Exponential stability of stochastic singular delay systems with general Markovian switchings

Guoliang Wang1,*,†, Qingling Zhang2,3 and Chunyu Yang4

1School of Information and Control Engineering, Liaoning Shihua University, 113001 Liaoning, China
2Institute of Systems Science, Northeastern University, 110004 Liaoning, China
3State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, 110004 Liaoning, China
4School of Information and Electrical Engineering, China University of Mining and Technology, 221116 Jiangsu, China

SUMMARY

In recent years, Markovian jump systems have received much attention. However, there are very few results on the stability of stochastic singular systems with Markovian switching. In this paper, the discussed system is the stochastic singular delay system with general transition rate matrix in terms of uncertain and partially unknown transition rate matrix. The aim is to answer the question whether there are conditions guaranteeing the underlying system having a unique solution and being exponentially admissible simultaneously. The proposed results show that all the features of the underlying system such as time delay, diffusion, and general Markovian switchings play important roles in the system analysis of exponential admissibility. A numerical example is used to demonstrate the effectiveness of the proposed methods. Copyright © 2014 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Because of the extensive applications in various kinds of practical systems such as electrical systems, economics, chemical processes, and mechanics, singular systems [1, 2] described by differential-algebraic equations have been widely studied. As we know, two additional modes named as impulsive and non-dynamic modes, respectively, are included in singular systems, which are not contained in normal systems. Because of such essential differences, singular systems are usually more complicated, in which the stability, regularity, and impulse elimination (for continuous case) or causality (for discrete case) should be considered simultaneously. On the other hand, many practical systems [3–6] have their system parameters or structures changed abruptly and are usually modeled into Markovian jump systems (MJSs). This kind of systems has two kinds of mechanisms simultaneously. The first one is the time-evolving mechanism which is related to state vector and is continuous in time; the second one named as operation mode or system mode is the event-driven mechanism and driven by the Markov process taking values in a finite set. During the past decades, many results on various kinds of MJSs have emerged, see, for example, [7–17]. In particular, it is known that singular MJSs (SMJSs) [18–20] have natural representation in the description of singular systems experiencing abrupt changes. Up to now, many research topics on SMJSs have been studied in [21–27].
As we know, stochastic systems are very suitable to describe systems having environmental noise. Many important applications of this kind of systems can be found in many branches of science and industry, such as air traffic management [28], communication networks [29], and biology systems [30]. During the past decades, stochastic systems have received much attention, see, for example, [31–35]. When such systems are with jump parameters, they are usually modeled into stochastic hybrid systems (SHSs). Up to now, various kinds of problems of SHSs have been considered in [36–41]. Especially, when there is time delay in SHSs, the exponential stability was considered in [42]. In these references, the discussed systems are all normal stochastic systems, whose derivative matrix is nonsingular and is always simplified as an identical matrix. When the derivative matrix of SHSs is singular, very few results are developed, even for the essential problems of singular systems such as the regularity and impulse elimination. It is mainly because the singular derivative matrix makes the analysis and synthesis quite different to normal SHSs. Recently, the authors in [43] considered the stability of stochastic SMJSs (SSMJSs) without any time delay. In this reference, non-convex assumptions on system matrices are needed to guarantee the uniqueness of solution. Based on the proportional-derivative state feedback control (PDSFC) method [44], the robust control problem of SSMJSs was discussed, where all the system matrices including the derivative matrix have uncertainties. It is said that the problems of regularity and impulse elimination of singular systems in addition to the uniqueness of solution are avoided, because the original SSMJS is transformed into a normal SHS. Moreover, from the existing references about SHSs, it is known that the transition rate matrix (TRM) is very important in system analysis and synthesis and is assumed to be known exactly. Unfortunately, this may be impossible in many practical applications. The first example is the networked control systems. When the underlying system is closed by the real time networks, due to the induced-time delay or packet dropout existing, it is very hard and higher cost to obtain the TRM precisely. Instead, we only have its estimation, or even we only have some elements. Another practical example is the vertical take-off landing helicopter system, which is modeled into an MJS [4]. Because of the external environment (like weather) changes, the switching probabilities among the multiple airspeeds will not be fixed, and not all the switching probabilities are easy to measure. The other practical systems with similar phenomena can also be found in [45, 46]. Based on the existing references [12, 15, 26], it is known that such general TRMs can lead to instability or degrade the performance of a system. Thus, it is necessary and important to consider the related problems of MJSs with general TRMs. More importantly, when the TRM is uncertain or partially unknown, how to study the stability of SSMJSs should be reconsidered. That is because some new problems will emerge because of time delay, diffusion, singular derivative matrix, and general Markovian switching existing simultaneously. To the best of the authors’ knowledge, such problems related to SSMJSs have not been fully investigated and still remain challenging, which need further more investigations.

In this paper, the exponential stability problem of SSMJSs with time-varying delays is considered whose TRM can be uncertain and partially unknown. By the results to be proposed in this paper, though the TRM satisfies the aforementioned general cases, the underlying system will have a unique solution and be exponentially admissible. All the effects of time delay, diffusion, and general TRMs are taken into account.

Notation: \( \mathbb{R}^n \) denotes the \( n \) dimensional Euclidean space, and \( \mathbb{R}^{m \times n} \) is the set of all \( m \times n \) real matrices. \( M^\dagger \) is the general inverse of matrix \( M \). \( \lambda_{\text{max}}(M) \) and \( \lambda_{\text{min}}(M) \) denote the maximum and minimum value of matrix \( M \) respectively. In symmetric block matrices, we use ‘*’ as an ellipsis for the terms induced by symmetry.

2. PROBLEM FORMULATION

Consider a class of SSMJSs described as

\[
\begin{align*}
E dx(t) = & \begin{bmatrix}
A(\eta_1) x(t) + B(\eta_1) x(t - \tau(t)) \\
C(\eta_1) x(t) + D(\eta_1) x(t - \tau(t))
\end{bmatrix} \, dt \\
& + \begin{bmatrix}
C(\eta_1) x(t) + D(\eta_1) x(t - \tau(t))
\end{bmatrix} \, d\omega(t) \\
\end{align*}
\]

\[
x_0 = \phi(t), \, \forall t \in [-\tau, 0]
\]
where \( x(t) \in \mathbb{R}^n \) is the state vector, and \( \omega(t) \) is a one-dimensional Brownian motion or Wiener process. The underlying complete probability space is \( (\Omega, \mathcal{F}, \mathbb{F}_t, \text{and } \mathbb{P}) \) with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions (i.e., it is right continuous and \( \mathcal{F}_0 \) contains the \( \mathbb{P} \)-null sets). Matrix \( E \in \mathbb{R}^{n \times n} \) may be singular, which is assumed to be \( \text{rank}(E) = r \leq n \). Matrices \( A(\eta_t), B(\eta_t), C(\eta_t), \) and \( D(\eta_t) \) are known matrices with compatible dimensions. \( \{\eta_t, t \geq 0\} \) is the Markov process and takes values in a finite set \( S \triangleq \{1, \cdots, N\} \), with TRM \( \Pi \triangleq (\pi_{ij}) \in \mathbb{R}^{N \times N} \) given by

\[
\Pr\{\eta_{t+h} = j | \eta_t = i\} = \begin{cases} 
\pi_{ij} h + o(h) & i \neq j \\
1 + \pi_{ii} h + o(h) & i = j
\end{cases}
\]  

(2)

where \( h > 0, \lim_{h \to 0^+} (o(h)/h) = 0 \), and \( \pi_{ij} \geq 0 \), if \( i \neq j \), \( \pi_{ii} = -\sum_{j \in S} \pi_{ij} \). Time delay \( \tau(t) \) is time-varying and satisfies

\[
0 \leq \tau(t) \leq \tau, \ i(t) \leq \mu < 1, \forall t \geq 0
\]

(3)

For system (1), the following definition and assumption are needed here.

**Definition 1**

System (1) is said to be

1. regular if \( \text{det}(sE - A(\eta_t)) \) is not identical zero for any \( \eta_t \in S \);
2. impulse-free if \( \deg(\text{det}(sE - A(\eta_t))) = \text{rank}(E) \) for any \( \eta_t \in S \);
3. exponentially stable in mean square, if

\[
\lim_{t \to \infty} \sup_{t} \frac{1}{t} \log(\mathbb{E}[x(t, x_0, \eta_0)^2]) \leq -\lambda < 0
\]

for any initial conditions \( x_0 \in \mathbb{R}^n \) and \( \eta_0 \in S \), where \( \lambda > 0 \);
4. exponentially admissible in mean square, if it is regular, impulse-free and exponentially stable in mean square.

**Assumption 1**

The pair \( (E C_i D_i), \forall i \in S \), satisfies \( \text{rank}(E C_i D_i) \leq \text{rank}(E) \).

It is worth mentioning that the additional assumption needed here is important to guarantee the uniqueness of the solution to system (1). When a system is analyzed and synthesized, the requirement of stability is a fundamental problem, while the uniqueness of solution is an essential precondition. As we know, for common singular systems including SMJSs such as [1, 18, 21–23, 25], the uniqueness of solution is usually determined by the regularity and impulse elimination of the underlying system. In this case, no additional assumption described in Assumption 1 is needed. Unfortunately, when there is Brownian motion in an SMJS, the corresponding problem is quite different from the ones in the references mentioned earlier. Due to Brownian motion existing, it makes the analysis of the unique solution to SSMJS (1) very difficult, where only the traditional requirements of regularity and impulse elimination are not enough. In order to ensure the uniqueness of solution, an additional assumption is needed here. However, it is seen that Assumption 1 is a non-convex condition, which cannot be solved easily. In order to make the obtained results solved by the standard softwares easily, system (1) in this paper is assumed with the following form:

\[
\begin{cases}
E \dot{x}(t) = [A(\eta_t)x(t) + B(\eta_t)x(t - \tau(t)) + C(\eta_t)x(t) + D(\eta_t)x(t - \tau(t)) + \omega(t) \\
x_0 = \phi(t), \forall t \in [-\tau, 0]
\end{cases}
\]

(4)

It is obtained that the condition about the corresponding matrices similar to the pair in Assumption 1 is always satisfied. That is, for any \( i \in S \), one has

\[
\text{rank}(E E C_i E D_i) \leq \min\{\text{rank}(E), \text{rank}(C_i), \text{rank}(D_i)\} \leq \text{rank}(E)
\]

(5)
Based on this fact, without loss of generality, we only consider system (4). In this paper, the actual TRM $\Sigma$ is first assumed to be obtained inexactly. It has admissible uncertainty $\Delta \Sigma = (\Delta \tilde{\pi}_{ij})$ described as

$$\Pi = \tilde{\Pi} + \Delta \tilde{\Pi} \text{ with } |\Delta \pi_{ij}| \leq \epsilon_{ij}, \epsilon_{ij} \geq 0, j \neq i \quad (6)$$

In (6), TRM $\tilde{\Pi} \triangleq (\tilde{\pi}_{ij})$ is a known constant estimation of $\Pi$ with $\tilde{\pi}_{ij} \geq 0$ satisfying (2). It is assumed that $\Delta \tilde{\pi}_{ij}, j \neq i$, takes any value in $[-\epsilon_{ij}, \epsilon_{ij}]$ with $\epsilon_{ij} \geq 0$. Then, it is concluded that $|\Delta \tilde{\pi}_{ii}| \leq -\epsilon_{ii}$, where $\epsilon_{ii} \triangleq -\sum_{j=1, j \neq i}^{N} \epsilon_{ij}$ and $\alpha_{ij} \triangleq \tilde{\pi}_{ij} - \epsilon_{ij}, \forall i, j \in S$.

### 3. MAIN RESULTS

#### Theorem 1
For any given initial conditions $x_0 \in \mathbb{R}^n$ and $\eta_0 \in S$, there is a unique solution $x(t)$ to system (4) with (6) on $[0, \infty)$ and is robust exponentially admissible in mean square, if there exist matrices $P_i, Q_i > 0$, $W_i = W_i^T$, $V_i = V_i^T$, $R_i > 0, T_i > 0$, and scalars $\epsilon_i > 0, \lambda_1 > 0, \lambda_2 > 0$ such that the following conditions hold for all $i \in S$,

$$E^T P_i = P_i^T E \geq 0 \quad (7)$$

$$\begin{bmatrix} \Omega_i & W_i \\ * & -T_i \end{bmatrix} < 0 \quad (8)$$

$$E^T P_j - E^T P_i - W_i \leq 0, \forall j \neq i \in S$$

$$\begin{bmatrix} \tilde{\Omega}_i & B_i^T P_i & D_i^T P_i^T E \\ * & -\epsilon_i I & 0 \\ * & * & -\epsilon_i I \end{bmatrix} < 0 \quad (10)$$

$$\begin{bmatrix} \hat{\Omega}_i & V_i \\ * & -R_i \end{bmatrix} < 0 \quad (11)$$

$$Q_j - Q_i - V_i \leq 0, \forall j \neq i \in S$$

$$\begin{bmatrix} \hat{\Omega}_i & 2P_i^T B_i & W_i \\ * & -Q_i & 0 \\ * & * & -T_i \end{bmatrix} < 0 \quad (13)$$

$$\tau < \frac{\lambda_1}{\lambda_2} \quad (14)$$

where

$$\Omega_i = \hat{\Omega}_i + C_i^T E^T P_i C_i + \epsilon_i I + \epsilon_i C_i^T C_i + \lambda_1 I$$

$$\hat{\Omega}_i = -(1 - \mu) Q_i + D_i^T E^T P_i D_i$$

$$\hat{\Omega}_i = \sum_{j \neq i} \alpha_{ij}(Q_j - Q_i) + 0.25\epsilon_i^2 R_i - \epsilon_{ii} V_i - \lambda_2 I$$

$$\hat{\Omega}_i = \text{sym}(A_i^T P_i) + Q_i + \sum_{j \neq i} \alpha_{ij} E^T (P_j - P_i) - \epsilon_{ii} W_i + 0.25\epsilon_i^2 T_i$$

In this case, the unique solution has the property that
\[
\lim_{t \to \infty} \sup_t \frac{1}{t} \log \left( \mathcal{E} |x(t; x_0)|^2 \right) \leq -\lambda < 0
\] (15)

The positive scalar \( \lambda \) is the unique root to
\[
\alpha \lambda + (\lambda_2 + \alpha_1 \lambda) \tau e^{\lambda t} = \lambda_1
\] (16)
with condition
\[
\lambda \leq \frac{1}{\tau} \ln 4
\] (17)

where \( \alpha = \max_{i \in S} \{ \lambda_{\max}(E^T P_i) \} \), \( \alpha_1 = \max_{i \in S} \{ \lambda_{\max}(Q_i) \} \).

**Proof**

Let \( x_t = x(t + s), -\tau \leq s \leq 0 \), it is known that \( \{x_t, \eta_t\}_{t \geq 0} \) is a Markov process. Choose the Lyapunov functional as
\[
V_1(x_t, \eta_t, t) = e^{\lambda t} V(x_t, \eta_t, t)
\] (18)

where
\[
V(x_t, \eta_t, t) = x^T(t) E^T P(\eta_t) x(t) + \int_{t-\tau(t)}^t x^T(s) Q(\eta_t) x(s) ds
\]

By the generalized Itô formula, one has
\[
\mathcal{E} V_1(x_t, \eta_t, t) = \mathcal{E} V_1(x_0, \eta_0, 0) + \mathcal{E} \int_0^t \mathcal{L} V_1(x_s, \eta_s, s) ds
\] (19)

where
\[
\mathcal{L} V_1(x_t, i, t) = e^{\lambda t} \left( \lambda V(x_t, i, t) + \mathcal{L} V(x_t, i, t) \right)
\]

and \( \mathcal{L} \) is the weak infinitesimal generator of random process \( \{x_t, \eta_t\}_{t \geq 0} \). For each \( \eta_t = i \in S \), one has
\[
\mathcal{L} V(x_t, i, t) = \lim_{h \to 0} \frac{1}{h} [\mathcal{E} (V(x_{t+h}, \eta_{t+h}, t+h) | x_t, \eta_t = i) - V(x_t, \eta_t = i, t)]
\]
\[
= 2x^T(t) \left( P_i^T A_i x(t) + P_i^T B_i y(t) \right) + x^T(t) \sum_{j \in S} \pi_{ij} E^T P_j x(t)
\]
\[
+ (C_i x(t) + D_i y(t))^T E^T (E^T)^\dagger E^T P_i E (C_i x(t) + D_i y(t))
\]
\[
+ x^T(t) Q_i x(t) - (1 - \tilde{\mu}(t)) y^T(t) Q_1 y(t)
\]
\[
+ \sum_{j \in S} \pi_{ij} \int_{t-\tau(t)}^t x^T(s) Q_j x(s) ds
\] (20)
\[
\leq 2x^T(t) (P_i^T A_i x(t) + P_i^T B_i y(t)) + x^T(t) \sum_{j \in S} \pi_{ij} E^T P_j x(t)
\]
\[
+ (C_i x(t) + D_i y(t))^T E^T P_i (C_i x(t) + D_i y(t))
\]
\[
+ x^T(t) Q_i x(t) - (1 - \mu) y^T(t) Q_1 y(t)
\]
\[
+ \sum_{j \in S} \pi_{ij} \int_{t-\tau(t)}^t x^T(s) Q_j x(s) ds
\]
where \( y(t) = x(t - \tau(t)) \). For any \( \varepsilon_i > 0 \), we have the following inequalities:

\[
2x^T(t)P_i^TB_iy(t) \leq \varepsilon_i x^T(t)x(t) + \varepsilon_i^{-1}y^T(t)B_i^TP_iP_i^TB_iy(t)
\]  

(21)

\[
(C_i x(t) + D_i y(t))^T E^TP_i(C_i x(t) + D_i y(t)) = x^T(t)C_i^T E P_i C_i x(t) + y^T(t)D_i^T E P_i D_i y(t)
\]

\[
+ 2x^T(t)C_i^T E P_i D_i y(t)
\]

\[
\leq x^T(t)C_i^T E P_i C_i x(t) + \varepsilon_i x^T(t)C_i^T C_i x(t) + y^T(t)D_i^T E P_i D_i y(t)
\]

\[
+ \varepsilon_i^{-1}y^T(t)D_i^T P_i E E^T P_i D_i y(t)
\]  

(22)

On the other hand, for uncertain TRM described in (6), we obtain that

\[
\sum_{j \in S} \pi_{ij} E^T P_j = \sum_{j \in S} (\tilde{\pi}_{ij} + \Delta \tilde{\pi}_{ij}) E^T P_j + \sum_{j \in S} \pi_{ij} W_i
\]

\[
= \sum_{j \neq i} \alpha_{ij} E^T (P_j - P_i) - \Delta \tilde{\pi}_{ii} W_i - \varepsilon_{ii} W_i
\]

\[
+ \sum_{j \neq i} (\Delta \tilde{\pi}_{ij} + \varepsilon_{ij})(E^T P_j - E^T P_i - W_i)
\]  

(23)

where \( W_i^T = W_i \). For any \( T_i > 0 \), it is concluded that

\[
- \Delta \tilde{\pi}_{ii} W_i \leq 0.25 \varepsilon_{ii}^2 T_i + W_i T_i^{-1} W_i
\]  

(24)

Similarly, for any matrices \( V_i^T = V_i \) and \( R_i > 0 \), one gets that

\[
\sum_{j \in S} \pi_{ij} Q_j = \sum_{j \neq i} \alpha_{ij}(Q_j - Q_i) - \Delta \tilde{\pi}_{ii} V_i - \varepsilon_{ii} V_i
\]

\[
+ \sum_{j \neq i} (\Delta \tilde{\pi}_{ij} + \varepsilon_{ij})(Q_j - Q_i - V_i)
\]

\[
\leq \sum_{j \neq i} \alpha_{ij}(Q_j - Q_i) + 0.25 \varepsilon_{ii}^2 R_i + V_i R_i^{-1} V_i - \varepsilon_{ii} V_i
\]

\[
\leq \lambda_2 I
\]  

(25)

is guaranteed by condition (12) and the following condition:

\[
\lambda_{\text{max}} \left( \sum_{j \neq i} \alpha_{ij}(Q_j - Q_i) + 0.25 \varepsilon_{ii}^2 R_i + V_i R_i^{-1} V_i - \varepsilon_{ii} V_i \right) < \lambda_2
\]  

(26)

By the Schur complement Lemma, it is known that (11) implies (26). Moreover, similar to [42], for any given corresponding eigenvector \( v \neq 0 \) of \( \lambda_{\text{min}}(Q_j) = \min_{x \in S}(\lambda_{\text{min}}(Q_j)) \), we have

\[
\lambda_{\text{max}} \left( \sum_{j \in S} \pi_{ij} Q_j \right) |v|^2 \geq v^T \sum_{j \in S} \pi_{ij} Q_j v \geq 0
\]  

(27)
Based on (26) and (27), one obtains $\lambda_2 > 0$. Taking into account (14), it is concluded that there is a unique root to (16). Based on these and taking into account (8)–(13), one concludes that

\[
\mathcal{L}V(x_t, i, t) \leq x^T(t)(\Omega_i - \lambda_1 I + W_i T_i^{-1} W_i)x(t) + y^T(t)(\hat{\Omega}_i + \epsilon_i^{-1} B_i^T P_i P_i^T B_i
\]

\[
+ \epsilon_i^{-1} D_i^T P_i^T E E^T P_i D_i)y(t) + \int_{t-\tau(t)}^{t} x^T(s) \left( \hat{\Omega}_i + \lambda_2 I + V_i R_i^{-1} V_i \right) x(s) ds
\]

\[
\leq -\lambda_1 |x(t)|^2 + \lambda_2 \int_{t-\tau(t)}^{t} |x(s)|^2 ds
\]

(28)

On the other hand, it is known that

\[
V(x_t, i, t) \leq \alpha |x(t)|^2 + \alpha_1 \int_{t-\tau}^{t} |x(s)|^2 ds
\]

(29)

Considering (18), (19), (28), and (29), we have

\[
\mathcal{E}V_1(x_t, \eta_t, t) \leq \mathcal{E}V_1(x_0, \eta_0, 0) + \mathcal{E} \left( \int_{0}^{t} \lambda e^{\lambda s} x^T(s) E^T P(\eta_0)x(s) ds 
\right.
\]

\[
+ \int_{0}^{t} \lambda e^{\lambda s} \int_{s-\tau}^{s} x^T(\theta) Q(\eta_0)x(\theta) d\theta ds
\]

\[
- \int_{0}^{t} \lambda_1 e^{\lambda s} |x(s)|^2 ds + \int_{0}^{t} e^{\lambda s} \int_{s-\tau}^{s} \lambda_2 |x(\theta)|^2 d\theta ds
\]

\[
\leq \mathcal{E}V_1(x_0, \eta_0, 0) - (\lambda_1 - \alpha \lambda) \mathcal{E} \left( \int_{0}^{t} e^{\lambda s} |x(s)|^2 ds 
\right.
\]

\[
+ (\lambda_2 + \lambda \alpha_1) \mathcal{E} \left( \int_{0}^{t} e^{\lambda s} \int_{s-\tau}^{s} |x(\theta)|^2 d\theta ds \right)
\]

(30)

It is very known that

\[
\int_{0}^{t} e^{\lambda s} \int_{s-\tau}^{s} |x(\theta)|^2 d\theta ds \leq \int_{t-\tau}^{t} |x(s)|^2 \int_{\theta}^{\theta+\tau} e^{\lambda s} ds d\theta 
\]

\[
\leq \tau e^{\lambda \tau} \int_{t-\tau}^{t} e^{\lambda s} |x(s)|^2 ds
\]

Then, we have

\[
\mathcal{E}V_1(x_t, \eta_t, t) \leq \mathcal{E}V_1(x_0, \eta_0, 0) - (\lambda_1 - \alpha \lambda) \mathcal{E} \left( \int_{0}^{t} e^{\lambda s} |x(s)|^2 ds 
\right.
\]

\[
+ (\lambda_2 + \lambda \alpha_1) \tau e^{\lambda \tau} \mathcal{E} \left( \int_{0}^{t} e^{\lambda s} |x(s)|^2 ds 
\right.
\]

\[
+ (\lambda_2 + \lambda \alpha_1) \tau e^{\lambda \tau} \mathcal{E} \left( \int_{0}^{t} e^{\lambda s} |x(s)|^2 ds 
\right.
\]

\[
\leq \mathcal{E}V_1(x_0, \eta_0, 0) + [\lambda_1 + \alpha \lambda + (\lambda_2 + \lambda \alpha_1) \tau e^{\lambda \tau}] \mathcal{E} \left( \int_{0}^{t} e^{\lambda s} |x(s)|^2 ds 
\right.
\]

\[
+ (\lambda_2 + \lambda \alpha_1) \tau e^{\lambda \tau} \mathcal{E} \left( \int_{-\tau}^{0} e^{\lambda s} |x(s)|^2 ds 
\right.
\]

\[
\leq \mathcal{E}V_1(x_0, \eta_0, 0) + (\lambda_2 + \lambda \alpha_1) \tau e^{\lambda \tau} \mathcal{E} \left( \int_{-\tau}^{0} |x(s)|^2 ds 
\right.
\]

(31)

On the other hand, it is known that there are always two nonsingular matrices $M$ and $N$ such that

\[
MEN = \begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix}, \quad \tilde{P}_i \triangleq M^{-T} P_i N = \begin{bmatrix}
\tilde{P}_{i1} & \tilde{P}_{i2} \\
\tilde{P}_{i3} & \tilde{P}_{i4}
\end{bmatrix}
\]

(32)
By pre-multiplying and post-multiplying both sides of (7) with $N^T$ and $N$ respectively, one has

$$N^T E^T M^T M^{-T} P_i N = N^T P_i^T M^{-1} M E N \geq 0$$

(33)

which implies $\tilde{P}_{i2} = 0$. Based on this, one has

$$e^{\lambda t} \mathbb{E} \left( x^T(t) N^T E^T M^T N^{-T} P_i N \tilde{x}(t) \right) \geq e^{\lambda t} \mathbb{E} \left( \min_{i \in S} \{ \tilde{x}^T(t) N^T E^T M^T P_i N \tilde{x}(t) \} \right)$$

$$\geq e^{\lambda t} \mathbb{E} \left( \min_{i \in S} \left\{ \left[ \begin{array}{c} \tilde{x}_1^T(t) \\ \tilde{x}_2^T(t) \end{array} \right] \left[ \begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right] \left[ \begin{array}{c} \tilde{P}_{11} \\ \tilde{P}_{13} \end{array} \right] \left[ \begin{array}{c} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{array} \right] \right\} \right)$$

$$\geq e^{\lambda t} \mathbb{E} \left( \min_{i \in S} \{ \lambda_{\text{max}}(\tilde{P}_{11}) |\tilde{x}_1(t)|^2 \} \right)$$

(34)

where $\tilde{x}(t) = N^{-1} x(t)$. Thus, $\tilde{x}_1(t)$ satisfies

$$\lim_{t \to \infty} \text{sup} \frac{1}{t} \log \mathbb{E} |\tilde{x}_1(t)|^2 \leq -\lambda < 0$$

Next, we will consider $\tilde{x}_2(t)$. In order to obtain the stability of $\tilde{x}_2(t)$, we should first prove that the system is regular and impulse-free. By conditions (9) and (13), similar to the method dealing with (23), it is obtained that

$$\left[ \begin{array}{cc} \Omega_i & 2 \tilde{P}_{i}^T B_i \\ * & -Q_i \end{array} \right] < 0$$

(35)

where

$$\Omega_i = \text{sym} (A_i^T P_i) + \sum_{j \in S} \pi_{ij} E^T P_j + Q_i$$

It is also obtained that $\Omega_i < 0$. Similarly, pre-multiplying and post-multiplying $\Omega_i < 0$ by $N^T$ and its transpose respectively, and taking into account (33) and (35), we conclude that

$$\tilde{A}_i^T \tilde{P}_i + \tilde{P}_i^T \tilde{A}_i + \sum_{j \in S} \pi_{ij} N^T E^T P_j N + \tilde{Q}_i < 0$$

(36)

where

$$\tilde{A}_i \triangleq MA_i N = \left[ \begin{array}{cc} \tilde{A}_{i1} & \tilde{A}_{i2} \\ \tilde{A}_{i3} & \tilde{A}_{i4} \end{array} \right], \quad \tilde{Q} \triangleq N^T Q_i N = \left[ \begin{array}{cc} \tilde{Q}_{i1} & \tilde{Q}_{i2} \\ \tilde{Q}_{i3} \end{array} \right] > 0$$

It is equivalent to

$$\left[ \begin{array}{cc} * & * \\ * & \tilde{P}_{i4}^T \tilde{A}_{i4} + \tilde{A}_{i4}^T \tilde{P}_{i4} + \tilde{Q}_{i3} \end{array} \right] < 0$$

(37)

where $*$ denotes the terms not used in (37). Then, we have $\tilde{P}_{i4}^T \tilde{A}_{i4} + \tilde{A}_{i4}^T \tilde{P}_{i4} < 0$ implying $[\tilde{A}_{i4}] \neq 0$. Thus, it is obtained that the pair $(E, A_i)$ is regular and impulse-free for any $i \in S$. Because system is regular and impulse-free, let

$$M_i \triangleq \left[ \begin{array}{cc} I & -\tilde{A}_{i2} \tilde{A}_{i4}^{-1} \\ 0 & \tilde{A}_{i4}^{-1} \end{array} \right] M$$

(38)

we obtain that

$$E_i \triangleq M_i EN = \left[ \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right], \quad \hat{A}_i \triangleq M_i A_i N = \left[ \begin{array}{cc} \hat{A}_{i1} & 0 \\ \hat{A}_{i3} & I \end{array} \right]$$

(39)
where $\hat{A}_{i1} = \tilde{A}_{i1} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\tilde{A}_{i3}$ and $\hat{A}_{i3} = \tilde{A}_{i4}^{-1}\tilde{A}_{i3}$. Similarly, one has
\[
\begin{align*}
\bar{C}_i \triangleq M_i C_i N &= \begin{bmatrix} \bar{C}_{i1} & \bar{C}_{i2} \\ \bar{C}_{i3} & \bar{C}_{i4} \end{bmatrix},
\bar{D}_i \triangleq M_i D_i N &= \begin{bmatrix} \bar{D}_{i1} & \bar{D}_{i2} \\ \bar{D}_{i3} & \bar{D}_{i4} \end{bmatrix},
\bar{B}_i \triangleq M_i B_i N &= \begin{bmatrix} \bar{B}_{i1} & \bar{B}_{i2} \\ \bar{B}_{i3} & \bar{B}_{i4} \end{bmatrix},
\bar{P}_i \triangleq M_i^{-T} P_i N &= \begin{bmatrix} \bar{P}_{i1} & \bar{P}_{i2} \\ \bar{P}_{i3} & \bar{P}_{i4} \end{bmatrix}
\end{align*}
\]
(40)

where $\bar{P}_{i2} = 0$ comes from $E^T P_i = P_i^T E$. Because $\dot{x}(t) = N^{-1}x(t)$, we have
\[
d\dot{x}_1(t) = [\hat{A}_{i1}\dot{x}_1(t) + \bar{B}_{i1}\dot{x}_1(t - \tau(t)) + \bar{B}_{i2}\dot{x}_2(t - \tau(t))]dt
+ \left[\bar{C}_{i1}\dot{x}_1(t) + \bar{C}_{i2}\dot{x}_2(t) + \bar{D}_{i1}\dot{x}_1(t - \tau(t)) + \bar{D}_{i2}\dot{x}_2(t - \tau(t))\right]d\omega(t)
\]
(41)

On the other hand, it is known that (35) is equivalent to
\[
\begin{bmatrix} N^T & 0 \\ * & N^T \end{bmatrix} \begin{bmatrix} \Omega_i & 2P_i^T B_i \\ * & -Q_i \end{bmatrix} \begin{bmatrix} N & 0 \\ * & N \end{bmatrix} < 0
\]
(42)

which implies
\[
\begin{bmatrix} \bar{P}_{i1} + \bar{P}_{i4}^T + \bar{Q}_{i1} & 2\bar{P}_{i4}^T B_{i4} \\ * & -\bar{Q}_{i3} \end{bmatrix} < 0
\]

Then, one has
\[
(2\bar{P}_{i4}^T) \bar{Q}_{i1} - \bar{Q}_{i3} < 0
\]
(43)

which implies
\[
2\|\bar{B}_{i4}\| < 1
\]
(44)

Obviously, it is seen that
\[
\|\bar{B}_{i4}\| \leq \rho < \frac{1}{2}
\]
(45)

where $\rho = \max_{i \in S}\{\|\bar{B}_{i4}\|\}$. Taking into account (41), one has
\[
\mathcal{E}|\hat{x}_2(t)| \leq \rho\mathcal{E}\left(\sup_{-\tau \leq s \leq t}|x_2(s)|\right) + \kappa e^{-\frac{1}{2}\tau}
\]
(46)

where $\kappa = \max_{i \in S}\left\{\|\hat{A}_{i3}\|, \|\bar{B}_{i3}\|\right\} \left(1 + e^{\frac{1}{2}\tau}\right)$. By condition (17), it is known that $e^{\frac{1}{2}\tau} \leq 2$ which implies
\[
\rho e^{\frac{1}{2}\tau} \leq 2\rho < 1
\]
(47)

By [47], it is concluded that
\[
\mathcal{E}|\hat{x}_2(t)| \leq \left(|\hat{x}_2(t)| + \frac{\kappa}{1 - \rho e^{\frac{1}{2}\tau}}\right) e^{-\frac{1}{2}\tau}
\]
(48)
where \(|\hat{x}_2(t)|_\epsilon = \sup_{-\tau \leq s \leq 0} |\hat{x}_2(s)|\). Then, we have
\[
\lim_{t \to \infty} \frac{1}{t} \log(\epsilon |\hat{x}(t)|^2) \leq -\lambda < 0
\] (49)

Considering \(\hat{x}(t) = N^{-1}x(t)\), one has that \(x(t)\) is exponentially stable in mean square.

Finally, we will prove that there is a unique solution on \([0, \infty)\). Let \(t_0 = 0\) and define a sequence of stopping time
\[
t_{k+1} = \inf \{ t > t_k : \eta_t \neq \eta_{t_k} \}
\]
for all \(k \geq 0\). It is concluded that for any \(k \geq 0\), \(\eta_t = \eta_{t_k}\) is constant for all \(t \in [t_k, t_{k+1})\) and \(t_k \to \infty\) as \(k \to \infty\). We first show that there is a unique solution to equation (7) with \(\eta_t = i\) on interval \([t_0, t_1)\). For the first equation of (41) with any given \(\hat{x}_2(t)\), it is very known that there is a unique solution on interval \([t_0, t_1)\). Meanwhile, by [48], it is known that the latter equation of (41) also has a unique solution on \([t_0, t_1)\), if \((E, A_i)\) with \(\eta_t = i\) on interval \([t_0, t_1)\) is regular and impulse-free. Similarly, we can also show that there is a unique solution on interval \([t_1, t_2)\) for any given initial condition \(\hat{x}(t_1)\) and so on. Then, there is a unique solution \(\hat{x}(t)\) on interval \([0, \infty)\). Finally, we have a unique solution \(x(t)\) on \([0, \infty)\). This completes the proof. \(\square\)

Remark 1
From Theorem 1, it is seen that time delay, diffusion, and uncertain TRM play important roles in the analysis of SSMJS (4), where the uniqueness and exponential admissibility of solution are guaranteed. Moreover, the theory developed here may be used to many different and complicated situations such as stabilization, signal estimation, and so on.

It is said that the method proposed in Theorem 1 is also applied to another general case of TRM partially unknown. For example, a partly unknown \(\Pi\) may be expressed as
\[
\Pi = \begin{bmatrix}
p_{11} & p_{12} & ? & ? \\
p_{21} & ? & ? & p_{24} \\
? & ? & p_{33} & ? \\
p_{41} & p_{42} & p_{43} & p_{44}
\end{bmatrix}
\]

where ‘?’ represents the unknown element. Then for any \(i \in S\), define \(S_i^i = S_k^i + \tilde{S}_k^i\) such that
\[
S_k = \{ j : \pi_{ij} \text{ is known} \} \quad \text{and} \quad \tilde{S}_k = \{ j : \pi_{ij} \text{ is unknown} \}
\] (50)

where \(\sigma = \min_{i \in S_k} \{ \pi_{ii} \}\) is known. Moreover, \(S_i^i\) and \(\tilde{S}_k^i\) are further described as
\[
S_k^i = \{ k^i_1, \cdots, k^i_m \} \quad \text{and} \quad \tilde{S}_k^i = \{ \tilde{k}^i_1, \cdots, \tilde{k}^i_{N-i} \}
\]

where \(k^i_j \in \mathbb{Z}^+\) is the column index of the \(j\)th known element in the \(i\)th row of \(\Pi\), and the column index of the \((N-j)\)th unknown element in the \(i\)th row of \(\Pi\) is represented as \(\tilde{k}^i_{N-j} \in \mathbb{Z}^+\). For this general case, similar to Theorem 1, we have the following result.

Theorem 2
For any given initial conditions \(x_0 \in \mathbb{R}^n\) and \(\eta_0 \in S\), there is a unique solution \(x(t)\) to system (4) with (50) on \([0, \infty)\) and is exponentially admissible in mean square, if there exist matrices \(P_i\), \(Q_i > 0\), \(W_i = W_i^T\), \(V_i = V_i^T\), and scalars \(\epsilon_i > 0\), \(\lambda_1 > 0\), \(\lambda_2 > 0\) such that (7), \(i \in S\), (9), \(j \neq i \notin S^i_k\), (10), \(i \in S\), (12), \(j \neq i \notin S^i_k\), (14), and the following conditions hold,
\[
\Phi^i_1 + C_i^T P_i C_i + \epsilon_i I + \epsilon_i C_i^T C_i + \lambda_1 I < 0, \quad i \in S_k^i
\] (51)
\[
\tilde{\Phi}^i_1 + C_i^T P_i C_i + \epsilon_i I + \epsilon_i C_i^T C_i + \lambda_1 I < 0, \quad i \in \tilde{S}_k^i
\] (52)
\[
\Phi^i_2 < 0, \quad i \in S_k^i
\] (53)
EXPONENTIAL STABILITY OF SSMJSs WITH GENERAL TRMs

\[ \Phi_i^2 < 0, \ i \in S_k^i \]  

\[ \begin{bmatrix} \Phi_i 2 P_i B_i \\ * -Q_i \end{bmatrix} < 0, \ i \in S_k^i \]  

where

\[ \Phi_i = \text{sym} \left( A_i^T P_i \right) + Q_i + \sum_{j \in S_k^i, j \neq i} \pi_{ij} E^T (P_j - P_i - W_i) - \pi_{ii} W_i \]

\[ \Phi_i = \text{sym} \left( A_i^T P_i \right) + Q_i + \sum_{j \in S_k^i, j \neq i} \pi_{ij} E^T (P_j - P_i - W_i) - \sigma W_i \]

\[ \Phi_i^2 = \sum_{j \in S_k^i, j \neq i} \pi_{ij} (Q_j - Q_i - V_i) - \pi_{ii} V_i - \lambda_2 I \]

\[ \Phi_i^2 = \sum_{j \in S_k^i, j \neq i} \pi_{ij} (Q_j - Q_i - V_i) - \sigma V_i - \lambda_2 I \]

In this case, the unique solution has property (15), and the positive scalar \( \lambda \) is the unique root to equation (16) with condition (17). The other parameters are given in Theorem 1.

**Proof**

Via the similar methods in (21)–(23) and (25) and taking into account (50), it is obtained that

\[ \mathcal{L} V(x_i, i, t) \leq x^T(t) \left[ \text{sym}(A_i^T P_i) + Q_i + C_i^T E^T P_i C_i + \varepsilon_i I + \varepsilon_i C_i^T C_i \right. \]

\[ + \sum_{j \in S_k^i, j \neq i} \pi_{ij} (E^T P_j - E^T P_i - W_i) - \pi_{ii} W_i \]

\[ + \sum_{j \in S_k^i, j \neq i} \pi_{ij} (E^T P_j - E^T P_i - W_i) \right] x(t) \]

\[ + y^T(t) \left( \hat{\Omega}_i + \varepsilon_i^{-1} B_i^T P_i P_i^T B_i + \varepsilon_i^{-1} D_i^T P_i E E^T P_i D_i \right) y(t) \]

\[ + \int_{t-\tau(t)}^{t} x^T(s) \left[ \sum_{j \in S_k^i, j \neq i} \pi_{ij} (Q_j - Q_i - V_i) - \pi_{ii} V_i \right] x(s) ds \]

\[ + \int_{t-\tau(t)}^{t} x^T(s) \sum_{j \in S_k^i, j \neq i} \pi_{ij} (Q_j - Q_i - V_i) x(s) ds \]
By conditions (9), (10), (12), one has (57) implied by
\[
L V(x_{t}, i, t) \leq x^T(t) \left[ \text{sym}(A^T_i P_i) + Q_i + C^T_i E^T P_i C_i + \varepsilon_i I + \varepsilon_i C^T_i C_i 
\right. 
\begin{aligned}
&+ \sum_{j \in S^i_k, j \neq i} \pi_{ij}(E^T P_j - E^T P_i - W_i) - \pi_{ii} W_i \bigg] x(t) 
&+ \int_{t_{-\tau(t)}}^{t} x^T(s) \left[ \sum_{j \in S^i_k, j \neq i} \pi_{ij}(Q_j - Q_i - V_i) - \pi_{ii} V_i \right] x(s) ds 
\end{aligned}
(58)
\]
Considering property (50) and taking into account (51)–(54), it is concluded that (58) is guaranteed by
\[
L V(x_{t}, i, t) \leq x^T(t) \left[ \text{sym}(A^T_i P_i) + Q_i + C^T_i E^T P_i C_i + \varepsilon_i I + \varepsilon_i C^T_i C_i 
\right. 
\begin{aligned}
&+ \sum_{j \in S^i_k, j \neq i} \pi_{ij}(E^T P_j - E^T P_i - W_i) - \pi_{ii} W_i \bigg] x(t) 
&+ \int_{t_{-\tau(t)}}^{t} x^T(s) \left[ \sum_{j \in S^i_k, j \neq i} \pi_{ij}(Q_j - Q_i - V_i) - \pi_{ii} V_i \right] x(s) ds 
\end{aligned}
(59)
\]
and
\[
L V(x_{t}, i, t) \leq x^T(t) \left[ \text{sym}(A^T_i P_i) + Q_i + C^T_i E^T P_i C_i + \varepsilon_i I + \varepsilon_i C^T_i C_i 
\right. 
\begin{aligned}
&+ \sum_{j \in S^i_k, j \neq i} \pi_{ij}(E^T P_j - E^T P_i - W_i) - \pi_{ii} W_i \bigg] x(t) 
&+ \int_{t_{-\tau(t)}}^{t} x^T(s) \left[ \sum_{j \in S^i_k, j \neq i} \pi_{ij}(Q_j - Q_i - V_i) - \pi_{ii} V_i \right] x(s) ds 
\end{aligned}
(60)
\]
On the other hand, as for (35), via similar method dealing with (50), one gets that it is implied by (55) and (56). The next process is similar to the proof of Theorem 1. This completes the proof. \[\square\]

Remark 2
Because of rank(E) = r \leq n, similar results for normal SHSs can be obtained directly. At the same time, if the TRM is known exactly, one also has the similar results for SSMJS (4) which are omitted here.
When the TRM is exact and completely known, system (4) without time delay $\tau(t)$ will be reduced to
\[
\begin{cases}
Edx(t) = A(\eta_t)x(t)dt + EC(\eta_t)x(t)d\omega(t) \\
x_0 = x(0)
\end{cases}
\]  
(61)

Then, the results obtained earlier will be reduced to the following corollary.

**Corollary 1**

For any given initial conditions $x_0 \in \mathbb{R}^n$ and $\eta_0 \in S$, there is a unique solution $x(t)$ to system (61) with exactly known TRM on $[0, \infty)$ and is exponentially admissible in mean square, if there exist matrix $P_i$ and scalar $\lambda_1 > 0$ such that (7) and the following condition hold for all $i \in S$,
\[
sym\left(A_i^TP_i\right) + \sum_{j \in S} \pi_{ij} E^T P_j + C_i^T E^T P_iC_i + \lambda_1 I < 0
\]  
(62)

In this case, the unique solution has property (15), and the positive scalar $\lambda$ is obtained by
\[
\lambda = \frac{\lambda_1}{\alpha}
\]  
(63)

**Remark 3**

It is worth mentioning that similar results were given in [43], which were obtained by exploiting the Kronecker product method and whose matrix $G$ in the Lyapunov function is a common matrix. In this corollary, the corresponding matrix $P_i$ depends on system mode $\eta_t$. Moreover, based on the developed stability results in this paper, we can also study the stabilization problem directly. Especially, a kind of stochastic controller only in the diffusion section may be designed, which is described as
\[
u_\omega(t) = EK(\eta_t)x(t)
\]  
(64)

For example, the closed-loop system (61) is described as
\[
\begin{cases}
Edx(t) = A(\eta_t)x(t)dt + u_\omega(t)d\omega(t) \\
x_0 = x(0)
\end{cases}
\]  
(65)

In the aforementioned reference, it is said that a linear state-dependent diffusion can be used to almost surely stabilize an SMJS, where the corresponding pencil should be regular. In order to satisfy this requirement, some non-convex conditions such as $\text{rank}([E \ H_t]) = k$ with $k$ equaling to the one in $\text{deg}(\text{det}(sE - A_t)) = k$ are needed in this reference. Unfortunately, no more detailed method or information on how to deal with such conditions in a convex way is presented. Based on the results in this paper, the stochastic stabilizing controller (64) without the rank equation constraint is proposed, whose existence condition will be proposed in terms of LMIs. In this sense, it is said that Corollary 1 may be seen as an extension and convex realization of [43].

4. NUMERICAL EXAMPLE

**Example 1**

Consider a two-dimensional SSMJS of form (4) with $\eta_t \in S = \{1, 2, 3\}$, and its parameters are given by

\[
A_1 = \begin{bmatrix}
-3 & -1 \\
2 & -2.5
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
-3.7 & 1.6 \\
3 & -2.5
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
-1 & 0.5 \\
1 & -3.5
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
-1 & -0.1 \\
0 & 0.2
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
-0.2 & 0.1 \\
0.3 & -0.3
\end{bmatrix}, \quad B_3 = \begin{bmatrix}
-0.4 & -0.1 \\
0 & -0.5
\end{bmatrix}
\]
Figure 1. The simulation of state $x(t)$ with uncertain $\Pi$.

$$
C_1 = \begin{bmatrix} 0 & -1 \\ 2 & -2.5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.7 & 1.6 \\ 0.3 & -0.5 \end{bmatrix}, \quad C_3 = \begin{bmatrix} -1 & 1.4 \\ 1 & -1.5 \end{bmatrix}
$$

$$
D_1 = \begin{bmatrix} 0 & 0.1 \\ -1 & 0.2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.2 & 0.3 \\ 0.7 & -0.7 \end{bmatrix}, \quad D_3 = \begin{bmatrix} -0.4 & 0.1 \\ -0.3 & -0.6 \end{bmatrix}
$$

where time delay $\tau(t) = 0.2 \sin(1.5t)$ satisfies $\tau(t) \in [0, \tau]$ with $\tau = 0.2$ and $\mu = 0.3$. The singular matrix $E$ is given as

$$
E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
$$

First, the TRM is assumed to be uncertain, which is described as (6). The estimation $\tilde{\Pi}$ is given as

$$
\tilde{\Pi} = \begin{bmatrix} -1.5 & 0.6 & 0.9 \\ 0.7 & -1.2 & 0.5 \\ 1.5 & 1.4 & -2.9 \end{bmatrix}
$$

with $\Delta \tilde{\Pi} \triangleq (\Delta \tilde{\pi}_{ij})$ satisfying $\Delta \tilde{\pi}_{ij} \leq \epsilon_{ij} \triangleq 0.6 \tilde{\pi}_{ij}$, $\forall i, j \in S$, and $i \neq j$. By Theorem 1, it is obtained that $\alpha = 3.4450$, $\alpha_1 = 7.9173$, $\lambda_1 = 0.5523$, and $\lambda_2 = 2.6375$. Then, by solving (16), it is obtained that $\lambda = 0.0048$, which also satisfies condition (17). Under the initial condition $x_0 = [-1 1]^T$, we have the state simulation shown in Figure 1.

When the TRM is partially unknown, it is described as

$$
\Pi = \begin{bmatrix} ? & 0.6 & ? \\ 0.7 & -1.2 & 0.5 \\ ? & ? & -2.9 \end{bmatrix}
$$

(66)

with $\sigma = -1.5$. Under the similar condition, by Theorem 2, one has $\alpha = 43.6061$, $\alpha_1 = 71.4210$, $\lambda_1 = 13.9564$, and $\lambda_2 = 51.4955$. By solving (16), we have $\lambda = 0.0608$. For this case, it is known that they also satisfy condition (17). The response curves of system state are given in Figure 2.

When TRM is uncertain with $\Delta \pi_{ij} \leq \epsilon_{ij} \triangleq \delta \pi_{ij}$, $\forall i, j \in S$, and $i \neq j$, scalar $\delta$ belongs to $[0, 1]$. By Theorem 1, the correlation between $\delta$ and $\lambda$ is shown in Figure 3. On the other hand, if TRM is partially unknown under (66), the correlation between the upper bound $\sigma$ of unknown element $\pi_{ii}$ and $\lambda$ is also given in Figure 4. Such simulations give the explicit correlations between scalar $\lambda$ and general TRMs.
Figure 2. The response of state $x(t)$ with partially unknown $\Pi$.

Figure 3. The correlation of $\delta$ and $\lambda$ under transition rate matrix (6).

Figure 4. The correlation of $\sigma$ and $\lambda$ under transition rate matrix (50).
5. CONCLUSIONS

In this paper, we have discussed the exponential stability of time-delayed SSMJSs, where the corresponding TRM may be uncertain or partially unknown. Up to now, there are very few results to report such problems for this kind of systems. Here, new results on exponential admissibility for time-delayed SSMJSs with such general TRMs have been established, which guarantee the underlying system having a unique solution. All the effects of time delay, diffusion, and general Markovian switchings are taken into account in the system analysis of exponential admissibility. The utility of the developed theories is illustrated by a numerical example.

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