quad v_i = O_m(f(x_i)) = O_m(\varepsilon).$$

Further we use the following presentation

$$f(y_i) = \prod_{j=1}^{n} \frac{(y_i - \alpha_j)}{(x_i - \alpha_j)} = \left(\frac{y_i - \alpha_i}{x_i - \alpha_i}\right) \prod_{j \neq i} \frac{(y_i - \alpha_j)}{(x_i - \alpha_j)} = \varepsilon_i - v_i \prod_{j \neq i} \frac{(y_i - \alpha_j)}{(x_i - \alpha_j)}. \quad (9)$$

Using that

$$\prod_{j \neq i} \frac{(y_i - \alpha_j)}{(x_i - \alpha_j)} = \prod_{j \neq i} \frac{(x_i - v_j - \alpha_j)}{(x_i - \alpha_j)} = \prod_{j \neq i} \left(1 - \frac{v_i}{(x_i - \alpha_j)} \right) = 1 - \sum_{j \neq i} \frac{v_i}{(x_i - \alpha_j)} P_{ij},$$

where $P_{ij}$ is a polynomial of $\frac{v_i}{(x_i - \alpha_j)}$. It follows that

$$\prod_{j \neq i} \frac{(y_i - \alpha_j)}{(x_i - \alpha_j)} = 1 - O(\varepsilon_i).$$

From the last estimation and (9) we find

$$\frac{f(y_i)}{f(x_i)} = \frac{\varepsilon_i - v_i}{\varepsilon_i} (1 - O(\varepsilon_i)) = 1 - \frac{v_i}{\varepsilon_i} \left(1 - \frac{v_i - \varepsilon_i}{\varepsilon_i} O(\varepsilon_i)\right) = 1 - \frac{v_i}{\varepsilon_i} \left(1 - O_m(\varepsilon)\right). \quad (10)$$

Hence, (10) implies

$$\frac{f(x_i)}{f(x_i) - f(y_i)} = \left(1 - \frac{f(y_i)}{f(x_i)}\right)^{-1} = \frac{\varepsilon_i}{v_i} (1 + O_m(\varepsilon)).$$

Further, from (7) it follows

$$|\hat{\varepsilon}_i| = |\hat{x}_i - \alpha_i| = \left|\varepsilon_i - \frac{v_i}{v_i} \varepsilon_i (1 + O_m(\varepsilon))\right| = O_m(\varepsilon^2).$$

Since $O_m(\varepsilon^2)$ is the dominant term we conclude that the order of convergence of the method (5) is two, which completes the proof of Theorem 3.
3.2 Application to nonlinear equations

In our consideration we make use of the following well-known theorem from the theory of iterative processes.

**Theorem 4** (Traub [1, Theorem 2.2]) Let \( \phi \) be an iterative function such that \( \phi \) and its derivatives \( \phi', \ldots, \phi^{(p)} \) are continuous in the neighborhood of a root \( \alpha \) of a given function \( f \). Then \( \phi \) defines an iterative method of order \( p \) if and only if

\[
\phi(\alpha) = \alpha, \phi'(\alpha) = \ldots = \phi^{(p-1)}(\alpha) = 0, \phi^{(p)}(\alpha) \neq 0. \tag{11}
\]

The following theorem is concerned with the order of convergence of methods (5) for a nonlinear sufficiently differentiable function \( f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R} \).

**Theorem 5** Let \( \alpha_1, \alpha_2, \ldots, \alpha_m \) be simple zeros of the nonlinear sufficiently differentiable function \( f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R} \). If \( x_1, x_2, \ldots, x_m \) are sufficiently close approximations to \( \alpha_1, \alpha_2, \ldots, \alpha_m \), then the order of convergence of the iterative method (5) is two.

**Proof.** Let us fix \( i \) and put \( x = \alpha = (\alpha_1, \ldots, \alpha_m) \). Considering \( y_i = y_i(x) = x_i - v_i(x) \) as a function of \( x_i \) and using the Lhopital’s rule we get

\[
f[x_i, y_i] = \left. \frac{f(x_i) - f(y_i)}{x_i - y_i} \right|_{x=\alpha} = \frac{f(x_i)}{\alpha_i - y_i(\alpha)} = \frac{f'(\alpha_i) - f'(y_i(\alpha))y'_i(\alpha)}{1 - y'_i(\alpha)} = f'(\alpha_i). \tag{12}
\]

Obviously, we have

\[
\hat{x}_i(\alpha) = \alpha_i - \frac{f(\alpha_i)}{f[\alpha_i, y_i(\alpha)]} = \alpha_i, \quad (i = 1, \ldots, m).
\]

The first derivative of the function \( \hat{x}_i \) regarding \( x_i \) is

\[
\hat{x}_i' = 1 - \frac{f'(x_i)}{f[x_i, y_i]} + \frac{f(x_i)f[x_i, y_i]}{f[x_i, y_i]^2}. \tag{13}
\]

Using the Taylor series

\[
f(y) = f(x) - f'(x)(x - y) + \frac{f''(x)}{2!}(x - y)^2 - \frac{f'''(x)}{3!}(x - y)^3 + O((x - y)^4),
\]

we get

\[
f[x_i, y_i] = \frac{f(x_i) - f(y_i)}{x_i - y_i} = f'(x_i) - \frac{f''(x_i)}{2}(x - y)^2 + \frac{f'''(x_i)}{3!}(x - y)^3 + O((x - y)^4). \tag{14}
\]

Then it follows

\[
f[x_i, y_i] = \frac{f''(\alpha_i)}{2}(1 + y'_i(\alpha)) \neq 0.
\]

Finally we obtain

\[
\hat{x}_i'(\alpha) = 0, \quad (i = 1, \ldots, m),
\]

and

\[
\hat{x}_i''(\alpha) = \frac{f''(\alpha_i)}{f'(\alpha_i)} y'_i(\alpha) \neq 0, \quad (i = 1, \ldots, m). \tag{15}
\]

From (12),(14) and Theorem 4 it follows that the iterative function (5) has the order of convergence two.
4 Generalizations for multiple zeros

In this section we suggest generalizations of the method (2) in the case of multiple zeros. Two cases arises in finding multiple zeros of nonlinear equations.

4.1 In the case of preliminary known multiplicities

We examine the equation (1) where the function $f$ has the form in the particular

$$f(x) = (x - \alpha_i)^s \prod_{j \neq i} (x - \alpha_j)^{s_j} g(x)$$

(15)

where $g : D \subset R \rightarrow R$ be such a function that $g(\alpha_i) \neq 0$ for $i = 1, \ldots, m$. In other words the $\alpha_1, \alpha_2, \ldots, \alpha_m$ are roots of $f$ respectively having multiplicity rates $s_1, s_2, \ldots, s_m$ and $s_1 + s_2 + \ldots + s_m = l$. In particular case of $f$ polynomial of $n$-th degree, the function $g(x)$ is a polynomial of degree $n - l$ (where $l \leq n$) and $g(\alpha_i) \neq 0$ for $i = 1, \ldots, m$.

We suggest the following iterative function

$$\begin{align*}
    y_i &= x_i - G_i, \\
    \hat{x}_i &= x_i - G_i \frac{f(x_i)^{1/s_i}}{f(x_i)^{1/s_i} - f(y_i)^{1/s_i}}, \quad i = 1, \ldots, m,
\end{align*}$$

(16)

where $G_i = G_i(x_1, \ldots, x_m) = \frac{f(x_i)}{\prod_{j \neq i} (x_i - x_j)}$.

The order of convergence of this methods is analyzed in the following Theorem.

**Theorem 6** Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be multiple roots of function $f$ defined by (15). If $x_1, x_2, \ldots, x_m$ are sufficiently close approximations to the respective zeros $\alpha_1, \alpha_2, \ldots, \alpha_m$, then the iterative function (16) is convergent with order of convergence two.

**Proof:** Let us fix $i$ and denote $\psi(x_i) = \frac{f(x_i)^{1/s_i}}{f(x_i)^{1/s_i} - f(y_i)^{1/s_i}}$. Observe that

$$f(x_i) = (x_i - \alpha_i)^s \prod_{j \neq i} (x_i - \alpha_j)^{s_j} g(x_i).$$

(17)

We will transform $\psi(x_i)$ in the following way

$$\psi(x_i) = \frac{1}{1 - \frac{\sqrt[1/s_i]{f(y_i)}}{f(x_i)}} = \frac{1}{1 - \frac{\prod_{j=1}^{m} (y_i - \alpha_j)^{s_j} g(y_i)}{\prod_{j=1}^{m} (x_i - \alpha_j)^{s_j} g(x_i)}}$$

$$= \frac{1}{1 - \sqrt[s_i]{\prod_{j \neq i} (x_i - \alpha_j)^{s_j} g(x_i)}}$$

$$= \frac{1}{1 - \frac{\varepsilon_i - G_i}{\varepsilon_i} \sqrt[s_i]{\prod_{j \neq i} (x_i - \alpha_j)^{s_j} g(x_i)}}.$$
Let us examine
\[ \sqrt{\prod_{j \neq i}^{m} \left( \frac{y_i - \alpha_j}{x_i - \alpha_j} \right)^{s_j}} = \sqrt{\prod_{j \neq i}^{m} \left( \frac{x_i - G_i - \alpha_j}{x_i - \alpha_j} \right)^{s_j}} = \sqrt{\prod_{j \neq i}^{m} \left( 1 - \frac{G_i}{x_i - \alpha_j} \right)^{s_j}} = \sqrt{1 - \sum_{j \neq i}^{m} M_{ij} \frac{G_i}{x_i - \alpha_j}}, \]

where \( M_{ij} \) are polynomials of \( \frac{G_i}{x_i - \alpha_j} \).

Using that \( G_i = O(\varepsilon_i) \) it is easy to obtain that
\[ \sum_{j \neq i}^{m} M_{ij} \frac{G_i}{x_i - \alpha_j} = O(\varepsilon_i). \]

Thus we have demonstrated that the following presentation is valid
\[ \sqrt{\prod_{j \neq i}^{m} \left( \frac{y_i - \alpha_j}{x_i - \alpha_j} \right)^{s_j}} = 1 + O(\varepsilon_i). \]

Using the Taylor series
\[ g(y_i) = g(x_i) - g'(x_i)(x_i - y_i) + \frac{g''(x_i)}{2!}(x_i - y_i)^2 + O \left( (x_i - y_i)^3 \right), \]
we get that
\[ \frac{g(y_i)}{g(x_i)} = 1 - \frac{g'(x_i)}{g(x_i)} G_i + \frac{g''(x_i)}{2!g(x_i)} G_i^2 + O \left( (x_i - y_i)^3 \right). \]

Then
\[ \sqrt{\frac{g(y_i)}{g(x_i)}} = \sqrt{1 - \frac{g'(x_i)}{g(x_i)} G_i + O(G_i^2)} = \sqrt{1 + O(\varepsilon_i)} = 1 + O(\varepsilon_i). \]

Hence, we get
\[ \psi(x_i) = \varepsilon_i \frac{G_i}{G_i}(1 + O(\varepsilon_i)). \]

Further, for the iterative process (16), we obtain
\[ \tilde{\varepsilon}_i = x_i - \alpha_i - G_i \psi(x_i) \]
\[ \tilde{\varepsilon}_i = \varepsilon_i - G_i \frac{\varepsilon_i}{G_i} (1 + O(\varepsilon_i)) \]
\[ \tilde{\varepsilon}_i = \varepsilon_i - \varepsilon_i (1 + O(\varepsilon_i)) \]
\[ \tilde{\varepsilon}_i = O(\varepsilon_i^2). \]

The proof is completed.

In this case we have used the acceleration technics presented in [26] to obtain the iterative function (16).
4.2 In the case of preliminary unknown multiplicities

In this section we suggest the following generalized version of method (2)

\[
\begin{align*}
    y_i &= x_i - \frac{F(x_i)}{\prod_{j \neq i} (x_i - x_j)}, \\
    \hat{x}_i &= x_i - \frac{F(x_i)}{F[x_i, y_i]}, \quad i \in I_m,
\end{align*}
\]

where \( F(x_i) = \frac{f(x_i)}{f[x_i, x_i - f(x_i)]} \).

**Theorem 7** Let \( \alpha_1, \alpha_2, \ldots, \alpha_m \) be multiple zeros of the nonlinear function \( f \) defined by (15). If \( x_1, x_2, \ldots, x_m \) are sufficiently close approximations to the respective zeros \( \alpha_1, \alpha_2, \ldots, \alpha_m \), then the order of convergence of the iterative method (18) is two.

**Proof.** Let us fix \( i \) and consider the divided difference

\[
f[x_i, x_i - f(x_i)] = \frac{f(x_i) - f(x_i - f(x_i))}{f(x_i)}.
\]

From Taylor expansion

\[
f(x_i - f(x_i)) = f(x_i) - f'(x_i) f(x_i) + \frac{f''(x_i)}{2} f(x_i)^2 - \frac{f'''(x_i)}{3!} f(x_i)^3 + \ldots,
\]

it follows that

\[
f[x_i, x_i - f(x_i)] = f'(x_i) - \frac{f''(x_i)}{2} f(x_i) + \frac{f'''(x_i)}{3!} f(x_i)^2 + \ldots.
\]

Then for the function \( F(x_i) \), we have

\[
F(x_i) = \frac{f(x_i)}{f[x_i, x_i - f(x_i)]} = \frac{f(x_i)}{f'(x_i)} \left( 1 - \frac{f''(x_i)}{2} \frac{f(x_i)}{f'(x_i)} + \frac{f'''(x_i)}{3!} \frac{f(x_i)^2}{f'(x_i)^2} + \ldots \right).
\]

Using the presentation (15) let us denote

\[
f(x_i) = (x_i - \alpha_i)^s h(x_i),
\]

where \( h(\alpha_i) \neq 0 \). Then

\[
f(x_i)^2 = s_i (x_i - \alpha_i)^{s_i-1} h(x_i) + (x_i - \alpha_i)^{s_i} h(x_i)^2 x_i.
\]

From last two equations it follows that

\[
u(x_i) \frac{f(x_i)}{f'(x_i)} = \frac{(x_i - \alpha_i)^s h(x_i)}{s_i (x_i - \alpha_i)^{s_i-1} h(x_i) + (x_i - \alpha_i)^{s_i} h(x_i)^2 x_i} = \frac{(x_i - \alpha_i) h(x_i)}{s_i h(x_i) + (x_i - \alpha_i) h(x_i)^2 x_i}
\]

and

\[
u(\alpha_i) = 0.
\]
Differentiating \( u(x_i) \) regarding \( x_i \), we get

\[
 u'(x_i) = \frac{s_i h(x_i)^2 - s_i (x_i - \alpha_i) h(x_i) h'(x_i) - (x_i - \alpha_i)^2 h(x_i) h''(x_i)}{(s_i h(x_i) + (x_i - \alpha_i) h'(x_i))^2},
\]

then it follows

\[
 u'(\alpha_i) = \frac{1}{s_i} \neq 0.
\] (21)

From (19), (20) and (21), it follows that

\[
 F(\alpha_i) = 0, \quad i = 1, \ldots, m
\]

and

\[
 F'(\alpha_i) = \frac{1}{s_i} \neq 0, \quad i = 1, \ldots, m,
\]

i.e. the function \( F(x) \) has only simple zeros at \( \alpha_1, \ldots, \alpha_m \).

Using (5) and Theorem 5 (when \( \varphi_i(x) = 1 \) for \( i = 1, \ldots, m \), it follows that the iterative function (18) is convergent of order two. Theorem is proved.

5 Numerical results

In this section, we employ the new methods (16) and (18) to solve some nonlinear equations. All computations were done using MATLAB 7.0. We accept an approximate solution rather than the exact root, depending on the precision (\( \epsilon \)) of the computer. We use the following stopping criteria for computer programs:

\[
 (i) \quad \| x^{(k+1)} - x^{(k)} \| < \epsilon,
 (ii) \quad \| F(x^{(k+1)}) \| < \epsilon,
\]

where we use the maximum norm \( \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\} \) and the notation \( F(x) = (f(x_1), \ldots, f(x_n))^T \). When the stopping criterion is satisfied, \( x^{(k+1)} \) is taken as the exact root computed. For numerical illustrations in this section we used the fixed stopping criterion \( \epsilon = 10^{-8} \).

Our consideration of the method is based upon the following criteria: the initial approximation \( x^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)}) \) (where \( m \) is the number of roots we are seeking), the number of iterations to approximate the zero(s) (Iterations) and the roots reached \( \alpha_m = (\alpha_{s_1}, \ldots, \alpha_{s_m}) \).

We tested the proposed methods (16) and (18) in the examples of several nonlinear equations to demonstrate their performance. We selected the following three examples for illustration.

**Example 1.** Consider the polynomial equation

\[
 (x + 2)^4(x - 3)^2x^3 = 0,
\]

with three multiple zeros \( \alpha_1 = -2, \alpha_2 = 0, \alpha_3 = 3 \) and the corresponding multiplicities \( s_1 = 4, s_2 = 3, s_3 = 2 \). See the results in Table 1.
Example 3. Consider the nonlinear equation:

\[
\left( e^{x(x-1)(x-2)(x-3)} - 1 \right)^4 = 0,
\]

Table 1. Numerical results for Example 1.

<table>
<thead>
<tr>
<th>Initial point ( x^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)}) )</th>
<th>Iterations (k) (16) (18)</th>
<th>Zeros reached ( \alpha = (\alpha_{s1}, \ldots, \alpha_{sm}) ) (16) (18)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^{(0)} = -2.1 )</td>
<td>4 4</td>
<td>( \alpha_1 ) ( \alpha_1 )</td>
</tr>
<tr>
<td>( x^{(0)} = -1.7 )</td>
<td>NaN 23</td>
<td>– ( \alpha_1 )</td>
</tr>
<tr>
<td>( x^{(0)} = -0.1 )</td>
<td>4 4</td>
<td>( \alpha_2 ) ( \alpha_2 )</td>
</tr>
<tr>
<td>( x^{(0)} = 2.9 )</td>
<td>NaN NaN</td>
<td>– –</td>
</tr>
<tr>
<td>( x^{(0)} = (-2.1, 0.1) )</td>
<td>4 5</td>
<td>( \alpha = (\alpha_1, \alpha_2) ) ( \alpha = (\alpha_1, \alpha_2) )</td>
</tr>
<tr>
<td>( x^{(0)} = (-2.2, -0.2) )</td>
<td>20 NaN</td>
<td>( \alpha = (\alpha_1, \alpha_2) ) –</td>
</tr>
<tr>
<td>( x^{(0)} = (-1.7, 0.2) )</td>
<td>283 12</td>
<td>( \alpha = (\alpha_1, \alpha_2) ) ( \alpha = (\alpha_1, \alpha_2) )</td>
</tr>
<tr>
<td>( x^{(0)} = (-1.8, 0.3, 3.1) )</td>
<td>NaN NaN</td>
<td>– –</td>
</tr>
</tbody>
</table>

The results presented in Table 1 show that for this example the iterative processes (16) and (18) have similar convergence behavior. For the both methods there exist initial points when the iterative process fail. For some initial approximations the presented methods converges linearly at the beginning of iterative procedure. This is the reason for larger number of iterations. For the initial approximation in neighbourhood of \( \alpha_3 = 3 \) the both methods are not convergent (denoted \( NaN \)).

Example 2. Consider the nonlinear equation:

\[
\left( \frac{\sin^2(x)}{\exp(x/2)} - 1 \right)^3 = 0,
\]

with infinite number multiple zeros. In the numerical experiments we approximate (and denote) only the following three zeros \( \alpha_1 = -3.564283758 \), \( \alpha_2 = -2.590959369 \) and \( \alpha_3 = -0.918673296 \), each of them with multiplicity 3. The results are given in Table 2.

Table 2. Numerical results for Example 2.

<table>
<thead>
<tr>
<th>Initial point ( x^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)}) )</th>
<th>Iterations (k) (16) (18)</th>
<th>Zeros reached ( \alpha = (\alpha_{s1}, \ldots, \alpha_{sm}) ) (16) (18)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^{(0)} = -0.6 )</td>
<td>NaN 4</td>
<td>– ( \alpha_3 )</td>
</tr>
<tr>
<td>( x^{(0)} = -1.1 )</td>
<td>3 4</td>
<td>( \alpha_3 ) ( \alpha_3 )</td>
</tr>
<tr>
<td>( x^{(0)} = -2.5 )</td>
<td>NaN 4</td>
<td>– ( \alpha_2 )</td>
</tr>
<tr>
<td>( x^{(0)} = -3.4 )</td>
<td>NaN 4</td>
<td>– ( \alpha_1 )</td>
</tr>
<tr>
<td>( x^{(0)} = (-3.6, -1.2) )</td>
<td>4 4</td>
<td>( \alpha = (\alpha_1, \alpha_3) ) ( \alpha = (\alpha_1, \alpha_3) )</td>
</tr>
<tr>
<td>( x^{(0)} = (-2.1, -0.1) )</td>
<td>NaN 9</td>
<td>– ( \alpha = (\alpha_2, \alpha_3) )</td>
</tr>
<tr>
<td>( x^{(0)} = (-3.5, -2.8) )</td>
<td>4 4</td>
<td>( \alpha = (\alpha_1, \alpha_2) ) ( \alpha = (\alpha_1, \alpha_2) )</td>
</tr>
<tr>
<td>( x^{(0)} = (-3.5, -2.3, -0.5) )</td>
<td>NaN 6</td>
<td>– ( \alpha = (\alpha_1, \alpha_2, \alpha_3) )</td>
</tr>
<tr>
<td>( x^{(0)} = (-3.8, -3, -1.2) )</td>
<td>30 30</td>
<td>( \alpha = (\alpha_2, \alpha_1, \alpha_3) ) ( \alpha = (\alpha_1, \alpha_2, \alpha_3) )</td>
</tr>
<tr>
<td>( x^{(0)} = (-3.4, -2.4, -0.5) )</td>
<td>NaN 5</td>
<td>– ( \alpha = (\alpha_1, \alpha_2, \alpha_3) )</td>
</tr>
</tbody>
</table>

In Table 2 we can see that the method (18) has better convergence properties than method (16). The iterative process (16) is convergent for few initial approximations.
with four multiple zeros $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 3$ with multiplicities 4. The results are given in Table 3.

Table 3. Numerical results for Example 3.

| Initial point $x^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)})$ | Iterations (k) $|Zeros reached \alpha = (\alpha_{s_1}, \ldots, \alpha_{s_m})$|
|----------------------------------------------------------|--------------------------------------------------|
| $x^{(0)} = 1.4$                                          | $x^{(0)} = 3.1$                                  |
| $x^{(0)} = 2.5$                                          | $x^{(0)} = (1.3, 2.6)$                           |
| $x^{(0)} = (1.1, 2.2)$                                   | $x^{(0)} = (0.2, 2.9)$                           |
| $x^{(0)} = (0.5, 2.5)$                                   | $x^{(0)} = (0.4, 1.3, 2.6)$                      |
| $x^{(0)} = (0.3, 0.8, 2.2)$                              | $x^{(0)} = (0.1, 0.9, 2.9)$                      |
| $x^{(0)} = (0.2, 0.8, 1.8, 3.1)$                         | $x^{(0)} = (0.2, 0.8, 1.8, 2.8)$                  |

For this example, again the method (18) has better convergence properties than method (16). It has wider neighbourhood of convergence for each of the roots. For initial approximations where the both methods are convergent, they have equal number of iterations to reach the zeros.

We have to denote that it is very important the choice of initial approximations. If they are chosen sufficiently close to the sought roots, then the expected (theoretical) convergence speed will be reached in practice. Otherwise the method (as many other iterative methods) shows slower convergence, especially at the beginning of the iterative process. For this reason, a special attention should be paid to finding good initial approximations.

6 Conclusion

In this work we presented two iterative methods of second order for extraction of more than one multiple zero of nonlinear equations, simultaneously. The new methods do not require the computation of the first-order or higher derivatives of the function $f$. The first modification (16) is applicable in the case of preliminarily known multiplicities of the extracting roots. The main advantage of the method (18) is that it is applicable for wide range of nonlinear functions $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ for computing arbitrary number of multiple zeros. For this method it is not necessary to know the multiplicities of the roots seeking. The numerical experiments confirm the theoretical results.

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References


