ADDENDUM TO "ISOMETRIC STUDY OF WASSERSTEIN SPACES – THE REAL LINE"

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Abstract. We show an example of a Polish metric space X whose quadratic Wasserstein space $\mathcal{W}_2(X)$ possesses an isometry that splits mass. This gives an affirmative answer to Kloeckner's question, [2, Question 2].

Let us denote the metric space $([0,1], |\cdot|)$, equipped with the usual distance, by Y. In our recent manuscript we determined the isometry group of the Wasserstein space $\mathcal{W}_1(Y)$, for details see [1, Subsection 2.1]. In particular, it turned out that the bijective map defined by

$$j\colon \mathcal{W}_1(Y) \to \mathcal{W}_1(Y), \qquad \mu \mapsto j(\mu), \quad F_{j(\mu)} = F_\mu^{-1}$$

is an isometry of $\mathcal{W}_1(Y)$, that is,

$$d_{\mathcal{W}_1(Y)}(\mu,\nu) = d_{\mathcal{W}_1(Y)}(j(\mu), j(\nu)) \quad (\mu,\nu \in \mathcal{W}_1(Y)).$$

Notice that we also have

$$j(\delta_t) = t\delta_0 + (1-t)\delta_1 \quad (t \in (0,1)),$$

that is, j does not preserve the set of all Dirac masses.

Very recently we realized that the above example can be easily manipulated in order to answer the following question of Kloeckner ([2, Question 2.]): *Does it exist a Polish space X whose quadratic Wasserstein space $\mathcal{W}_2(X)$ possesses an isometry that does not preserve the set of all Dirac masses?*

Example. Let us equip the set $[0,1]$ with an other metric: $\rho(x,y) = \sqrt{|x-y|}$. (It is easy to see that ρ indeed defines a metric.) Let the symbol X stand for the Polish space $([0,1], \rho)$. Obviously $\mu,\nu \in \mathcal{W}_2(X)$ if and only if $\mu,\nu \in \mathcal{W}_1(Y)$. Notice that we have

$$\begin{aligned} d_{\mathcal{W}_2(X)}(\mu,\nu) &= \sqrt{\inf_{\pi \in \Pi(\mu,\nu)} \int\limits_{[0,1]\times[0,1]} \big(\rho(x,y)\big)^2 \,\mathrm{d}\pi(x,y)} \\ &= \sqrt{\inf_{\pi \in \Pi(\mu,\nu)} \int\limits_{[0,1]\times[0,1]} \big|x-y\big| \,\mathrm{d}\pi(x,y)} = \sqrt{d_{\mathcal{W}_1(Y)}(\mu,\nu)}. \end{aligned}$$

Therefore the map j is also an isometry of $\mathcal{W}_2(X)$:

$$d_{\mathcal{W}_2(X)}(j(\mu), j(\nu)) = \sqrt{d_{\mathcal{W}_1(Y)}(j(\mu), j(\nu))} = \sqrt{d_{\mathcal{W}_1(Y)}(\mu,\nu)} = d_{\mathcal{W}_2(X)}(\mu,\nu).$$

This answers Kloeckner's question affirmatively.

We note that a similar construction works for any $p > 1$: if we choose Z to be $([0,1], |\cdot|^{1/p})$, then j defines an isometry of $\mathcal{W}_p(Z)$.

Finally, observe that the isometry j does not preserve the existence of a transport map between μ and ν in general.

References

[1] Gy. P. Gehér, T. Titkos, D. Virosztek, *Isometric study of Wasserstein spaces – the real line*, Trans. Amer. Math. Soc., to appear (arXiv:2002.00859).

[2] B. Kloeckner, *A geometric study of Wasserstein spaces: Euclidean spaces,* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze IX, 2 (2010), 297–323.

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