Applied stochastic programming models and computation

H.I. Gassmann∗

Revision of March 22, 2007

Abstract

Stochastic programming has been used in applications for fifty years. Different model types and solution algorithms were developed, often taking advantage of increases in computing power. This paper studies 144 applications of stochastic programming from the past. It classifies the papers along several dimensions and demonstrates historic trends and other developments.

1 Introduction

Stochastic programming has been in existence for more than half a century. The first algorithms for the two-stage stochastic program with recourse were developed independently by Dantzig (1955) and Beale (1955), and the first application of this type of problem appeared in the following year (Ferguson and Dantzig 1956).

A competing model using probabilistic constraints was formulated by Charnes and Cooper (1959). Despite some theoretical shortcomings (Higle and Sen 1995; Blau 1974), this model was preferred at first because a lack of computing power made solutions of large-scale recourse problems impractical. However, rapid advances in computing power as well as algorithmic development, especially since the mid 1980s, have seen larger and larger problems being formulated and solved successfully (see, e.g., Gondzio and Kouwenberg 2001; Sen et al. 1994; Linderoth and Wright 2003).

The first computationally viable algorithm for two-stage recourse problems, a special application of Benders decomposition, was developed by van Slyke and Wets (1969). Birge (1985) extended the algorithm to multistage problems, and a general implementation was described by Gassmann (1990).

The first successful commercial application of multistage stochastic programming with recourse is the Russell-Yasuda Kasai model developed for a Japanese insurance

∗School of Business Administration, Dalhousie University, Halifax, Canada, e-mail: Horand.Gassmann@dal.ca
company (Cariño et al. 1994; Cariño et al. 1998; Cariño and Ziemba 1998). This work was a finalist for the Edelman prize in 1989.

There have been several commercial implementations for general stochastic programming (Infanger 1997; King et al. 2005; Dash Optimization 2005), and specialized systems (Mulvey and Thorlacius 1998; Edirisinghe and Patterson 2007; Valente et al. 2005). Kall and Mayer (2005) developed a work-bench incorporating a wide variety of algorithms for a great number of different problem classes, intended mostly as a learning tool.

The stochastic programming research bibliography maintained by van der Vlerk at the University of Groningen (van der Vlerk 2003) contained 3840 entries in 2003. It is worth mentioning lastly that an application of stochastic programming for personal finance won the EURO application prize in 2006 (Consiglio et al. 2004).

The purpose of this paper is to review computational approaches and applications of stochastic programming that have appeared in the literature over the last fifty years. Several stochastic programming models are formulated and presented in Section 2. Section 3 describes the origin of the applications and the scope of the study. Results are tabulated in section 4, and some brief conclusions in section 5 round out the paper.

2 Model types

This section reviews several types of stochastic programs. They all have in common that decision problems under uncertainty need to deal with nonanticipativity, that is, decisions made today cannot anticipate any particular future outcome or incorporate it into the decision. Recourse problems allow the decision maker to take corrective action after the randomness has been revealed, while chance-constrained problems limit the probability with which violations of constraints can be tolerated.

The two-stage recourse model has the mathematical form

$$\begin{align*}
\min & \quad c'x + E_\omega q(\omega)'y(\omega) \\
\text{s.t.} & \quad Ax & \leq b \\
& \quad H(\omega)x + W(\omega)y(\omega) & \leq h(\omega) \text{ a.e. } \omega \\
& \quad x \geq 0, y(\omega) \geq 0,
\end{align*}$$

(1)

where $\omega$ is a random variable on some probability space $(\Omega, \mathcal{F}, P)$. The subdiagonal matrix $H(\omega)$ is called the technology matrix; $W(\omega)$ is called the recourse matrix. $x$ is a set of first-stage decisions made before the value of the random variable $\omega$ is revealed; $y(\omega)$ is a set of second-stage decisions permitting corrective action after the outcome of $\omega$ has been observed.

This idea can be extended to the multistage recourse model, whose mathematical
form is

\[
\min_{x_0} c'_0 x_0 + \mathbb{E}_{\omega_1} \left\{ \min_{x_1} c_1(\omega_1)' x_1 + \mathbb{E}_{\omega_2|\omega_1} \left[ \min_{x_2} c_2(\eta_2)' x_2 \right. \right.
\]
\[
\left. + \cdots + \mathbb{E}_{\omega_T|\eta_{T-1} \cdots \omega_1} (\min_{x_T} c_T(\eta_T)' x_T) \right\} 
\]
\[
\text{s.t. } A_{00} x_0 \leq b_0, \quad A_{10}(\omega_1)x_0 + A_{11}(\omega_1)x_1 \leq b_1(\omega_1) \text{ a.s.} \quad (2)
\]
\[
A_{20}(\eta_2)x_0 + A_{21}(\eta_2)x_1 + A_{22}(\eta_2)x_2 \leq b_2(\eta_2) \text{ a.s.}
\]
\[
\vdots \quad \vdots \quad \ddots \quad \vdots
\]
\[
A_{T0}(\eta_T)x_0 + A_{T1}(\eta_T)x_1 \cdots A_{TT}(\eta_T)x_T \leq b_T(\eta_T) \text{ a.s.}
\]
\[
l_0 \leq x_0 \leq u_0, l_t(\eta_t) \leq x_t \leq u_t(\eta_t), t = 1, \ldots, T.
\]

Here \(\omega_1, \omega_2, \ldots, \omega_T\) must be adapted to a filtration \(\emptyset, \Omega = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_T \subseteq \mathcal{F}\) of some probability space \((\Omega, \mathcal{F}, P)\), and constraints involving induced decisions \(x_1, x_2, \) etc. need hold only almost surely (a.s.), that is, they may be violated on a set of probability measure 0. The notation \(\eta_t, t = 1, \ldots, T\) is used to denote the history \(\eta_t = (\omega_1, \ldots, \omega_t)\).

If the random variables have finite discrete distributions, the expected values can be written as finite sums, which allows the problem to be reformulated as deterministic equivalent large-scale ordinary linear programs. (Details may be found in standard texts on stochastic programming, such as Birge and Louveaux 1997 or Kall and Wallace 1994.)

Chance-constrained models take the form

\[
\min c' x
\]
\[
\text{s.t. } A x \leq b, \quad \text{Pr} \left\{ R_i(\omega) x \leq r_i(\omega) \right\} \geq \alpha_i, i = 1, \ldots, I
\]
\[
l \leq x \leq u.
\]

They are usually single-period models, although some multistage models have also been used (see, e.g., Loucks 1980).

Klein Haneveld and van der Vlerk (2002) formulated problems with integrated chance constraints of the form

\[
\min c' x
\]
\[
\text{s.t. } A x \leq b, \quad \mathbb{E} \{ R_i(\omega) x | R_i(\omega) x > r_i(\omega) \} \leq d_i, i = 1, \ldots, I
\]
\[
l \leq x \leq u.
\]

A closely related form of constraints, the so-called CVaR constraints, has seen some use in financial applications (see, e.g., Rockafellar and Uryasev 2000).
Problems that combine recourse actions and chance constraints are a relatively recent innovation. Special algorithms are possible (see, e.g., Steinbach 2003) if the random variables have finite discrete distributions, in which case a two-stage model with a single chance constraint can be given in deterministic equivalent form as

$$\min \ c'x$$
$$\text{s.t.} \ Ax \leq b$$
$$R_s x - My_s \leq r_s, s = 1, \ldots, S$$
$$\sum_{s=1}^{S} p_s y_s \geq \alpha$$
$$l \leq x \leq u, y_s \in \{0, 1\}, s = 1, \ldots, S,$$

where \((R_s, r_s), s = 1, \ldots, S\) are the possible realizations of the data process, \(p_s\) are the corresponding probabilities and \(M\) is a suitably large constant.

The problem formulations above show the full generality. Not every problem has random elements in every problem component (cost vector, right-hand sides, bounds, constraint coefficients, etc.). This will be studied in more detail in section 4.

3 Scope of the study and sources

The study was intended to look at real-life applications and novel models only. All types of stochastic programs are represented, using a mixture of illustrative and “industrial-strength” models. Papers with a primary focus on computations were included, as long as the models considered referred to specific applications. On the other hand, I excluded purely theoretical papers, even if they introduced a new model type not considered previously. I also excluded papers describing new algorithms for previously published applications, as well as computational studies using randomly generated problems.

Data were gathered from a number of research bibliographies (Birge 1984; Stancu-Minasian and Wets 1976; van der Vlerk 2003), supplemented by consultation of citation indices, existing problem collections (Ahmed (undated); Ariyawansa and Felt 2004; Holmes 1994) and online searches. Altogether this effort yielded 144 papers. The earliest application is the airline problem of Ferguson and Dantzig (1956) mentioned in the introduction, the latest papers are from 2005. Other early applications include Charnes et al. (1958), Midler and Wollmer (1969), Resh (1970/71), Bradley and Crane (1972) and two models of economic planning (Tintner and Ferraghi 1967; Tintner and Raghaven 1970). The latter two are also distribution problems, in which the aim is to describe the distribution of the objective value rather than to find a single set of “optimal” decisions.

A number of the papers pre-1985 are simple recourse problems, two-stage problems in which the recourse matrix has the special form \(T(\omega) = [I | - I]\). Problems of this type include Thurner (1974), Martel and Al-Nuaimi (1973), Mirás Amor (1973), Peters et al. (1977), Cleef and Gaul (1982), Kallberg et al. (1982), Kusy
and Ziemba (1986), and the product mix problem from the collection in King (1988; p.554).

Recourse problems with random recourse are a relatively recent innovation. The first application that uses this feature seems to be Guldmann (1983): other early examples include Bienstock and Shapiro (1988), Klaassen et al. (1990), Dentcheva and Römis ch (1998) and Mulvey and Vladimirou (1992).

A surprising number of the early multistage problems have non-staircase format, including Charnes et al. (1958), Ellis and Rishel (1974), Lyons (1975), Lockett and Muhlemann (1978), Bogle and O'Sullivan (1979), Prékopa and Szántai (1980), Lane and Hutchinson (1980), Guldmann (1983).

Integer variables are a frequently used feature after about 2000, but even some of the early models had them, including Midler and Wollmer (1969), Lockett and Muhlemann (1978), Carbone (1974), as well as Martel (1977). Ruszczyński (2002) and Lulli and Sen (2004) use integer variables in conjunction with linking constraints to model chance constraints inside a recourse problem, as described in the previous section.

Some problems have unique features not found elsewhere in the literature, such as the probabilistic objectives of Kelle (1986) and Prékopa and Szántai (1980), or the decision-dependent distributions used by Popova and Morton (1998). Takriti, Birge, and Long (1996) has a quadratic objective, while Helgason and Wallace (1991) and de Groote et al. (1988) are nonlinear problems. Sakawa and Kato (2002) is a multi-objective problem, Huh and Roundy (2005) uses continuous time. Louveaux (1982) is an unconstrained problem, and Ellis and Rishel (1974) is the only problem I could find that uses stochastic bounds.


Another source of applications is energy planning and power generation. This includes papers such as Louveaux (1980), Louveaux and Smeers (1988), Birge and Rosa (1995), Jacobs et al. (1995), Hobbs and Ji (1999), Fleten et al. (1999), Nowak


Applications in personnel scheduling are described in Duffuaa and Al-Sultan (1999) and Morton and Popova (2004); Gröwe (1997), Shih and Frey (1995) and Armstrong and Balintfy (1980) describe blending problems. The last of these is a stochastic version of the diet problem, one of the first linear programs ever formulated (see Dantzig 1990).

The management of renewable resources forms the basis of Gassmann (1989) (Forestry), Helgason and Wallace (1991) (Fishery) and Maatman et al. (2002) (Agriculture). Tomasgard and Høeg (2005) deals with the production of meat and meat products.

Kao and Queyranne (1985) and Fragnière and Haurie (1996) apply stochastic programming in health care and public policy, respectively; Watanabe and Ellis (1993) is concerned with assessing and maintaining air quality. Other applications of stochastic programming include Lai and Ng (2005), who study hotel yield management, and Petkov and Maranas (1997), concerned with the design of metabolic pathways, a problem in biochemistry. Philpott (2005), finally, applies stochastic programming in sports, describing various problems and applications in sailing, specifically the preparation of New Zealand’s entry in the America’s cup.

4 Analysis of the data

As mentioned previously, the works were published between 1956 and 2005. This made it convenient to break the observation period into five ten-year periods. The number of papers published in each period is given in table 1. This table shows a
4 Analysis of the data

<table>
<thead>
<tr>
<th>Decade</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1956-1965</td>
<td>2</td>
</tr>
<tr>
<td>1966-1975</td>
<td>13</td>
</tr>
<tr>
<td>1976-1985</td>
<td>28</td>
</tr>
<tr>
<td>1986-1995</td>
<td>30</td>
</tr>
<tr>
<td>1996-2005</td>
<td>71</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>144</strong></td>
</tr>
</tbody>
</table>

**Tab. 1:** Distribution of publications by decade

<table>
<thead>
<tr>
<th>Source</th>
<th>Number of Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Operations Research</td>
<td>12</td>
</tr>
<tr>
<td>Management Science</td>
<td>11</td>
</tr>
<tr>
<td>Annals of Operations Research</td>
<td>6</td>
</tr>
<tr>
<td>European Journal of Operational Research</td>
<td>6</td>
</tr>
<tr>
<td>Mathematical Programming</td>
<td>4</td>
</tr>
<tr>
<td>Other OR journal</td>
<td>24</td>
</tr>
<tr>
<td>Other journal</td>
<td>33</td>
</tr>
<tr>
<td>Book / collection</td>
<td>32</td>
</tr>
<tr>
<td>SPEPS</td>
<td>8</td>
</tr>
<tr>
<td>Other web source</td>
<td>4</td>
</tr>
<tr>
<td>Technical report</td>
<td>4</td>
</tr>
</tbody>
</table>

**Tab. 2:** Avenues of publication

Healthy growth in output and provides clear testimony to the gradual acceptance of stochastic programming by practitioners.

Since the data were available, I also tracked where the papers had been published originally. This information is gathered in table 2. The usual journals in mathematical programming are well represented (39 out of 144 papers were published in the top five journals in the field), but there are also many journal publications in other areas, ranging from *Forest Science* (Boychuk and Martell 1996) to *Ecological Modelling* (Tung and Hathhorn 1990). SPEPS is an electronic collection of papers in stochastic programming, maintained by Werner Römisch (Römisch 2006).

Table 3 shows the number of papers in different areas. Finance applications clearly dominate, although they make up less than 25% of all papers. One paper (Pereira and Pinto 1985) was concerned with the management of water resources for the generation of electricity. This paper could conceivably have been classified as either energy planning or water management applications, but I decided to count it among the former only.
### Application areas

<table>
<thead>
<tr>
<th>Application</th>
<th>Number of Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finance</td>
<td>30</td>
</tr>
<tr>
<td>Energy planning</td>
<td>16</td>
</tr>
<tr>
<td>Scheduling</td>
<td>14</td>
</tr>
<tr>
<td>Water management</td>
<td>12</td>
</tr>
<tr>
<td>Capacity expansion</td>
<td>9</td>
</tr>
<tr>
<td>Agriculture/Fishery/Forestry</td>
<td>8</td>
</tr>
<tr>
<td>Transportation</td>
<td>8</td>
</tr>
<tr>
<td>Operations Management</td>
<td>8</td>
</tr>
<tr>
<td>Supply chain management</td>
<td>7</td>
</tr>
<tr>
<td>Blending problems</td>
<td>4</td>
</tr>
<tr>
<td>Telecommunications</td>
<td>3</td>
</tr>
<tr>
<td>Other</td>
<td>25</td>
</tr>
</tbody>
</table>

**Tab. 3:** Application areas

One very important facet of the study was to track what features were used in the different applications. Table 4 shows the number of problems in which some random coefficients appeared in the right-hand sides, the cost vectors, etc. It further proved insightful to track the use of features in each ten-year period. (Since problems can have many features, the columns do not total to 144.)

Some interesting patterns emerge. Two thirds of the problems have some right-hand side coefficients that are random, while cost coefficients are random in only about one third of the problems. Moreover, random cost coefficients are much more common now than they were prior to 1975. By contrast only one problem uses random bounds. (This information may be useful in the design of data structures for stochastic programming solvers, when computational efficiency must be contrasted with use of memory.)

Chance constraints were popular prior to 1975 but fell out of favour slightly before making a come-back in the last decade. Also remarkable is the large number of problems in each decade that had integer variables. Stochastic programming with integer recourse is clearly a very active area of research now, with many new and exciting results (for instance the works by Schultz and Tiedemann 2002; Morton et al. 2002; Ahmed et al. 2003; Ahmed and Garcia 2003; Alonso-Ayuso et al. 2003), but ad hoc approaches were used as early as 1969 (Midler and Wollmer 1969).

I was further interested in the number of stage and the type of linkages between stages. This can be read off from table 5. One of the problems was set up as a continuous-time problem; since it does not fit any of the problem classifications, it was omitted from this table. It is interesting to note that the first multistage problems appeared well before efficient algorithms were available to solve them routinely.

Finally I tried to classify the application by their degree of “seriousness”. I
### Tab. 4: Features by decade

<table>
<thead>
<tr>
<th>Feature</th>
<th>'56-'65</th>
<th>'66-'75</th>
<th>'76-'85</th>
<th>'86-'95</th>
<th>'96-'05</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random RHS</td>
<td>2</td>
<td>9</td>
<td>21</td>
<td>17</td>
<td>44</td>
<td>93</td>
</tr>
<tr>
<td>Random Cost</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>11</td>
<td>34</td>
<td>51</td>
</tr>
<tr>
<td>Random Bounds</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Random Subdiagonal Blocks</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>14</td>
<td>22</td>
<td>40</td>
</tr>
<tr>
<td>Random Blocks on Diagonal</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>12</td>
<td>20</td>
<td>39</td>
</tr>
<tr>
<td>Integer Variables</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>21</td>
<td>26</td>
</tr>
<tr>
<td>Chance Constraints</td>
<td>1</td>
<td>4</td>
<td>12</td>
<td>5</td>
<td>12</td>
<td>34</td>
</tr>
<tr>
<td>Integrated Chance Constraints</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Linking Constraints</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Total problems</td>
<td>2</td>
<td>13</td>
<td>28</td>
<td>30</td>
<td>71</td>
<td>144</td>
</tr>
</tbody>
</table>

### Tab. 5: Number of stages per decade

<table>
<thead>
<tr>
<th>Number of stages</th>
<th>'56-'65</th>
<th>'66-'75</th>
<th>'76-'85</th>
<th>'86-'95</th>
<th>'96-'05</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single stage</td>
<td>0</td>
<td>5</td>
<td>7</td>
<td>4</td>
<td>12</td>
<td>28</td>
</tr>
<tr>
<td>Two stages</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>10</td>
<td>21</td>
<td>40</td>
</tr>
<tr>
<td>Multiple stages</td>
<td>1</td>
<td>3</td>
<td>18</td>
<td>16</td>
<td>37</td>
<td>75</td>
</tr>
<tr>
<td>Staircase</td>
<td>0</td>
<td>1</td>
<td>12</td>
<td>12</td>
<td>26</td>
<td>51</td>
</tr>
<tr>
<td>Non-staircase</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>4</td>
<td>11</td>
<td>24</td>
</tr>
</tbody>
</table>

Tab. 5: Number of stages per decade
5 Conclusions

The first obvious fact is that the number of papers produced is rising steadily, which points to a growing acceptance of the field. The mantra often repeated that Finance is the only outside area in which stochastic programming has been accepted is not supported by the data. While financial applications are the most numerous, their
fraction of the total is quite constant over time. In addition there is considerable breadth of other applications.

It is interesting to note, however, that the proportion of financial models among the serious applications is actually decreasing over time: Only two of the sixteen serious applications published between 1996 and 2005 were financial applications. There are many possible explanations for this fact. Application areas outside of finance may be using stochastic programming with increasing frequency, but it is also possible that I did not find everything. One might even argue that the use of stochastic programming has become so routine in the Finance area that new papers and models cannot be published any longer. Whatever the reason, it is clear that stochastic programming is being used in many different fields, and the interest in stochastic programming shows no sign of slowing down.

Acknowledgements

This research was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada. I am grateful to Mark Stoddard for helping with the original search and for setting up the database.

References


