On the density of the minimal subspaces generated by discrete linear Hamiltonian systems

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ABSTRACT

This paper focuses on the density of the minimal subspaces generated by a class of discrete linear Hamiltonian systems. It is shown that the minimal subspace is densely defined if and only if the maximal subspace is an operator; that is, it is single valued. In addition, it is shown that, if the interval on which the systems are defined is bounded from below or above, then the minimal subspace is non-densely defined in any non-trivial case.

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1. Introduction

Consider the following discrete linear Hamiltonian system:

\[ J \Delta y(t) = (P(t) + \lambda W(t))R(y(t)), \quad t \in I, \]

where \( I := \{t_{i}\}_{i=a}^{b} \) is an integral interval, \( a \) is a finite integer or \( a = -\infty \), and \( b \) is a finite integer or \( b = +\infty \), \( b - a \geq 1 \); \( J \) is the canonical symplectic matrix, i.e.,

\[ J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \]

and \( I_n \) is the \( n \times n \) unit matrix; \( \Delta \) is the forward difference operator, i.e., \( \Delta y(t) = y(t + 1) - y(t) \); \( W(t) \) and \( P(t) \) are \( 2n \times 2n \) Hermitian matrices, and the weight function \( W(t) \geq 0 \), i.e. \( W(t) \) is positive semi-definite for \( t \in I \); the partial right shift operator \( R(y)(t) = (u^T(t + 1), v^T(t))^T \) with \( y(t) = (u^T(t), v^T(t))^T \) and \( u(t), v(t) \in \mathbb{C}^n \); and \( \lambda \) is a complex spectral parameter.

Throughout the whole paper, we assume that \( W(t) \) is of the block diagonal form,

\[ W(t) = \text{diag}[W_1(t), W_2(t)], \]

where \( W_j(t) \geq 0 \) is an \( n \times n \) matrix, \( j = 1, 2 \). Let \( P(t) \) be blocked as

\[ P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}. \]
where $A(t)$, $B(t)$, and $C(t)$ are $n \times n$ matrices, $B(t)$ and $C(t)$ are Hermitian matrices, and $A^*(t)$ is the complex conjugate transpose of $A(t)$. Then (1.1) can be written as
\[
\Delta u(t) = A(t)u(t + 1) + (B(t) + \lambda W_2(t))v(t),
\]
\[
\Delta v(t) = (C(t) - \lambda W_1(t))u(t + 1) - A^*(t)v(t), \quad t \in I.
\]

To ensure the existence and uniqueness of the solution of any initial value problem for (1.1), we always assume that
\[
(A_1) \quad I_n - A(t) \text{ is invertible in } I.
\]

It is known that (1.1) contains the following formally self-adjoint vector difference equation of order $2m$ [1]:
\[
\sum_{j=0}^{m} (-1)^j \Delta^j [p_j(t) \Delta^j z(t - j)] = \lambda w(t) z(t), \quad t \in I,
\]
where $w(t)$ and $p_j(t)$, $0 \leq j \leq m$, are $l \times l$ Hermitian matrices, $w(t) \geq 0$, and $p_m(t)$ is invertible in $I$.

The spectral theory of self-adjoint operators and self-adjoint extensions of symmetric operators (i.e., densely defined Hermitian operators) in Hilbert spaces has been well developed (see [2,3]). It has been widely applied to study spectral problems of differential operators. For differential system
\[
Jy'(x) = (\lambda W(x) + P(x))y(x), \quad x \in I = [a, b),
\]
to ensure that the maximal operator is densely defined and single valued, Krall [4] first proposed the following assumption without proof:
\[
Jy'(x) - P(x)y(x) = W(x)f(x) \quad \text{and} \quad W(x)y(x) \equiv 0 \quad \text{implies} \quad f(x) \equiv 0 \text{ on } [a, b).
\]

We mention here that this assumption is reasonable. For details, see Remark 3.1.

In the study of spectral theory of difference expression (1.1) as well as (1.2), it was found that the minimal operator may be a non-densely defined, and the maximal subspace may not be well defined as an operator; that is, it is multi-valued [5]. This fact is ignored in previous literature, including [1,6,8]. More recently, to guarantee the maximal subspace to be single valued, Behncke imposed condition (2.8) in [8], imitating (1.4). But the main result of this paper shows that condition (2.8) in [7] is not satisfied, unless $W(t) \equiv 0$ on $I$.

Since the adjoint operator is not well defined, the spectral theory of symmetric operators cannot be applied to study spectral problems of these difference operators. So it is necessary and timely for us to study the density of the minimal subspace generated by (1.1) and the conditions for the maximal subspace to be an operator.

Note that the graph $G(T)$ of a linear operator or multi-valued linear operator $T$ in a Hilbert space $X$ is a linear subspace (briefly, subspace) in the product space $X^2$. A subspace in $X^2$ is also called a linear relation. In 1961, Arens initiated the study of linear relations [9]. He introduced the concept of adjoint subspace for a subspace in $X^2$, which is a general subspace whose domain is not required to be dense in $X$. In addition, he decomposed a closed subspace in $X^2$ as an operator part and a purely multi-valued part. This decomposition provides a bridge between closed subspaces in $X^2$ and linear operators in $X$.

The rest of the paper is organized as follows. In Section 2, some basic concepts about subspaces and some fundamental results about the maximal and minimal subspaces generated by (1.1) are briefly recalled. In Section 3, it is shown that the minimal subspace is densely defined if and only if the maximal subspace is single valued. In addition, it is shown that, if the interval on which the systems are defined is bounded from below or above, the minimal subspace is non-densely defined in any non-trivial case.

2. Preliminaries

In this section, we first recall some basic concepts and useful results about subspaces. For more results about non-densely defined Hermitian operators or Hermitian subspaces, we refer to [9–11] and some of references cited therein.

Let $X$ be a complex Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$, $T$ and $S$ be two subspaces in $X^2$, and $\lambda \in \mathbb{C}$. Denote
\[
D(T) := \{ x \in X : (x, f) \in T \text{ for some } f \in X \},
\]
\[
R(T) := \{ f \in X : (x, f) \in T \text{ for some } x \in X \},
\]
\[
T^* := \{ (x, f) \in X^2 : (x, g) = \langle f, y \rangle \text{ for all } (y, g) \in T \}.
\]

If $T \cap S = \{ 0 \}$, we write
\[
T + S := \{ (x + y, f + g) : (x, f) \in T, (y, g) \in S \},
\]
which is denoted by $T \oplus S$ in the case that $T$ and $S$ are orthogonal.

Denote
\[
T(x) := \{ f \in X : (x, f) \in T \}.
\]
It can be easily verified that $T(0) = \{ 0 \}$ if and only if $T$ can determine a unique linear operator from $D(T)$ into $X$ whose graph is just $T$. For convenience, we will identify a linear operator in $X$ with a subspace in $X^2$ via its graph.
Definition 2.1. Let $T$ be a subspace in $X^2$.

(1) $T$ is said to be a Hermitian subspace if $T \subseteq T^*$. Further, $T$ is said to be a Hermitian operator if it is an operator; i.e., $T(0) = \{0\}$.

(2) $T$ is said to be a self-adjoint subspace if $T = T^*$. Further, $T$ is said to be a self-adjoint operator if it is an operator; i.e., $T(0) = \{0\}$.

In 1961, Arens [9] introduced the following important decomposition for a closed subspace $T$ in $X^2$:

$$T = T_s \oplus T_\infty,$$

where

$$T_\infty := \{(0, g) \in X^2 : (0, g) \in T\}, \quad T_s := T \ominus T_\infty.$$

It can be easily verified that $T_s$ is an operator, and $T$ is an operator if and only if $T = T_s$, $T_s$ and $T_\infty$ are called the operator and pure multi-valued parts of $T$, respectively. This decomposition will play an important role in our study. It is evident that $R(T_\infty) = T(0)$; $T_\infty = (0 \cap D(T)) \times T(0)$.

Lemma 2.1 ([9]). Let $T$ be a closed Hermitian subspace in $X^2$. Then $D(T_s) = D(T)$ is dense in $T^*(0) \subseteq T(0)^\perp$.

Next, we discuss the maximal and minimal subspaces generated by (1.1). Since $b$ may be finite or infinite, we introduce the following convention for brevity in the subsequent discussion: $b + 1$ means $+\infty$ in the case of $b = +\infty$. Denote

$$L^2_T(I) := \left\{ y = \{y(t)\}_{t=0}^{b+1} \subseteq \mathbb{C}^{2n} : \sum_{t \in I} R(y)^*(t) W(t) R(y)(t) < +\infty \right\}$$

with the semi-inner product

$$\langle y, z \rangle := \sum_{t \in I} R^*(z(t)) W(t) R(y)(t).$$

Further, denote $\|y\| := (\langle y, y \rangle)^{1/2}$ for $y \in L^2_T(I)$. Since the weighted function $W(t)$ may be singular in $I$, $\|\cdot\|$ is a semi-norm. Introduce the quotient space

$$L^2_T(I) := L^2_T(I) / \{y \in L^2_T(I) : \|y\| = 0\}.$$

Then $L^2_T(I)$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. For a function $y \in L^2_T(I)$, denote the corresponding class in $L^2_T(I)$ by $\tilde{y}$. And for any $\tilde{y} \in L^2_T(I)$, denote a representative of $\tilde{y}$ by $y \in L^2_T(I)$. It is evident that $\langle \tilde{y}, \tilde{z} \rangle = \langle y, z \rangle$ for any $\tilde{y}, \tilde{z} \in L^2_T(I)$.

The natural difference operator corresponding to system (1.1) is denoted by

$$\mathcal{L}(y)(t) := J A y(t) - P(t) R(y)(t).$$

Further, we denote

$$L^2_{T,0}(I) := \{ y \in L^2_T(I) : \text{there exist two integers } s, k \in I \text{ with } s \leq k \text{ such that } y(t) = 0 \text{ for } t \leq s \text{ and } t \geq k + 1\},

H := \{\tilde{y}, \tilde{g} \in (L^2_{T,0}(I))^2 : \text{there exists } y \in \tilde{y}, g \in \tilde{g} \text{ such that } \mathcal{L}(y)(t) = W(t) R(g)(t), \ t \in I\},

H_{00} := \{\tilde{y}, \tilde{g} \in H : \text{there exists } y \in \tilde{y}, g \in \tilde{g} \text{ such that } y \in L^2_{T,0}(I) \text{ and } \mathcal{L}(y)(t) = W(t) R(g)(t), \ t \in I\}.$$

It can be easily verified that $H$ and $H_{00}$ are both linear subspaces in $(L^2_T(I))^2$. Here, $H$ and $H_{00}$ are called the maximal and pre-minimal subspaces corresponding to (1.1) in $(L^2_T(I))^2$, and $H_{00} := \overline{H_{00}}$ is called the minimal subspace corresponding to (1.1) in $(L^2_T(I))^2$.

The following lemma shows that $H_{00}$ is a closed Hermitian subspace in $(L^2_T(I))^2$.

Lemma 2.2 ([12]). Assume that $(A_1)$ holds. $H_{00}^* = H^* = H$.

We now point out that the definiteness condition for system (1.1), which was given in [1], cannot guarantee the maximal subspace to be single valued. The definiteness condition for (1.1) given in [1] is of the following form.

(A2) There exists a finite subinterval $I_0 := [0, t_0] = [t]_{t=0}^{t_0} \subset I$ such that, for any $\lambda \in \mathbb{C}$ and any non-trivial solution, $y(t)$ of (1.1) satisfies

$$\sum_{t \in I_0} R(y)^*(t) W(t) R(y)(t) > 0.$$

Consider the second-order difference equation

$$-\Delta(p(t) \nabla x(t)) + q(t) x(t) = \lambda w(t) x(t), \quad t \in I,$$

where $p(t)$, $q(t)$ are real valued, $p(t) \neq 0$, and $w(t) > 0$ in $I$. By [12, Theorem 4.5], the definiteness condition $(A_2)$ is satisfied, while it has been shown in [5] that the maximal subspace is not an operator.
3. Main results

In this section, we discuss the density of $D(H_0)$ in $L^2_W(I)$.

**Theorem 3.1.** Assume that (A1) holds. Then $D(H_0)$ is dense in $L^2_W(I)$ if and only if $H$ is an operator; that is, $H(0) = \{0\}$.

**Proof.** We first show the sufficiency. If $H$ is an operator, then it follows from Lemma 2.2 that $H_0^*(0) = H(0) = \{0\}$. So, by Lemma 2.1, one has that $D(H_0)$ is dense in $L^2_W(I)$.

We now show the necessity. Let $D(H_0)$ be dense in $L^2_W(I)$. It follows from Lemma 2.2 and that $D(H_0)$ is dense in $H(0)^\perp$. It is clear that $H(0)^\perp$ is a closed subspace in $L^2_W(I)$. Hence, $H(0)^\perp = L^2_W(I)$, and consequently $H(0) = \{0\}$. The proof is complete.

It is clear that $H_0$ is an operator if $H$ is, and that $H$ is densely defined if $H_0$ is. So the following result is obtained.

**Corollary 3.1.** Assume that (A1) holds. Then $H_0$ is densely defined operator in $L^2_W(I)$ if and only if $H$ is an operator. If $H$ is an operator, then it must be a densely defined operator.

**Remark 3.1.** Theorem 3.1 can also apply to differential system (1.3). If condition (1.4) holds, then it follows that $Wf \equiv 0$. Thus $f = 0$ in the meaning of equivalence in the Hilbert space $L^2_W(a, b)$. This yields that the maximal operator is single valued. Then the minimal operator is single valued, and, by Theorem 3.1, it is densely defined, and consequently the maximal operator is also densely defined. In addition, it is worth noting that condition (1.4) can be satisfied regardless of the form of the interval $I$ on which the system is defined. For example, when $W(t) > 0$ on $I$, then condition (1.4) holds.

Next, we want to show that when $I = [a, +\infty)$, $I = (-\infty, b)$, or $I = [a, b]$, $H$ is not an operator in any non-trivial case.

**Theorem 3.2.** Let $I = [a, +\infty)$, $I = (-\infty, b)$, or $I = [a, b]$, and let (A1) hold. If there exists $k_0 \in I$ such that $W(k_0) \neq 0$, then $H$ is not an operator.

**Proof.** Since the proofs are similar, we only give the proof for the case that $I = [a, +\infty)$.

Let $k_0 \in I$ be the minimal point in $I$ such that $W(k_0) \neq 0$; that is, $W(t) = 0$ for $a \leq t \leq k_0 - 1$. The proof is divided into two cases.

**Case 1.** $k_0 = a$. This proof is divided into two subcases.

If $W_2(a) \neq 0$, then there exists $\xi \in \mathbb{C}^n$ such that $W_2(a)\xi \neq 0$. Take

$$y(a) = (-W_2(a)\xi)^T, \quad y(t) = 0, \quad t \geq a + 1;$$

$$R(g)(a)(0, 0)^T, \quad R(g)(t) = 0, \quad t \geq a + 1.$$ 

Then $\tilde{y} = 0, 0 \neq \tilde{g} \in L^2_W(I)$, and $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for $t \in I$. This yields $H(0) \neq \{0\}$. So $H$ is not an operator in this case.

If $W_2(a) = 0$, then $W_1(a) \neq 0$. There exists $\xi \in \mathbb{C}^n$ such that $W_1(a)\xi \neq 0$. Take

$$v(a) = (l_n - A^*(a))^{-1}W_1(a)\xi, \quad y(a) = (-B(a)v(a))^T, v(a)^T;$$

$$y(t) = 0, \quad t \geq a + 1;$$

$$R(g)(a)(\xi^T, 0)^T, \quad R(g)(t) = 0, \quad t \geq a + 1.$$ 

Then it can be easily verified that $\tilde{y} = 0, 0 \neq \tilde{g} \in L^2_W(I)$, and $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for $t \in I$. This yields $H(0) \neq \{0\}$. So $H$ is not an operator in this case.

**Case 2.** $k_0 > a$. This proof is also divided into two subcases.

If $W_2(k_0) \neq 0$, then there exists $\xi \in \mathbb{C}^n$ such that $W_2(k_0)\xi \neq 0$. Take $y = (u^T, v^T)^T$ and $g$ with

$$u(k_0) = -W_2(k_0)\xi, \quad v(k_0) = 0,$$

$$v(k_0 - 1) = -(l_n - A^*(k_0 - 1))^{-1}C(k_0 - 1)u(k_0),$$

$$u(k_0 - 1) = (l_n - A^*(k_0 - 1))u(k_0) - B(k_0 - 1)v(k_0 - 1),$$

$$y(t) = 0, \quad t \neq k_0, k_0 - 1;$$

$$R(g)(k_0)(0, 0)^T, \quad R(g)(t) = 0, \quad t \neq k_0.$$ 

Then $\tilde{y} = 0, 0 \neq \tilde{g} \in L^2_W(I)$, and $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for $t \in I$, which implies that $\tilde{y} \in H(0) \neq \{0\}$. So $H$ is not an operator in this case.

If $W_2(k_0) = 0$, then $W_1(k_0) \neq 0$. There exists $\xi \in \mathbb{C}^n$ such that $W_1(k_0)\xi \neq 0$. Take

$$v(k_0) = (l_n - A^*(k_0))^{-1}W_1(k_0)\xi, \quad u(k_0) = -B(k_0)v(k_0),$$

$$v(k_0 - 1) = (l_n - A^*(k_0 - 1))^{-1}(v(k_0) - C(k_0 - 1)u(k_0)),$$

$$u(k_0 - 1) = (l_n - A^*(k_0 - 1))u(k_0) - B(k_0 - 1)v(k_0 - 1),$$

$$y(t) = 0, \quad t \neq k_0, k_0 - 1;$$

$$R(g)(k_0)(0, 0)^T, \quad R(g)(t) = 0, \quad t \neq k_0.$$ 

Then $\tilde{y} = 0, 0 \neq \tilde{g} \in L^2_W(I)$, and $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for $t \in I$, which yields that $\tilde{y} \in H(0) \neq \{0\}$. So $H$ is not an operator in this case. The whole proof is complete.
The following result can be directly derived from Theorems 3.1 and 3.2.

**Corollary 3.2.** Let $I = [a, +\infty)$, $I = (-\infty, b]$, or $I = [a, b]$, and let $(A)$ hold. If there exists $k_0 \in I$ such that $W(k_0) \neq 0$, then $D(H_0)$ is not dense in $L^2_W(I)$.

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**References**