Chapter 1
Topological aspects of dynamics of pairs, tuples and sets

Piotr Oprocha and Guohua Zhang

1.1 Introduction

Topological theory of dynamical systems concentrates on study of qualitative aspects of dynamics of continuous transformations by various topological tools (many of them are borrowed from other fields such as algebra or measure theory). The aim of this chapter is to present, by no means complete, an exposition on local aspects of dynamics of pairs, or more generally tuples and sets. It is motivated by a strong believe of the authors that very often global properties of dynamics can be deduced from its local behavior. A detailed analysis of local properties of dynamics can provide an additional insight into related global properties. In some other cases, properly stated local properties can help to draw some conclusions on the dynamics that could be derived from a global property, if this global property was present in the studied system. We hope that the facts collected in this chapter will convince the reader that it is really the case and that the local properties can carry lots of information about studied dynamical systems.

We try to provide a representative sample of relatively new results, most of them published less than 10 years ago. We definitely want to put recent advances in a wider historical context, so the reader will find in our survey some classical results on dynamical properties of pairs as well. All notions we are going to consider have purely topological definitions, usually expressed in terms of relations between consecutive iterations and some open sets like neighborhoods of points forming the pair, neighborhood of the diagonal and the like.

Piotr Oprocha
AGH University of Science and Technology, Faculty of Applied Mathematics, al. A. Mickiewicza 30, 30-059 Kraków, Poland – and – Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland
e-mail: oprocha@agh.edu.pl

Guohua Zhang
School of Mathematical Sciences and LMNS, Fudan University, Shanghai 200433, China
e-mail: chiaths.zhang@gmail.com
In this survey we focus mainly on recent advances on topological characterization of evolution of points (pairs, tuples and sets) under iteration of a continuous map. However the reader should be aware that many of these results could be obtained mainly because of somehow parallel and intensive studies on measure-theoretic aspects of dynamical systems, which we present very briefly. We do not discuss any recent results on smooth dynamical systems on compact manifolds and only slightly mention some results for actions of topological semigroups. In fact, we are not even able to present all important aspects of discussed topological properties. One of such topics is the theory of Ellis semigroup (also called the enveloping semigroup) developed in particular in [AAB98, AAG08, Aus88, Bro79, Ell69] for flows, and extended recently in [BGKM02] to so-called adherence semigroup in the context of continuous surjections. These semigroups are powerful tools for classification of some dynamical properties, in particular characterization of systems without Li-Yorke pairs [BGKM02]. All the above mentioned topics, while certainly important and interesting, exceed capacity of this chapter.

Studies of dynamics of pairs are present in the field of dynamical systems more than 50 years. One of the first, still very important notions were proximal pairs and regionally proximal pairs. They have very strong connections with distality and equicontinuity, which are another two fundamental properties in the dynamical systems theory. There are many beautiful theorems related to these notions, e.g. Furstenberg’s structure theorem of distal systems.

Broad interest in the study of dynamics of pairs was generated by the article [LY75] by Li and Yorke, where they observed a phenomenon presently known as chaos in the sense of Li and Yorke. The observation made by Li and Yorke is very important in our context, because it highlights the fact that complicated interactions between trajectories of points can be a prerequisite for chaos. Therefore, studies on dynamics of pairs can be useful in the description of two extremal situations: order and chaos. In this chapter we will discuss various concepts, which describe different gradations of regularity or complexity of trajectories. We will also study dependencies and implications between considered properties.

As we said before, we focus mainly on topological properties of considered notions, however as usual, it is impossible to work in one field of mathematics without connections with other fields. Here we are most interested in topology, however we cannot neglect influence of mathematical analysis, combinatorics, ergodic theory, etc. In some cases these other fields were real motivations for topological analogs presented here (e.g. it is the case of weak mixing). In some other situations no known “purely” topological or combinatorial proof exists. For example, all the proofs known to the authors which relate entropy pairs with the closure of the set of asymptotic pairs, strongly rely on methods from ergodic theory.

Throughout this chapter, by a (topological) dynamical system, without any further qualification, we mean a pair consisting of a surjective map \( f : X \to X \) and a compact metric space \((X,d)\) on which the map \( f \) acts continuously. In the special case, when \( f \) is a homeomorphism, we will say that \((X,f)\) is an invertible dynamical system or simply a flow. Note that every continuous surjective map \( f : X \to X \) induces in a natural way an inverse limit space.
Orb of all transitive points the shortest arc joining \( x \) with \( G \) isomorphic to \( G \), such that there is a one dimensional complex \( G \) which is homeomorphic to a polyhedron (a geometric realization) of some finite.

In this section we provide basic terminology used later in this chapter.

1.2 Preliminaries

In this section we provide basic terminology used later in this chapter.

Denote by \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{Z} \) the set of all positive integers, non-negative integers and integers, respectively. A topological graph is a compact connected metric space which is homeomorphic to a polyhedron (a geometric realization) of some finite one-dimensional complex. In other words, a topological graph is a continuum \( G \) such that there is a one dimensional complex \( \mathbb{R} \) whose geometric carrier \( |\mathbb{R}| \) is homeomorphic to \( G \). We always endow \( G \) with the metric \( d(x, y) \) given by the length of the shortest arc joining \( x, y \) in \( G \) (induced on \( G \) from \( |\mathbb{R}| \)). Another important set is the hyperspace \( 2^X \), that is the set of all nonempty compact subsets of \( X \) endowed with Hausdorff metric \( \mathcal{H}_d \). Since \( X \) is a compact metric space, it is not hard to show that \( 2^X \) is also a compact metric space.

A closed set without isolated points is called perfect. A set \( X \) is totally disconnected if the only connected subsets of \( X \) are singletons. By a Cantor set we mean any compact, perfect and totally disconnected set. Any set that can be presented as an at most a countable union of Cantor sets is called a Mycielski set. A subset of \( X \) is non-trivial if it has at least two points. The Borel \( \sigma \)-algebra of \( X \) is denoted by \( \mathcal{B}_X \).

Let \( x \in X \). Denote by \( \text{Orb}^+(x, f) = \{ f^n(x) : n \in \mathbb{Z}_+ \} \), the positive orbit of \( x \) with respect to \( f \). If \( f \) is a homeomorphism, then we can define its full orbit \( \text{Orb}(x, f) = \{ f^n(x) : n \in \mathbb{Z} \} \). A set \( A \subset X \) is invariant if \( f(A) \subset A \). Denote by \( \text{Tran}(X, f) \) the set of all transitive points of \( (X, f) \), that is points \( x \in X \) satisfying \( \text{Orb}^+(x, f) = X \); and denote by \( \text{Rec}(X, f) \) the set of all recurrent points of \( (X, f) \), that is points \( x \in X \) such that there is an increasing sequence \( \{ n_k \}_{k=0}^{\infty} \) satisfying \( \lim_{k \to \infty} f^{n_k}(x) = x \). Observe that \( x \) is a recurrent point if and only if \( x \in \text{Orb}^+(f(x), f) \).

For every integer \( n \geq 2 \) we denote \( X^n = \{ (x_1, \cdots, x_n) : x_1, \cdots, x_n \in X \} \) and \( A_n(X) = \{(x, \cdots, x) \in X^n : x \in X \} \). We also define a dynamical system \( (X^n, f^{(n)}) \), where \( f^{(n)} \) is induced on \( X^n \) by the formula \( f^{(n)}(x_1, \cdots, x_n) = (f(x_1), \cdots, f(x_n)) \).

A dynamical system \( (X, f) \) is transitive if for any pair of nonempty open subsets \( U, V \) of \( X \) there exists \( n \in \mathbb{N} \) such that \( U \cap f^{-n}(V) \neq \emptyset \); minimal if for each nonempty open subset \( U \) of \( X \) there exists \( N \in \mathbb{N} \) such that \( \bigcup_{i=0}^{N} f^{-i}(U) = X \); weakly mixing if
(X^2, f^{(2)}) is transitive; mildly mixing if the product system \((X \times Y, f \times g)\) is transitive for every transitive system \((Y, g)\); and is strongly mixing if for any pair of nonempty open subsets \(U, V\) of \(X\) there exists \(N > 0\) such that \(U \cap f^{-n}(V) \neq \emptyset\) for every \(n \geq N\).

Recall that through this chapter the map \(f: X \to X\) is at least surjective, and so \((X, f)\) is transitive if and only if \(\text{Tran}(X, f)\) is a dense \(G_\delta\) subset of \(X\) if and only if \(\text{Tran}(X, f) \neq \emptyset\) (a more detailed exposition on transitivity and related topics can be found in the survey [KS97]). It is also not hard to verify that \((X, f)\) is minimal if and only if it contains no nonempty closed proper invariant subsets \(M \subset X\), if and only if \(\text{Tran}(X, f) = X\). A point \(x \in X\) is minimal if \((\text{Orb}^+(x, f), f)\) is a minimal dynamical system. Immediately from the definition we see that strong mixing implies mild mixing, mild mixing implies weak mixing, and weak mixing implies transitivity. Additionally, it is well known that \((X, f)\) is weakly mixing if and only if \((X^n, f^{(n)})\) is transitive for each integer \(n \geq 2\) (see [Fur67]). The reader can also check that if \((X, f)\) is weakly mixing then \(X\) is either perfect or a singleton.

We say that \(x \in X\) is an equicontinuous point of \((X, f)\) if for each \(\epsilon > 0\) there exists \(\delta > 0\) such that \(d(f^n(x), f^n(y)) < \epsilon\) for every integer \(n \geq 0\) and any point \(y\) such that \(d(x, y) < \delta\). Denote by \(\text{Eq}(X, f)\) the set of all equicontinuous points of \((X, f)\). A dynamical system \((X, f)\) is equicontinuous if \(\text{Eq}(X, f) = X\), or equivalently, for each \(\epsilon > 0\) there exists \(\delta > 0\) such that \(d(f^n(x), f^n(y)) < \epsilon\) for every integer \(n \geq 0\) and every points \(x, y \in X\) such that \(d(x, y) < \delta\).

Let \((X, f), (Y, g)\) be dynamical systems and let \(J \subset X \times Y\) be a closed subset with projections onto the first and second coordinate equal to, respectively, \(X\) and \(Y\). If additionally \(J\) is an \(f \times g\)-invariant subset then we say that \(J\) is a joining of \((X, f)\) and \((Y, g)\). Dynamical systems \((X, f)\) and \((Y, g)\) are disjoint if \(J = X \times Y\) is their only joining. Clearly if \((X, f)\) is disjoint from itself then \(X\) is a singleton, and if \(X_0, Y_0\) are closed and invariant nonempty subsets for \((X, f)\) and \((Y, g)\), respectively, then \(J = (X_0 \times Y) \cup (X \times Y_0)\) is a joining of \((X, f)\) and \((Y, g)\). Therefore, if systems \((X, f)\) and \((Y, g)\) are disjoint then at least one of them is minimal as observed by Furstenberg in [Fur67].

Denote by \(\pi: (X, f) \to (Y, g)\) a factor map between dynamical systems \((X, f), (Y, g)\), that is \(\pi: X \to Y\) is a continuous surjection and \(\pi \circ f = g \circ \pi\). In the above case we say that \((X, f)\) is an extension of \((Y, g)\) or \((Y, g)\) is a factor of \((X, f)\). For any factor map \(\pi\) we also define the following relation \(R_\pi = \{(x_1, x_2) \in X^2 : \pi(x_1) = \pi(x_2)\}\). If \((Y, g)\) is a trivial system (i.e. \(Y\) is a singleton) then we say that \(\pi\) is trivial.

The factor map can be defined equivalently by providing a special relation. Let \(R \subset X^2\) be an \(f^{(2)}\)-invariant and closed equivalence relation (ICER for short) over \(X\). The quotient space \(X/R\) equipped with the quotient topology is a compact metrizable space. Additionally \(f\) induces naturally a continuous surjection \(f_R: [x]_R \to [fx]_R\), where \([x]_R\) is the equivalence class of \(x\) under \(R\), thus \((X/R, f_R)\) is a topological dynamical system. Note that the quotient map \(\pi_R: X \to X/R\) is a factor map between dynamical systems \((X, f)\) and \((X/R, f_R)\) and moreover \(R_{\pi_R} = R\).

Denote by \(\mathcal{M}(X)\) the set of all Borel probability measures on \(X\) and by \(\mathcal{M}(X, f) \subset \mathcal{M}(X)\) the set of all \(f\)-invariant elements. The set of all \(f\)-invariant ergodic probability measures is denoted by \(\mathcal{M}^e(X, f)\). It is known that both \(\mathcal{M}(X)\) and \(\mathcal{M}(X, f)\) are
convex compact metrizable spaces when endowed with the weak-star topology, and that $M(X) \supset M(X, f) \supset M^e(X, f) \neq \emptyset$.

We say that $(X, f)$ is: 
- **uniquely ergodic** if $M(X, f)$ is a singleton, equivalently, $M^e(X, f)$ is a singleton; 
- **strictly ergodic** if it is uniquely ergodic and minimal. 

The support of a measure $\mu$, denoted by $\text{supp}(\mu)$, is the complement of the union of all open sets of $\mu$ measure zero. A measure $\mu \in M(X, f)$ is **fully supported** if $\text{supp}(\mu) = X$ and $(X, f)$ is **fully supported** if there exists fully supported $\mu \in M(X, f)$.

### 1.3 Entropy pairs and sets

In early 1990s Blanchard initiated in [Bla92, Bla93] a search for satisfactory topological analogue of Kolmogorov systems which are very important class of systems in ergodic theory (see [Wal82] for definition and basic properties of Kolmogorov systems). After publication of these papers, Blanchard’s ideas became widely known and accepted, developing to a field of research known presently as the local entropy theory. In this section we will discuss some of recent results on that field, viewing them only from topological dynamics perspective, that is without touching too many measure-theoretic aspects of dynamics. The reader should be aware that in the proofs of many results presented in this section, very strong tools from ergodic theory are employed.

For the reader’s convenience, to avoid ambiguity, all dynamical systems $(X, f)$ considered in this section will be flows, i.e. $f$ will be a homeomorphism. The reader is encouraged to check, e.g. using the natural extension $(X_f, \sigma_f)$, that most of the results hold also in non-invertible setting. Since many facts presented here were published only in the version for flows, we decided to keep invertibility condition as otherwise it would be necessary to present at least a sketch of proof in the extended case of surjections.

Ergodic theory and topological dynamics exhibit a remarkable parallelism. In ergodic theory there exists a rich and powerful entropy theory initiated in 1958 by Kolmogorov [Kol58]. A few years later, an analogous notion of topological entropy was introduced by Adler, Konheim and McAndrews [AKM65]. Their definition of topological entropy uses open covers, but later some equivalent definitions (on compact metric spaces) using properties of metric were introduced independently by R. Bowen [Bow71] and Dinaburg [Din71]. Here we will present only the basic concepts needed in this chapter, but the reader can find a more extensive study of topological (and measure-theoretic) entropy in books [Wal82] by Walters and [Gla03] by Glasner. Most recent results on entropy are collected in the monograph [Dow11] by Downarowicz.

Let $(X, f)$ be a dynamical system. Denote by $\mathcal{C}_X$ the set of all finite open covers of $X$. If $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$ then we say that $\mathcal{U}$ is **finer** than $\mathcal{V}$ (denoted by $\mathcal{V} \leq \mathcal{U}$ or $\mathcal{U} \geq \mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$.

For $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$ we define $\mathcal{U} \vee \mathcal{V} = \{ U \cap V : U \in \mathcal{U}, V \in \mathcal{V} \} \in \mathcal{C}_X$. Given non-negative integers $m \leq n$ we denote $\mathcal{U}^m_n = \bigvee_{k=m}^n f^{-k}(\mathcal{U})$, where $f^{-k}(\mathcal{U}) = \{ f^{-k}(U) : U \in \mathcal{U} \}$. 
$U \in \mathcal{U}$. Fix a compact (but not necessarily $f$-invariant) set $K \subset X$, a cover $\mathcal{U} \in \mathcal{C}_X$, and define
\[
    h_{\text{top}}(f, K, \mathcal{U}) = \limsup_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}, K),
\]
where $N(\mathcal{U}, K)$ is the minimal cardinality of family $\mathcal{U} \subset \mathcal{U}$ covering $K$. We put $N(\mathcal{U}, \emptyset) = 1$ by convention. Observe that when $K = X$ the sequence $[\log N(\mathcal{U}_0^{n-1}, X) : n \in \mathbb{N}]$ is sub-additive and so the upper limit in (1.1) is in fact a limit. The topological entropy $h_{\text{top}}(f, K)$ of $K$ with respect to $f$ is the supremum of values $h_{\text{top}}(f, K, \mathcal{U})$ taken over all $\mathcal{U} \in \mathcal{C}_X$. When $K = X$ we simply write $h_{\text{top}}(f, \mathcal{U}) = h_{\text{top}}(f, X, \mathcal{U})$, $N(\mathcal{U}) = N(\mathcal{U}, X)$ and $h_{\text{top}}(f) = h_{\text{top}}(f, X)$. The topological entropy of $\mathcal{U}$ and $f$ are, respectively, $h_{\text{top}}(f, \mathcal{U})$ and $h_{\text{top}}(f)$.

For a long time, the level of development of topological entropy theory lagged behind that of measure-theoretic entropy theory, but recently this situation changed rapidly. A turning point occurred with the pioneering papers of Blanchard [Bla92, Bla93] in the 1990s. To seek topological analogues of the important notion of Kolmogorov system in ergodic theory and better understanding how topological entropy is woven into the general pattern of topological dynamics, Blanchard introduced the properties of uniformly positive entropy and completely positive entropy [Bla92]. The main ingredient of these properties was the notion of topological entropy pair [Bla93].

The standard definition of topological entropy pair is the following. A pair $(x, y)$ of points is a topological entropy pair, if for any open cover $\mathcal{U}$ consisting of open sets $U, V \neq X$ such that $x \in \text{int} U^c$, $y \in \text{int} V^c$ we have $h_{\text{top}}(f, \mathcal{U}) > 0$. Note that it immediately follows from the definition that if $(x, y)$ is a topological entropy pair then $x \neq y$ and $h_{\text{top}}(f) > 0$. It was proved in [Bla93] that if $h_{\text{top}}(f) > 0$ then there exists a topological entropy pair and if $\mathcal{U} = \{U, V\}$ is an open cover satisfying $h_{\text{top}}(f, \mathcal{U}) > 0$ then there exists a topological entropy pair $(x, y) \in U^c \times V^c$.

Denote by $E_2(X, f)$ the set of all topological entropy pairs for $(X, f)$. Some basic properties of $E_2(X, f)$ proved in [Bla93] are summarized as follows.

**Proposition 1** $E_2(X, f)$ is an $f^{(2)}$-invariant subset of $E_2(X, f) \cup \Delta_2(X)$.

Note that since $(X, f)$ is a flow, the set $E_2(X, f)$ is also $f^{(2)}$-invariant.

**Proposition 2** For any factor map $\pi: (X, f) \to (Y, g)$ the following inclusion holds: $E_2(Y, g) \subset (\pi \times \pi)(E_2(X, f)) \subset E_2(Y, g) \cup \Delta_2(Y)$.

One of the basic facts in ergodic theory is that a maximal factor with zero measure-theoretic entropy exists for any measurable dynamical system. This factor can be determined as a quotient space in the measurable setting via the so-called Pinsker $\sigma$-algebra. It was shown in [BL93] that topological entropy pairs can be used to obtain a similar result in topological dynamics. Strictly speaking, Blanchard and Lacroix proved the following:

**Theorem 3** Each flow $(X, f)$ admits a maximal factor $(\overline{X}, \overline{f})$ with zero topological entropy in the sense that

3.1. $(\overline{X}, \overline{f})$ is a factor of $(X, f)$ via a factor map $\pi: (X, f) \to (\overline{X}, \overline{f})$ and
3.2. If $\phi: (X, f) \to (Y, g)$ is a factor map, where $h_{\text{top}}(g) = 0$, then there exists a factor map $\psi: (\tilde{X}, \tilde{f}) \to (Y, g)$ such that $\phi = \psi \circ \pi$.

Moreover, $(\tilde{X}, \tilde{f}) = (X/R, f_R)$, where $R \subset X^2$ is the smallest ICER containing $E_2(X, f) \cup \Delta_2(X)$.

Following [BL93], we say that $(X, f)$ has uniform positive entropy (u.p.e. for short) if any cover by two non-dense open sets has positive topological entropy, and has completely positive entropy (c.p.e. for short) if any non-trivial factor of $(X, f)$ has positive topological entropy. Note that $(X, f)$ has u.p.e. if and only if $E_2(X, f) = X^2 \setminus A_2(X)$. It was proved by Blanchard in [Bla92] that u.p.e. implies weak mixing, and so if $(X, f)$ has u.p.e. then either $X$ is a singleton or a perfect set. By Theorem 3, $(X, f)$ has c.p.e. if and only if its maximal factor with zero topological entropy is trivial, or equivalently, $X^2$ is the smallest ICER containing $E_2(X, f) \cup A_2(X)$. Observe that u.p.e. implies c.p.e. by Proposition 2.

Blanchard observed in [Bla93] that the following property, while weaker than u.p.e., can be of some interest. We say that $(X, f)$ is diagonal if $\{(x, f(x)) : x \in X\} \subset E_2(X, f)$. One of the nice properties of Kolmogorov systems is that they are disjoint (in measure-theoretic sense, e.g. see [Gla03] for the definition) from all measurable dynamical systems with zero measure-theoretic entropy. Blanchard proved that in the topological setting a similar property (i.e. topological version of disjointness) is represented by diagonal flows [Bla93].

**Theorem 4** Every diagonal flow is disjoint from minimal flows with zero topological entropy.

As a direct corollary of Theorem 3 each transitive diagonal flow has c.p.e. It is not known if the result can be reversed. In fact the following problem was proposed in [GY09].

**Problem 5** Does there exist a minimal flow with c.p.e. which is not diagonal?

In [HY06] the authors constructed a transitive diagonal flow which is not u.p.e. Next, a minimal flow having c.p.e. but without u.p.e. was provided in [SY09]. It was claimed in [SY09] that H. F. Li was able to construct a minimal c.p.e. system which is not diagonal as a modification of the example in [SY09]; however this result has not been published so far.

Every Kolmogorov system is mixing of all orders in measurable setting [Par81]. Therefore, in the topological setting it is natural to expect some range of mixing from its topological analogue. The result by Blanchard [Bla92] that u.p.e. implies weak mixing was later extended in [HSY05], where the following result was proved.

**Theorem 6** Each transitive diagonal flow is mildly mixing. In particular, u.p.e. implies mild mixing.

The above result may be the strongest possible, since in [Bla92] Blanchard provided an example of a weakly mixing flow with c.p.e. which does not have u.p.e.
and a transitive flow having c.p.e. which is not weakly mixing. He also constructed a flow which has u.p.e. but is not strongly mixing.

In the same paper Blanchard proved that every c.p.e. flow has a measure with full support, but unfortunately this measure does not have to be ergodic. Huang and Ye were able to construct in [HY06] an example of a flow \((X, f)\) having u.p.e. such that the support of any ergodic measure is a proper subset of \(X\). The following result of Kamiński, Siemaszko and Szymański [KSS05] shows that u.p.e. flows with fully supported ergodic measures may be very special.

**Proposition 7** If \((X, f)\) is a uniquely ergodic u.p.e. flow, then there exists a closed equivalence relation \(R \subset X^2\) such that

\[
f^{(2)}(R) \subset R, \quad \bigcap_{n=0}^{\infty} (f^{(2)})^n(R) = \Delta_2(X) \quad \text{and} \quad \bigcup_{n=0}^{\infty} (f^{(2)})^{-n}(R) = X^2.
\]

Dynamical systems admitting a relation \(R\) with the above properties were introduced and studied by the same authors in [KSS03]. By an analogy to sub-\(\sigma\)-algebras of Kolmogorov systems, the authors [KSS03] introduced the name **Kolmogorov flow**. Unfortunately, connections between relation satisfying (1.2) and topological entropy are not as strong as in the case of Kolmogorov systems and related sub-\(\sigma\)-algebras. Namely, [KSS03] provided an example of a non-trivial flow \((X, f)\) and a relation \(R\) satisfying (1.2) such that \(h_{\text{top}}(f) = 0\). Later in [GY09] Glasner and Ye constructed an example of a flow with c.p.e. which does not have u.p.e. but still satisfies (1.2). Moreover, it was proved in [GW94] that any ergodic measurable dynamical system \((\Omega, m, T)\) with positive measure-theoretic entropy admits a strictly ergodic u.p.e. model \((X, f)\), that is \((X, f)\) is a minimal u.p.e. flow, has a unique ergodic \(f\)-invariant Borel probability measure \(\mu\) and the measurable dynamical systems \((\Omega, m, T)\) and \((X, \mu, f)\) are measure-theoretically isomorphic. In other words, we cannot expect any other measure-theoretic aspects of dynamics of a flow in Proposition 7 beyond ergodicity. This clearly demonstrates that, while there are a few similarities with ergodic theory, it is not easy to provide good topological analogs of ergodic properties.

As we have seen, u.p.e. is a quite strong property of systems with positive topological entropy. However, it is still possible to make it a little bit stronger. To obtain a better understanding of the topological version of a Kolmogorov system, the definition of entropy pair was extended to entropy \(n\)-tuple by Glasner and Weiss in the topological setting in [GW95b], and later further extended by Huang and Ye in both topological and measurable settings in [HY06]. Finally, the definitions of entropy sets and points were introduced in [DYZ06] and [YZ07] respectively (see also [BH08]). As we will see, these extensions were not only illusionary and that entropy tuples (or sets) are a more sensitive tool than entropy pairs.

Before we go further let us make one important historical remark. Present understanding of entropy pairs probably wouldn’t be that good, if not parallel theory that has been built for measurable dynamical systems. Blanchard et al. introduced in [BHM*95] the concept of measure-theoretic entropy pairs, and then in [Gla97]
Glasner connected these objects with support of some special invariant measure over $X^2$. A strong relationship between these two kinds of entropy pairs was explored in [BGH97], where the key point was local variational principles concerning topological and measure-theoretic entropy. The local variational principles are generalizations of the classical variational principle bringing a new better insight into properties of topological entropy (e.g. see [BGH97, GW06, HY06, Rom03]). Since in our exposition we focus mainly on topological aspects of dynamics, we are not going to introduce formal definition of measure-theoretic entropy pairs, tuples or sets, neither will we discuss local variational principles in details. The reader interested in this topic is referred to the recent survey article [GY09] by Glasner and Ye and references therein. Further extensions of local variational principles and topological analogues of a Kolmogorov system are their relative analogs defined over fibres of factor maps and studied by Huang, Ye and Zhang [HYZ06, HYZ07] (see also [GW95b, LS01, PS01]).

Let us now turn to formal definitions of topological properties mentioned before. A nonempty set $K$ with at least two points is an entropy set if $h_{\text{top}}(f, U) > 0$ for every open cover $\mathcal{U}$ of $X$ which satisfies additional condition that $K \setminus \bigcup U \neq \emptyset$ for each $U \in \mathcal{U}$. We say that $(x_1, \ldots, x_n) \in X^n$ is an entropy $n$-tuple, where $n \geq 2$, if $(x_1, \ldots, x_n)$ is an entropy set. Note that every entropy pair is an entropy 2-tuple according to the above definition. It is easy to verify that any set with at least two points contained in an entropy set is again an entropy set, and that the closure of an entropy set is also an entropy set. It can also be proved (see [DYZ06]) that a set $K$ with at least two points is an entropy set if and only if for any distinct $n \geq 2$ points $x_1, \ldots, x_n$ from $K$, the tuple $(x_1, \ldots, x_n)$ is an entropy $n$-tuple.

So far we have seen that entropy pairs exist in a dynamical system if its entropy is positive and vice-versa. Entropy sets make such connection more tight [DYZ06].

**Theorem 8** $h_{\text{top}}(f) = \sup h_{\text{top}}(f, K) : K$ is an entropy set).

Recall that a set $\mathcal{D} = \{d_1 < d_2 < \cdots\} \subset \mathbb{N}$ has positive density if the following limit exists and is positive:

$$\lim_{n \to \infty} \frac{\#(\mathcal{D} \cap \{1, \ldots, n\})}{n} > 0,$$

where as usual $\#A$ denotes the cardinality of a set $A$. An interesting characterization of entropy tuples was obtained in [HY06], providing a good motivation for combinatorial approach to entropy, as we will see later in this section.

**Proposition 9** Fix an integer $n \geq 2$ and $(x_1, \ldots, x_n) \in X^n \setminus \Delta_n(X)$. Then $(x_1, \ldots, x_n)$ is an entropy $n$-tuple if and only if for all open neighborhoods $U_i \ni x_i$, $i = 1, \ldots, n$, there exists a set $J = \{d_1 < d_2 < \cdots\} \subset \mathbb{N}$ with positive density such that $\bigcap_{i \in J} f^{-i}(U_{s(i)}) \neq \emptyset$ for every function $s : J \to \{1, \ldots, n\}$.

It is worth emphasizing that the motivation for Proposition 9 can be found in [Bla92, §3], where Blanchard introduced and discussed Property P, which implies u.p.e. (hence mild mixing) and is strictly weaker than strong mixing.
We say that \( K \) is a \textit{maximal entropy set} if \( K \) is an entropy set and for each \( x \in X \setminus K \) the set \( K \cup \{x\} \) is not an entropy set. The existence of a maximal entropy set (in the sense of inclusion) for a system with positive entropy is ensured by the Kuratowski-Zorn Lemma. It is not hard to construct a dynamical system \((X, f)\) such that \( h_{\text{top}}(f, K) < h_{\text{top}}(f) \) for each maximal entropy set \( K \). An example of a transitive system admitting a maximal entropy set which consists of exactly two points is provided in [DYZ06]. It is also proved in [DYZ06] that each system with positive topological entropy admits a maximal entropy set with uncountably many points. Note that for every dynamical system \((X, f)\) it is possible to construct a countable compact subset \( K \) satisfying \( h_{\text{top}}(f, K) = h_{\text{top}}(f) \) (e.g. see [YZ07]).

When we have the definition of \( n \)-tuples at hand, it is natural to state the following definition which comes from [HY06]. For any \( n \geq 2 \) we say that a dynamical system \((X, f)\) has \textit{u.p.e. order } \( n \) if each tuple \((x_1, \ldots, x_n) \in X^n \setminus \Delta_n(X)\) is an entropy \( n \)-tuple. A dynamical system \((X, f)\) is \textit{topological } \( K \) if it has u.p.e. of order \( n \) for every integer \( n \geq 2 \). Obviously, u.p.e. of order \( n + 1 \) implies u.p.e. of order \( n \), however this implication cannot be reversed in general, since there are known examples of u.p.e. flows which do not have u.p.e. of order 3 (see [HY06]). With the help of the variational relationship between topological and measure-theoretic entropy tuples, the following result can be proved (see [Gla97, HY06] or [HYZ07] for particular cases, respectively).

**Proposition 10** Let \((X_1, f_1)\) and \((X_2, f_2)\) be flows and fix an integer \( n \geq 2 \). The product system \((X_1 \times X_2, f_1 \times f_2)\) has c.p.e. (has u.p.e. of order \( n \), is topological } \( K \), respectively) if and only if both \((X_1, f_1)\) and \((X_2, f_2)\) have c.p.e. (have u.p.e. of order \( n \), are topological } \( K \), respectively).

By Theorem 6 any topological } \( K \)-system is mildly mixing; however in some cases we may hope for even stronger amount of mixing. A particular case of such situation is the following result, stated in [HSY05] and implicitly proved in [HY04] with the help of results from ergodic theory. A topological proof of it was later presented in [HLY12, §7].

**Theorem 11** Each minimal topological } \( K \)-system is strongly mixing.

Examples provided in [HY06] show that in general we can not remove the assumption of minimality from Theorem 11.

We say that \( \mu \in \mathcal{M}(X, f) \) is a } \( K \)-\textit{measure} if the measurable dynamical system \((X, \mathcal{B}^\mu_X, \mu, f)\) is a Kolmogorov system, where \( \mathcal{B}^\mu_X \) is the } \( \mu \)-completion of \( \mathcal{B}_X \) (i.e. a subset \( A \subset X \) belongs to \( \mathcal{B}^\mu_X \) if and only if there exist \( B, C \subset \mathcal{B}_X \) such that \( A \setminus (B \cup C) \subset C \) and \( \mu(C) = 0 \)). Clearly if \((X, f)\) admits a } \( K \)-measure \( \mu \in \mathcal{M}(X, f) \) with \( \text{supp}(\mu) = X \) then it is c.p.e. It was strengthened firstly in [GW94] and then in [HY06] as follows.

**Proposition 12** If a flow \((X, f)\) admits a } \( K \)-measure \( \mu \in \mathcal{M}(X, f) \) with \( \text{supp}(\mu) = X \) then it is a topological } \( K \)-system.

The above mentioned properties of } \( K \)-measure were probably the main inspiration for Blanchard et al. [BHM’95], when they were introducing measure-theoretic
entropy pairs. Later, this became an important ingredient for a progress in the topological theory of entropy pairs.

Most results on local entropy theory presented so far have purely topological statements, however a huge part of their proofs relies strongly on facts from ergodic theory. It is a little bit surprising situation and it is natural to expect that there should exist another approach by mostly topological arguments. Such approach, if possible should give another insight into the topic of local entropy theory. Recently such a study was initiated by Kerr and Li in [KL07]. They were able to obtain new proof of some known results (and prove many new results as well), using mainly combinatorial and topological methods. It was possible by studying special types of tuples, which are responsible for some type of topological independence.

There is a strong evidence of relations between positive topological entropy and combinatorial aspects of dynamics. Two particular examples of such situation are the probabilistic notion of independence obtained in [GW95a, §3] via the celebrated Shannon-McMillan-Breiman Theorem, and independent behavior along subsets of positive density in each neighborhood of any entropy tuple reflected in Proposition 9. Observations of this kind gave Kerr and Li a good motivation for a systematic study of independence in topological dynamics. Let us present a very fruitful approach developed by them in [KL07].

Fix any integer \( n \geq 2 \). We say that \((x_1, \cdots, x_n) \in X^n\) is an IE-tuple, if for all open sets \( U_i \ni x_i, i = 1, \cdots, n \) there exists \( D \subset \mathbb{N} \) with positive density such that, \( \bigcap_{i \in J} f^{-1} U_{s(i)} \neq \emptyset \) for each finite nonempty \( J \subset D \) and any map \( s: J \to \{1, \cdots, n\} \). It is worth emphasizing that sets \( D \) with positive density are one of various possibilities considered in [KL07]. While IE-tuples are strongly related to positive topological entropy, other types of sets \( D \) can be used to characterize e.g. topological sequence entropy, etc.

Kerr and Li presented a combinatorial proof of the following result in [KL07, §3]. Let us highlight again the fact that Huang and Ye proved Proposition 9 by applications of the local variational principles, so despite of similarities to previous results, the approach in [KL07] provides a completely new methodology to studies on topological entropy.

**Proposition 13** Fix any tuple \((x_1, \cdots, x_n) \in X^n \setminus \Delta_n(X)\), where \( n \geq 2 \). Then \((x_1, \cdots, x_n)\) is an entropy \( n \)-tuple if and only if \((x_1, \cdots, x_n)\) is an IE-tuple.

The following lemma, inspired by [MV02], is a very important tool in the Kerr and Li approach to independence in topological dynamics (e.g. it is one of the important ingredients in the proof of Proposition 13). It allows us to select coordinates where independence takes place, provided that the given set of functions is big enough. This result is a little bit technical, so we have to fix some terminology.

Let \( k \geq 2 \), let \( Z \) be a nonempty finite set and let \( \mathcal{U} \) be a cover of \([0, 1, \cdots, k]^Z\) consisting of subsets of the form \( \prod_{z \in Z} [i_z]^\pm \), where \( 1 \leq i_z \leq k \) for each \( z \in Z \). For any \( S \subset [0, 1, \cdots, k]^Z \) we denote by \( F_S \) the minimal number of sets in \( \mathcal{U} \) sufficient to cover \( S \).
Lemma 14 Let $k \geq 2$ and $b > 0$. There exists $c = c(k, b) > 0$ such that for every finite nonempty set $Z \subset \mathbb{N}$ and any $S \subset \{0, 1, \ldots, k\}^Z$ with $F_S \geq k^{\#Z/b}$ there exists $W \subset Z$ such that $\#W \geq (\#Z)c$ and $S|_W \supset \{1, \ldots, k\}^W$.

This immediately implies the following combinatorial lemma, proved first by Karpovsky and Milman in [KM78].

Lemma 15 Given $k \geq 2$ and $\lambda > 1$ there is a constant $c > 0$ such that for all $n \in \mathbb{N}$, if $S \subset \{1, 2, \ldots, k\}^{[1,2,\ldots,n]}$ satisfies $\#S \geq ((k-1)\lambda)^n$ then there is a set $I \subset \{1, 2, \ldots, n\}$ with $\#I \geq cn$ and such that $S|_I = \{1, 2, \ldots, k\}^I$.

The above lemmas allow us, for a given dynamical system, to pick up points visiting simultaneously distinct sets by counting the total number of visits distinguished by these sets. This removes necessity of use of invariant measures and ergodic theorems.

Proposition 13 provides another view on entropy pairs. This approach enabled Kerr and Li to obtain new proofs of many results proved first by deep arguments from ergodic theory. Another advantage of Kerr and Li approach, is that it was possible to provide new characterizations of various other dynamical objects in terms of topological independence, initiating a completely new approach in the study of dynamics of pairs. Later we will see how results on entropy pairs can be used to answer some questions on chaotic pairs, e.g. Theorem 39 discussed later in this chapter, will be a nice example of such applications of this theory.

We finish this section with a few comments on further research and some open questions. First note, the following question is a natural extension of Theorem 11.

Problem 16 Does every minimal u.p.e. system have to be strongly mixing?

It is well known that in general, topological strong mixing or weak mixing does not have to imply measurable strong mixing or weak mixing, respectively. Simply, it is possible to construct uniquely ergodic proximal systems which are topologically weakly mixing or strongly mixing. For example, a construction of such a topologically weakly mixing system can be done by [LZ73, Theorem 1.3], since if symbol 1 is sufficiently rare in any sequence of the space (say it has zero density) then any invariant measure of the system will be supported on its unique minimal set $\{0^\omega\}$. In such a case, the only invariant measure is concentrated on the unique fixed point in the system, so from the point of view of ergodic theory such a system is not interesting. But in these systems intensive deformation of open sets take place, so from topological point of view dynamics is definitely non-trivial. This simple example shows that it may be hard to describe the space of invariant measures looking only on topological structure of orbits. Nevertheless, some topological properties seem to be strong enough to influence also measurable aspects of dynamics. One potential situation of this kind is described in the following question.

Problem 17 When does a topological K-system admit nice measure-theoretic properties?
It is also worth emphasizing that recently in [GLW11] the authors related mixing properties of invariant measures of a measurable dynamical system \((X, \mathcal{B}, \mu, f)\) to topological properties of some special associated topological dynamical system \((\tilde{X}, \tilde{f})\). It is interesting that this is a complete characterization. For example, measure \(\mu\) is weakly mixing if and only if \((\tilde{X}, \tilde{f})\) is transitive, etc. Unfortunately, it exceeds the capacity of this chapter to provide all definitions from [GLW11], which are needed to make the above statement mathematically precise. The reader is referred to [GLW11] and references therein for more details on the so-called topological lens of \((X, \mathcal{B}, \mu, f)\).

As we remarked before, in this survey we emphasize mainly topological aspects of dynamical systems. We apologize in advance to anyone whose contributions were overlooked. Here, we shall mention only some of these skipped results, while the full list of references contributing to this theory is far beyond. The reader interested in measurable aspects of entropy is referred to the survey [GY09] and reference therein (some other aspects are also discussed in [GW06] or [Zha12b]). We only mention that measurable aspects of local entropy theory, including the local variational principles were studied among others in [BGH97, BHM*95, DYZ06, Gla97, GW06, HMRY04, HY06, Rom03, Sha07, YZ07, Zha09]; topological analogue of a Kolmogorov system in the relative setting of factor maps were explored in [GW95b, HYZ06, HYZ07, LS01, PS01]; sequence entropy tuples and complexity tuples were discussed in [ALP05, BHM00, HLSY03, HMY04, HY04, HY07, HY09, Zha07]; many results on combinatorial independence, both topological and measure-theoretic, can be found in [Gla07, GW95a, HLY12, Hua06, KL05, KL07, KL09].

When speaking about entropy it is worth emphasizing that from the early beginning of the theory of dynamical systems, various authors were interested not only in the study of dynamics of a single transformation but more often in a general setting of groups of homeomorphisms acting on \(X\). Since publication of the pioneering paper [OW87] by Ornstein and Weiss in 1987, studies on action of amenable groups became quite popular, mainly because of many analogues to the case of iteration of a single homeomorphism. Recently, L. Bowen introduced a notion of entropy for measure-preserving actions of a countable discrete sofic group admitting a generating measurable partition with finite entropy [Bow10, Bow12]. Very soon after [Bow10], in the spirit of L. Bowen’s measure-theoretic entropy, Kerr and Li developed an operator-algebraic approach to entropy [KL11a, KL11b] which applies not only to continuous actions of countable sofic groups on compact metric spaces but also to all measure-preserving actions of countable sofic groups on standard probability measure spaces. Due to lack of space, the definition of the amenable or sofic group is omitted here. We only mention that the class of countable discrete amenable groups includes all finite groups and solvable groups, and the class of countable sofic groups includes all countable discrete amenable groups and residually finite groups. Recently, results on entropy pairs and tuples, including local variational principles, were generalized to dynamical system defined by a countable discrete amenable group action in [CL12, HYZ11, KL07, KL09] and to random dynamical system of...
a countable discrete amenable group action in [DZ12]. Part of this theory works well also in the case of a countable sofic group action (see [Zha12a] for more details).

1.4 Proximal pairs and distal points

Proximality and regional proximality are two fundamental notions in topological dynamics. A pair of points \(x, y \in X\) is proximal if \(\lim \inf_{n \to \infty} d(f^n(x), f^n(y)) = 0\) and a point \(x \in X\) is distal if for every \(y \in \text{Orb}^+(x, f)\) \(\{x\}\) the pair \((x, y)\) is not a proximal pair. If for every \(\varepsilon > 0\) the set \(\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \varepsilon\}\) is syndetic, then \(x, y\) are syndetically proximal. Recall that a set \(S \subset \mathbb{N}\) is syndetic if there exists \(M \in \mathbb{N}\) such that \(S\) intersects any block of consecutive \(M\) integers in \(\mathbb{N}\). Denote by \(\text{Prox}(f)\) and \(\text{SynProx}(f)\), respectively, the set of all proximal pairs and syndetically proximal pairs for \((X, f)\). Given \(x \in X\) we define its proximal cell \(\text{Prox}(f)(x) = \{y \in X : (x, y) \in \text{Prox}(f)\}\). It is easy to see that both \(\text{Prox}(f)\) and \(\text{Prox}(f)(x)\) are \(G_\delta\) subsets of respective state spaces.

The following theorem is due to Auslander [Aus60] and Ellis [Ell60]. Since every dynamical system contains a minimal sub-system, this implies that every dynamical system containing a point which is not minimal must contain two distinct points which are proximal.

**Theorem 18** For every \(x \in X\) there is a minimal point \(y \in \text{Orb}^+(x, f)\) such that \((x, y) \in \text{Prox}(f)\). In particular, if \(x\) is a distal point then it is minimal.

Obviously \(\text{Prox}(f)\) is reflexive and symmetric. It is also not hard to provide an example showing that \(\text{Prox}(f)\) does not have to be an equivalence relation, e.g. any transitive system with at least two distinct periodic orbits. The following theorem, which combines results of [Cla63] and [Wu65], characterizes the case when \(\text{Prox}(f)\) is an equivalence relation. It was first proved for the case of homeomorphisms (or more general, actions of groups). A shorter proof, working in the case of surjective maps, was later presented in [Sha06].

**Theorem 19** For any dynamical system \((X, f)\) the relation \(\text{SynProx}(f)\) is an equivalence relation and the following conditions are equivalent:

19.1. \(\text{Prox}(f)\) is an equivalence relation,
19.2. \(\text{Prox}(f) = \text{SynProx}(f)\),
19.3. the closure of orbit of any point \((x, y) \in X^2\) in the dynamical system \((X^2, f^{(2)})\) contains exactly one minimal set.

Furthermore, by results of [Cla63] if \(\text{Prox}(f) \subset X^2\) is a closed subset then \(\text{Prox}(f) = \text{SynProx}(f)\), but it is also easy to see that the converse implication does not hold.

When we analyze dynamical systems from the point of view of proximal relation then we have two extrema. First, we have distal systems, i.e. systems where all points are distal. In this case we have \(\text{Prox}(f) = A_2(X)\). On the other side, we have
the so-called \textit{proximal} systems, i.e. dynamical systems satisfying $\text{Prox}(f) = X^2$. By the above facts, in such a case all pairs are syndetically proximal. It can also be proved (e.g. see [AK03]) that $(X, f)$ is proximal if and only if $(X, f)$ has the unique fixed point, which is the only minimal point of $(X, f)$.

As we said before, if $\text{Prox}(f)$ is closed then it is an equivalence relation (cf. [Aus60]). The converse is not true, even under strong assumption such as transitivity, because as shown by Shapiro in [Sha70], there exists a minimal dynamical system $(X, f)$ where $\text{Prox}(f)$ is an equivalence relation, but is not closed. Since there is no hope to have $\text{Prox}(f)$ closed in general, it is natural to consider its extension which is closed. In this way we come to the following classical definition.

A pair $(x, y)$ is \textit{regionally proximal} if there exist $x_k \to x$ and $y_k \to y$ such that $\lim_{k \to \infty} d(f^{n_k}(x_k), f^{n_k}(y_k)) = 0$ for some sequence of positive integers $n_k$. The \textit{regionally proximal relation}, that is the set of all regionally proximal pairs, is denoted by $\text{RP}(f)$. It is clear that $\text{RP}(f)$ is a closed $f^{[2]}$-invariant subset and that $\text{Prox}(f) \subset \text{RP}(f)$, however it is also not hard to construct an example of $(X, f)$ where $\text{RP}(f) \neq \text{Prox}(f)$ or where $\text{RP}(f)$ is not an equivalence relation. Any minimal distal system $(X, f)$ which is not equicontinuous is an example satisfying $\text{RP}(f) \neq \text{Prox}(f)$, because in that case $\text{Prox}(f) = A_2(X)$ while $\text{RP}(f) \neq A_2(X)$ (see a discussion below on relations between $\text{RP}(f)$, $S_{\text{eq}}(f)$ and equicontinuous factors). A simple example where $\text{RP}(f)$ is not an equivalence relation can be obtained by considering the following map:

$$f: [-1, 1] \ni x \mapsto \text{sign}(x) \sqrt{|x|} \in [-1, 1]$$

where $\text{sign}(x) = 1$ for $x \geq 0$ and $\text{sign}(x) = -1$ for $x < 0$. Namely, $\text{RP}(f) = [0, 1]^2 \cup [-1, 0]^2$ in the above case.

The following basic result shows the utility of regionally proximal relation (see [EG60] or [Aus88, Kür03] and the references therein for a historical context).

\textbf{Proposition 20} Let $\pi: (X, f) \to (Y, g)$ be a factor map, where $(X, f)$ is minimal and $(Y, g)$ is equicontinuous. If $(x, y) \in \text{RP}(f)$ then $\pi(x) = \pi(y)$.

For any dynamical system $(X, f)$ there exists the smallest ICER, denoted by $S_{\text{eq}}(f)$, such that the quotient system $(X/S_{\text{eq}}(f), f_{S_{\text{eq}}(f)})$ is equicontinuous (see [Aus88, Kür03] for more details). It was first proved for flows by Ellis and Gottschalk [EG60] that if $R \subset X^2$ is an ICER and the quotient dynamical system $(X/R, f_R)$ is equicontinuous then $R \supset \text{RP}(f)$. Furthermore, for any minimal flow we have $\text{RP}(f) = S_{\text{eq}}(f)$, so in particular $\text{RP}(f)$ is an equivalence relation (see [Aus88]). By [HY02] the same result is also true in the non-invertible case. The question when $\text{RP}(f)$ is an equivalence relation turns out to be a difficult one, especially when we consider transformation groups in place of homeomorphisms. During last 40 years this question was considered by leading researchers in the field of topological dynamics, including Veech [Vee68], Ellis and Keynes [EK71], McMahon [McM78], Auslander and Guerin [AG97] or Auslander, Grechonig and Nagar [AGN12] to name only a few. In [HKM10] Host, Kra and Maass provided an interesting extension of regionally proximal relation (so-called \textit{regionally proximal relation of order}
denoted by $\text{RP}^{[d]}$). Their approach can be used in the process of construction of the maximal nilfactor of any order for a distal minimal flow (a definition of nilfactor can be found in [HKM10] and [SY12]). The case of order 1 (i.e. $d = 1$) corresponds to the standard construction of the maximal equicontinuous factor with the help of $\text{RP}(f)$. Recently it was proved in [SY12] by Shao and Ye that $\text{RP}^{[d]}$ is an equivalence relation for every minimal flow and every positive integer $d$.

The case of weakly mixing systems is probably the nicest case from the point of view of regionally proximal relation. Simply, directly from the definition of weak mixing we have $\text{RP}(f) = X^2$. It is also not surprising that there are various useful characterizations of proximal relation for these systems.

It is not hard to prove that if $(X, f)$ is a weakly mixing system or if $(X, f)$ is a transitive system with a fixed point, then $\text{Prox}(f) \subset X^2$ is residual. In view of the above, we can ask about the structure of proximal cells $\text{Prox}(f)(x) = \{y \in X : (x, y) \in \text{Prox}(f)\}$. If we take a transitive system $f$ on $[0, 1]$ which is not totally transitive, that is $(0, 1, f^n)$ is not transitive for some $n \in \mathbb{N}$, say $f((0, \frac{1}{2})) = [\frac{1}{2}, 1]$ and $f((\frac{1}{2}, 1)) = [0, \frac{1}{2}]$, then we see that for most of points, $\text{Prox}(f)(x)$ is not residual. The case of weakly mixing systems is completely different. First, it was proved in [KR69] that for weakly mixing systems the set $\{x \in X : \text{Prox}(f)(x) \text{ is residual in } X\}$ is residual in $X$, that is, $\text{Prox}(f)$ has a residual set of parameters where sections are also residual. Later it was proved by Furstenberg in [Fur81] that $\text{Prox}(f)(x)$ is residual for any $x$, provided that $(X, f)$ is minimal and weakly mixing. Finally, Akin and Kolyada provided in [AK03] the following nice result:

**Theorem 21** If a dynamical system $(X, f)$ is weakly mixing then for every $x$ its proximal cell $\text{Prox}(f)(x)$ is dense in $X$.

Since both $\text{Prox}(f)$ and $\text{Prox}(f)(x)$ are $G_\delta$ subsets of respective state spaces, we can equivalently say in the above theorem (and all other theorems on density of these sets) that $\text{Prox}(f)(x)$ is a residual subset of $X$ for every $x$. It is also proved in [AK03] that for minimal systems the above property provides full characterization of weak mixing (see also [HSY04]). This extends result of Auslander [Aus88] who proved it for invertible dynamical systems (note that book [Aus88] considers dynamical properties of actions of groups).

**Theorem 22** For every minimal dynamical system $(X, f)$ the following conditions are equivalent:

22.1. $(X, f)$ is weakly mixing,
22.2. there exists $x \in X$ such that $\text{Prox}(f)(x)$ is dense in $X$,
22.3. $\text{Prox}(f)(x)$ is dense in $X$ for every $x$,
22.4. $\text{Prox}(f)$ is dense in $X^2$.

As a consequence we get the following fact (proved first for minimal flows in a more general setting in [Aus88]):

**Corollary 23** Assume that $(X, f)$ is weakly mixing and $A \subset X$ is at most countable. Then the set $\bigcap_{x \in A} \text{Prox}(f)(x)$ is residual. In particular there exists $z \in X$ proximal with every $x \in A$. 
Syndetically proximal pairs are almost the strongest possible type of proximality, but obviously there is one yet stronger property, that is asymptotic pair. A pair \((x, y)\) is asymptotic when \(\lim_{n \to \infty} d(f^n(x), f^n(y)) = 0\). Denote by \(\text{Asy}(f)\) the set of all asymptotic pairs of \((X, f)\), and for any \(x \in X\) define its asymptotic cell \(\text{Asy}(f)(x) = \{y \in X : (x, y) \in \text{Asy}(f)\}\).

While entropy pairs are “intuitively” connected with complicated dynamics, we are likely to connect asymptotic pairs with “regular” dynamical behavior. Therefore, it is a little bit surprising that these two notions are strongly related with each other, as shown by the following theorem, first proved in [BHR02].

**Theorem 24** For every flow \((X, f)\) we have the inclusion \(E_2(X, f) \subset \text{Asy}(f)\). In particular, if \((X, f)\) is a flow with u.p.e. then \(\text{Asy}(f)\) is dense in \(X^2\).

Combined with Theorem 3, Theorem 24 immediately implies that if a factor map collapses all asymptotic pairs then the entropy of resulting factor is zero. A system with zero entropy can still contain numerous asymptotic pairs, however it was proved in [DL12c] that it always admits a symbolic extension without proper asymptotic pairs. Recently this result was generalized in [DL12a] by showing that there is also an extension without proper forward mean proximal pairs, where a pair \((x, y)\) is forward mean proximal if \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(x), f^i(y)) = 0\). In the view of the above, it also seems likely that if a factor map decreases topological entropy, it must collapse some entropy pairs, and so also some asymptotic pairs. Indeed, this intuition is true as proved in [Zha06] (again by measure-theoretic arguments).

**Theorem 25** Let \(\pi : (X, f) \to (Y, g)\) be a factor map between flows. If \(R_\pi \cap \text{Asy}(f) = \Delta_2(X)\) then \(h_{top}(f) = h_{top}(g)\).

While the closure of the set of asymptotic pairs can be very large, \(\text{Asy}(f)\) itself can never be residual in \(X^2\) as shown by the following theorem proved in [HY02].

**Proposition 26** If \((X, f)\) is a transitive flow and \(X\) is infinite then \(\text{Asy}(f)\) is a first category subset of \(X^2\), and furthermore, for any \(x \in X\) its asymptotic cell \(\text{Asy}(f)(x)\) is a first category subset of \(X\).

While distal systems do not contain proximal pairs by definition, in every system we can find two points which stay close during all iterations. Strictly speaking, we have the following result, which is attributed to S. Schwartzman. Its proof can be found in [GH55] (see also [Kin90]):

**Proposition 27** If \(X\) is infinite, then for every \(\varepsilon > 0\) there are points \(x \neq y\) in \(X\) such that \(d(f^n(x), f^n(y)) < \varepsilon\) for all \(n \in \mathbb{N}\).

Recall that a flow \((X, f)\) is expansive if there exists an expansive constant \(\beta > 0\) such that for any points \(x, y \in X\), if the condition \(d(f^n(x), f^n(y)) < \beta\) holds for every \(n \in \mathbb{Z}\) then \(x = y\).

The notion of expansive flow can be extended to surjective dynamical systems in at least two ways. The first possibility is the following: a dynamical system \((X, f)\) is positively expansive if there exists \(\beta > 0\) such that if \(x, y \in X\) are distinct then
there exists \( n \geq 0 \) such that \( d(f^n(x), f^n(y)) \geq \beta \). Second possible definition is \( c\)-
expansive dynamical systems, that is dynamical systems \( (X, f) \) with expansive natural
extension \( (\mathbb{Z}_f, \sigma_f) \). Obviously, every positively expansive dynamical system is
\( c \)-expansive, but not vice-versa. Simply, every expansive flow is \( c \)-expansive, since
for homeomorphical dynamical systems \( (X, f) \) and \( (\mathbb{Z}_f, \sigma_f) \) are conjugate but an
expansive flow need not to be positively expansive, e.g. two-sided full shift is a par-
icular example of such a situation. In fact, in the case of homeomorphism there is no
an interesting example of positively expansive dynamical system, since every such
system has to operate on a finite space (this result was probably first proved by S.
Schwartzman in 1952 in his PhD thesis; see [CK06] and references therein for a
simple proof and historical comments).

As an immediate consequence of Proposition 27 we obtain that every expansive
(or more generally, \( c \)-expansive) dynamical system has a proper asymptotic pair, in
particular \( \text{SynProx}(f) \setminus A_2(X) \neq \emptyset \). For example, for an expansive flow \( (X, f) \) with
an expansive constant \( \beta > 0 \), by Proposition 27 there exists a pair \( (x, y) \in (\mathbb{X}_f^2 \setminus A_2(X) \)
such that \( d(f^n(x), f^n(y)) < \beta \) for all \( n \in \mathbb{N} \). Now, for any \( \delta > 0 \) from the expansivity
of the flow \( (X, f) \), there exists an integer \( n(\delta) > 0 \) such that if \( d(f^n(p), f^n(q)) < \beta \) for all
\( i = n(\delta), \ldots, n(\delta) \) then \( d(p, q) < \delta \). Therefore, we obtain that \( d(f^n(x), f^n(y)) < \delta \) for all
\( i \geq n(\delta) + 1 \). Since \( \delta > 0 \) was arbitrary, this immediately implies that
\( (x, y) \) is asymptotic. For \( c \)-expansive dynamical systems the situation is the same,
since for every proper asymptotic pair for the natural extension \( (\mathbb{Z}_f, \sigma_f) \) we can
find coordinate with distinct values, and projection on that coordinate produces an
asymptotic pair for \( (X, f) \).

Additionally, King provided in [Kin90] an example of a minimal two-sided subshift \( (X, \sigma) \) such that if \( x, y \) have pairwise disjoint full orbits under \( \sigma \), then
\( \{(x^n, y^n) : n \in \mathbb{Z}\} \) is dense in \( X^2 \). It is not clear if such a system exists for
non-invertible case, that is, if density of full orbits can be replaced by density of
forward orbits, therefore we state the following question.

**Problem 28** Is there a system \( (X, f) \) satisfying one of the following conditions:

28.1. \( (X, f) \) is transitive and \( \text{SynProx}(f) = A_2(X) \neq \text{Prox}(f) \);

28.2. \( X \) is infinite, \( (X, f) \) is minimal and \( (x_1, x_2) \in \text{Tran}(X^2, f^{(2)}) \) for all \( x_1 \neq x_2 \)
belonging to different orbits (i.e. \( \text{Orb}^+(x_1, f) \cap \text{Orb}^+(x_2, f) \neq \emptyset \) and \( \text{Orb}^+(x_1, f) \neq \emptyset \)).

Observe that a positive answer to Problem 28.2 answers also Problem 28.1 affirmatively. To see this, let us start with a system \( (X, f) \) which satisfies Condition 28.2.
Note that there exists a minimal sub-system \( Y \) of \( (X^2, f^{(2)}) \) such that \( Y \cap A_2(X) = \emptyset \).
Obviously \( \text{Prox}(f) \neq A_2(X) \) and any pair (outside \( A_2(X) \)) from the same orbit will
be not proximal, as \( (X, f) \) is an infinite minimal system. Additionally, if \( x_1, x_2 \) are
from different orbits, then \( (x_1, x_2) \in \text{Tran}(X^2, f^{(2)}) \) by Condition 28.2, and so there
is a thick set \( A \) such that \( A \subset \{ n \in \mathbb{Z} : d(f^n(x_1), f^n(x_2)), Y < \gamma \} \), where \( \gamma > 0 \) is at
most half of the distance between \( Y \) and \( A_2(X) \), in particular, \( (x_1, x_2) \notin \text{SynProx}(f) \).

Indeed, \( \text{SynProx}(f) = A_2(X) \).

There is a strong connection between the example of [Kin90] and systems with zero entropy. First let us state a formal definition which classifies systems
like the example in [Kin90]. A flow \((X, f)\) is doubly minimal (cf. [Wei98]) or has topologically minimal self-joining (cf. [DJ87]) if for every \(x \in X\) and every \(y \neq f^n(x) : n \in \mathbb{Z}\) the full orbit \(\{(f^n(x), f^n(y)) : n \in \mathbb{Z}\}\) is dense in \(X^2\). Observe that every doubly minimal system \((X, f)\) is minimal, since if \(x \in X\) is minimal and \(y\) is not, then the set \(\{(f^n(x), f^n(y)) : n \in \mathbb{Z}\}\) cannot be dense in \(X^2\). It is worth mentioning that Weiss proved in [Wei98] that every ergodic measurable dynamical system \((\Omega, m, T)\) with zero measure-theoretic entropy has a uniquely ergodic model \((X, f)\) which is double minimal, that is the flow \((X, f)\) is doubly minimal, has a unique ergodic \(f\)-invariant Borel probability measure \(\mu\) and the measurable dynamical systems \((\Omega, m, T)\) and \((X, \mu, f)\) are measure-theoretically isomorphic. Furthermore, Weiss proved by topological methods in [Wei98] that doubly minimal systems must have zero topological entropy, and one way to see this is through the notion of p-points in [AB90] by Auslander and Berg. As a direct application of the above results we can obtain a non-trivial doubly minimal system (and so with zero topological entropy) which is topologically strongly mixing. Simply it is enough to start from a non-trivial ergodic dynamical system which has measure theoretic entropy zero and is measurably strongly mixing (there are numerous examples of such systems, for example a system constructed in [Ger81]).

As a direct corollary of Theorem 24, if every point of \(X^2\) is recurrent under \(f^2\), that is \(X^2 = \text{Rec}(X^2, f^2)\), then the flow \((X, f)\) has zero topological entropy, which was first proved by Weiss in [Wei98] using ergodic theoretic ideas. However, the following question from [Wei98] still remains open.

**Problem 29** Let \((X, f)\) be a flow and assume that for every pair of points \(x, y \in X\) there exists a sequence \(\{n_k\}_{k=0}^\infty \subset \mathbb{Z}\) such that \(\lim_{k \to \infty} (f^{n_k}(x), f^{n_k}(y)) = (x, y)\) and \(\lim_{k \to \infty} |n_k| = \infty\). Can entropy of \((X, f)\) be positive?

While Theorem 24 and related facts refer to purely topological notions, all the proofs known to the authors rely on methods from ergodic theory. This motivates us to state the following question.

**Problem 30** How to prove the following results by solely topological or combinatorial arguments, i.e. without using properties of invariant measures:

30.1. \(\text{Asy}(f) \supset E_2(X, f)\).
30.2. If a factor map \(\pi : (X, f) \to (Y, g)\) satisfies \(\text{Asy}(f) \subset R_\pi\) then \(h_{\text{top}}(g) = 0\).

The proof of Theorem 3 does not involve measurable aspects of dynamical systems, hence, similar to Problem 28, a positive answer to Problem 30.1 answers also Problem 30.2 affirmatively with the help of Theorem 3.

By Theorem 24, if a flow \((X, f)\) has positive entropy then it has proper asymptotic pairs (i.e. \(\text{Asy}(f) \neq A_2(X)\)) and the same is true for its inverse, as \(h_{\text{top}}(f) = h_{\text{top}}(f^{-1})\). It does not imply, however, that there are proper bi-asymptotic pairs, that is, we cannot guarantee that if \(\text{Asy}(f) \cap \text{Asy}(f^{-1}) \neq A_2(X)\). While not much is known on relations between such pairs and entropy for a general dynamical system, it is possible to construct a flow \((X, f)\) with positive topological entropy which is not expansive, where all bi-asymptotic pairs are trivial, that is \(\text{Asy}(f) \cap \text{Asy}(f^{-1}) = A_2(X)\), e.g. see [LS99, Example 3.4].
Another flavor of this problem can be seen in [Kin90], where we have an example of two-sided minimal and infinite subshift (so both $\text{Asy}(f)$ and $\text{Asy}(f^{-1})$ contain pairs outside the diagonal), in which all points from different orbits are dense in $X^2$, in particular cannot be bi-asymptotic. But distinct points from the same orbit cannot by asymptotic as well, since $(X, f)$ is minimal and $X$ is infinite. Therefore, this system is expansive and satisfies $\text{Asy}(f) \cap \text{Asy}(f^{-1}) = A_2(X)$, however by the method of construction it also has entropy zero (see [Kin90]).

Then the following question was left open in [CL12] (see also [Zha12b]).

**Problem 31** Is there an expansive flow $(X, f)$ with positive topological entropy such that $\text{Asy}(f) \cap \text{Asy}(f^{-1}) = A_2(X)$?

Actions of a countable discrete group on a compact metrizable group by continuous automorphisms are a rich class of dynamical systems, and have drawn much attention since the beginning of ergodic theory. In a particular case of so-called algebraic actions there are many tools that help to deal with the relationship between asymptotic pairs and topological entropy if the action is additionally expansive, as shown by Lind and Schmidt [LS99] in the case of algebraic actions of $\mathbb{Z}^d$ or by Chung and Li [CL12] in the case of algebraic actions of polycyclic-by-finite groups by a deep investigation of a concept of homoclinic groups for algebraic actions.

In [HKM10] using some deep results from ergodic theory Host, Kra and Maass proved that if $(X, f)$ is distal and minimal and $d \in \mathbb{N}$ then the quotient system $(X/\mathbb{R}P^d, f_{\mathbb{R}P^d})$ is an inverse limit of $d$-step minimal nilsystems (for more details see [HKM10, §5 and §6]). The following question was suggested to the authors by X. Ye.

**Problem 32** Is there a topological proof avoiding measure-theoretic arguments for the result that $(X/\mathbb{R}P^d, f_{\mathbb{R}P^d})$ is an inverse limit of $d$-step minimal nilsystems when $(X, f)$ is a distal minimal system and $d \in \mathbb{N}$?

We presented here only two types of proximal pairs, however some further generalizations are possible. For example [Sha06] studies proximality in terms of Furstenberg families other than syndetic or infinite sets. Similar to the case of entropy pairs, it is also clear that proximality and regional proximality can be defined for more general group actions. Good place to start are the books by Glasner [Gla76] and Auslander [Aus88]. Especially the structure of proximal and regionally proximal relations in minimal group actions were deeply studied in recent years (e.g. see [AEE95, Aus01, Aus04, Pen98]).

### 1.5 Chaotic pairs

Before we start our exposition on chaotic pairs, let us first present a result which recently became a popular tool for finding large sets of chaotic pairs. It originated from papers of Mycielski [Myc64] and Kuratowski [Kur73], but probably it was Iwanik [Iwa89], who first observed utility of this technique in the construction of scrambled
sets. Recent paper by Akin [Aki04] collects and extends results on Mycielski sets, providing numerous applications to dynamical systems at the same time. Results of Kuratowski and Mycielski are valid on a perfect complete and separable metric space; however we present them in the context of compact metric spaces only, since it is enough for our considerations.

It is well known that the set of all Cantor subsets of $X$ forms a residual subset of $2^X$ when $X$ is perfect (e.g. see [Kur73]). By the above and the Baire Category Theorem, if we have a residual subset $Q$ of $2^X$ and $X$ is perfect then for any given subset $A$ of $X$ the set $Q$ contains a Cantor set $C$ which approximates $A$ well, that is for any fixed $\varepsilon > 0$ we can find a Cantor set $C \in Q$ such that Hausdorff distance $H_d(A, C) < \varepsilon$. Unfortunately, we cannot guarantee that the union of a sequence of Cantor sets constructed in that way will be also in $Q$. This difficulty can be removed easily by some rather mild assumption on $Q$. We say that $Q$ is hereditary if for every $A \in Q$ all nonempty compact subsets of $A$ are also in $Q$, i.e. $2^A \subset Q$. With this assumption it is possible to prove the following result (see [Aki04]).

**Theorem 33** If $Q$ is a hereditary residual subset of $2^X$, where $X$ is a perfect compact metric space, then there are Cantor sets $C_i$ such that $\{C_i : i \in \mathbb{N}\}$ is dense in $2^X$ and furthermore $\bigcup_{i=1}^{\infty} C_i \in Q$ for every $n$.

Let $R \subset X^n$ for some $n \geq 2$ and denote by $J(R) \subset 2^X$ the family of all nonempty compact sets $A \subset X$ such that if points $x_1, \cdots, x_n \in A$ are pairwise distinct (i.e. $x_i \neq x_j$ when $i \neq j$) then $(x_1, \cdots, x_n) \in R$. Note that from the definition it is direct that $J(R)$ is hereditary, $J(\bigcap_k R_k) = \bigcap_k J(R_k)$ and if $R_1 \subset R_2$ then $J(R_1) \subset J(R_2)$.

**Remark 34** We emphasize here, that some authors prefer to work with complements of relations $R$, which is originally in [Kur73]. In particular $J(R)$ defined in [Kur73] is just $J(X \setminus R)$ as defined here. However, when dealing with dynamical properties sometimes it is more natural to consider residual relations rather than their first category complements, e.g. it is the case of Prox($f$).

The following theorem is the main result of [Kur73].

**Theorem 35** Let $X$ be a perfect compact metric space and fix $n \geq 2$. If $R \subset X^n$ is a dense open set then $J(R)$ is residual (and so dense) in $2^X$.

As an immediate consequence of the above theorem we have the following result (see also [Myc64]).

**Corollary 36** Let $X$ be a perfect compact metric space, and assume that $R_k$ is a residual subset of $X^n$, where $n_k \geq 2$ for each $k \in \mathbb{N}$. Then the set $Q = \bigcap_{k=1}^{\infty} J(R_k)$ is a hereditary residual subset of $2^X$. In particular, there exists a Mycielski set $M$ dense in $X$ such that for each $k \in \mathbb{N}$ if points $x_1, \cdots, x_{n_k} \in M$ are pairwise distinct then $(x_1, \cdots, x_{n_k}) \in R_k$.

The following definition of Li-Yorke chaos is based on the ideas in [LY75]. After [BGKM02] we say that points $x, y \in X$ form a Li-Yorke pair of modulus $\delta > 0$ if
\[ \lim_{n \to \infty} d(f^n(x), f^n(y)) > \delta \quad \text{and} \quad \lim_{n \to \infty} d(f^n(x), f^n(y)) = 0. \]

Then by a \textit{Li-Yorke pair} we mean a Li-Yorke pair of modulus \( \delta \) for some \( \delta > 0 \).

Now, we are going to present one of possible extensions of the definition of Li-Yorke pair, known presently under the common name of distributional chaos. It was introduced in [SS94] as a property which fully characterizes positive topological entropy in the case of dynamical systems on the unit interval (cf. Theorem 43). For any \( n \in \mathbb{N} \), points \( x, y \in X \) and \( t \in \mathbb{R} \) let

\[ \Phi^{(n)}_{xy}(t) = \frac{1}{n} \# \left\{ i : d(f^i(x), f^i(y)) < t, 0 \leq i < n \right\}. \]

We denote by \( \Phi_{xy} \) and \( \Phi^*_{xy} \) the following two functions:

\[ \Phi_{xy}(t) = \lim_{n \to \infty} \Phi^{(n)}_{xy}(t), \quad \Phi^*_{xy}(t) = \lim_{n \to \infty} \Phi^{(n)}_{xy}(t). \]

Clearly both functions \( \Phi_{xy} \) and \( \Phi^*_{xy} \) are nondecreasing, \( \Phi_{xy}(t) = \Phi^*_{xy}(t) = 0 \) for \( t \leq 0 \) and \( \Phi_{xy}(t) = \Phi^*_{xy}(t) = 1 \) for \( t \) which is strictly larger than the diameter of \( X \).

We say that a pair \( x, y \in X \) is:

- DC1: if \( \Phi^*_{xy}(t) = 1 \) for all \( t > 0 \) and there is \( s > 0 \) such that \( \Phi_{xy}(s) = 0 \),
- DC2: if \( \Phi^*_{xy}(t) = 1 \) for all \( t > 0 \) and there is \( s > 0 \) such that \( \Phi_{xy}(s) < 1 \),
- DC3: if there are \( a < b \) such that \( \Phi^*_{xy}(t) > \Phi_{xy}(t) \) for every \( t \in (a, b) \).

While all the above definitions of DC1, DC2 and DC3 appeared implicitly in the first paper on distributional chaos [SS94], the terminology was evolving slowly to the present form (for more details, see [BSˇS05] and references therein).

Let \( \delta > 0 \). We say that a set \( S \) is \textit{scrambled}, \( \delta \)-\textit{scrambled}, \textit{distributionally scrambled} (of type 1, 2 or 3), if every distinct points \( x, y \in S \) form, respectively, a Li-Yorke pair, Li-Yorke pair of modulus \( \delta \), DC1, DC2 or DC3 pair.

Usually, systems with an uncountable scrambled set are referred to as \textit{chaotic in the sense of Li and Yorke} or \textit{Li-Yorke chaotic}. The first motivation for studying this notion comes from the theory of interval transformations. For such maps the existence of a Li-Yorke pair implies the existence of an uncountable scrambled set [KS89], which is not the case in the general setting (more comments on cardinality of scrambled sets the reader can find in [BDM04, BHS08]). A dynamical system \((X, f)\) is \textit{distributionally chaotic}, if there exists an uncountable distributionally scrambled set of type 1. As we mentioned earlier, definition of distributional chaos originated from paper [SS94] by Schweizer and Smítal, however the name distributional chaos was introduced later.

Observe that if a system contains no Li-Yorke pairs then every pair of points is either distal or asymptotic. Because of this, the authors of [BGKM02] called such systems \textit{almost distal}. It is not hard to see that any product of almost distal systems is still almost distal. The following additional properties of almost distal systems (first proved in [BGKM02]) are not that obvious.
Theorem 37 Let \((X, f)\) be almost distal and \(\pi: (X, f) \to (Y, g)\) a factor map. Then \((Y, g)\) is also almost distal.

Theorem 38 Any transitive almost distal system is minimal.

By Proposition 27 we immediately see that for any \(\varepsilon > 0\) the whole space \(X\) cannot be an \(\varepsilon\)-scrambled set. However the situation is completely different when we consider scrambled sets.

We say that \((X, f)\) is completely scrambled if the whole space \(X\) is a scrambled set. The first explicit examples of completely scrambled dynamical systems on infinite compact spaces (or even on continua) were obtained in [HY01]. But it is also worth emphasizing that some of the constructions from [HY01] can be derived from much older works of Katznelson and Weiss [KW81]. Namely, in [KW81] the authors, inspired by much older construction of Nemyckii from [Nem49], among other results, sketched an example of a proximal system where all points are recurrent. Later this technique was extended in [AAB96], together with the proof (see [AAB96, Theorem 4.1]) that the above construction always leads to a proximal system \((X, f)\) which is uniformly rigid, that is \(f^n\) converges uniformly to the identity transformation \(id_X\) for a properly chosen increasing sequence of integers \(n_i\). Clearly these systems are completely scrambled, since they are not only uniformly rigid but also proximal. Observe that by Theorem 24 a uniformly rigid system always has zero topological entropy, since it cannot have proper asymptotic pairs.

But it may also happen that \((X, f)\) has a quite large scrambled set, while it is not completely scrambled. For example [BHS08] described substitutive (but non-minimal) dynamical systems possessing a residual scrambled set and without being completely scrambled. Other examples are provided by results of L. Snoha who fully characterized maps on the unit interval with a residual set of Li-Yorke pairs [Sno92] (see also [Sno90]) combined with results of Mai [Mai97] who proved that scrambled sets on topological graphs always have an empty interior (this result was later strengthened in [BHS08]).

It is natural to ask which topological properties are sufficient for the existence of Li-Yorke pairs or a non-trivial scrambled set. It is also natural to ask what are relations between Li-Yorke pairs and other types of pairs considered so far. While proximality is a prerequisite for a Li-Yorke pair, the relation between Li-Yorke pairs and entropy pairs (or positive entropy in general) is not that clear. These relations were decided very quickly in the original setting of interval transformations. There were constructed examples of interval maps with zero topological entropy and non-trivial scrambled sets [Smi86], while it was also known that every such map with positive topological entropy always has an uncountable scrambled set [JS86]. Extending this implication to a general dynamical system turned out to be a hard task, and the question remained open for many years. Finally, development made on properties of entropy pairs brought a new insight into this topic and finally an affirmative answer to the question was provided. First, in [BGKM02], Blanchard et al. proved that every system with positive topological entropy is Li-Yorke chaotic and later Kerr and Li in [KL07] used the notion of IE-tuple to extend this result as follows.
Theorem 39  Fix any \( n \geq 2 \), any \((x_1, \cdots, x_n) \) \( \in X^n \setminus \Delta_n(X) \) and any open sets \( x_i \in U_i \), where \( i = 1, \cdots, n \). If \((x_1, \cdots, x_n)\) is an IE-tuple then for each \( i = 1, \cdots, n \) there exists a Cantor set \( Z_i \subset U_i \) such that:

39.1. every nonempty finite tuple of points from \( Z = \bigcup_{i=1}^n Z_i \) is an IE-tuple, and
39.2. for each \( m \in \mathbb{N} \) and distinct \( y_1, \cdots, y_m, z_1, \cdots, z_m \in Z \) we have

\[
\liminf_{k \to \infty} \max_{1 \leq j \leq m} d(f^k(y_j), z_j) = 0.
\]

As a direct corollary of the above result we obtain that the set of topological entropy pairs contains a dense subset consisting of Li-Yorke pairs. This was first observed by exploring properties of measure-theoretic entropy pairs in [BGKM02], giving Kerr and Li a good motivation for their combinatorial approach.

As we saw before, even in the context of interval maps topological entropy is not the only condition sufficient for Li-Yorke chaos. In fact, Li-York pairs are strongly related to various types of transitivity. First, Iwanik observed in [Iwa91] that weak mixing is such a sufficient condition, and next in [HY02] Huang and Ye made a similar observation for transitive but not minimal dynamical systems with periodic points. These results can be summarized as follows:

Theorem 40  Let \((X, f)\) be a transitive system containing at least two points.

40.1. If \((X, f)\) has a fixed point then it is Li-Yorke chaotic. If additionally the set of all periodic points is dense in \( X \) then there exist \( \varepsilon > 0 \) and a dense Mycielski subset \( S \subset X \) which is \( \varepsilon \)-scrambled.
40.2. If \((X, f)\) is weakly mixing then there exists a dense Mycielski subset \( S \subset X \) which is \( \varepsilon \)-scrambled, for every \( \varepsilon > 0 \) smaller than the half of the diameter of \( X \).

By Periodic Decomposition Theorem of Banks [Ban97] and the fact that any totally transitive dynamical system with dense periodic points is weakly mixing we obtain easily the following result.

Corollary 41  Let \((X, f)\) be a transitive system with dense periodic points. If \( X \) is infinite then there exists an \( \varepsilon > 0 \) and a Mycielski \( \varepsilon \)-scrambled set \( S \subset X \) such that \( \bigcup_{i=0}^n f^i(S) \) is dense in \( X \) for some \( n \in \mathbb{N} \).

Pioneering works mentioned before initiated intensive studies on relations between transitivity and scrambled sets. The following criterion for chaos obtained in [AGH+10] is a nice example of possible extensions of work from [HY02].

Theorem 42  Let \((X, f)\) be a transitive system without isolated points. Assume that \( Y \) is a closed, invariant and nonempty subset of \( X \) such that the system \((X \times Y, f^{(2)})\) is transitive. Then there are Cantor sets \( C_1 \subset C_2 \subset \cdots \) such that:

42.1. the set \( K = \bigcup_{i=1}^\infty C_i \) is a dense subset of \( \text{Tran}(X, f) \);
42.2. for every \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) there is \( k > 0 \) such that \( d(f^k(x), x) < \varepsilon \) for every \( x \in C_n \) and
42.3. for every \( n \in \mathbb{N} \), every continuous function \( \phi: K \to Y \) and every \( \varepsilon > 0 \) there is \( k > 0 \) such that \( d(f^k(x), \phi(x)) < \varepsilon \) for every \( x \in C_n \).

By the above theorem we have another evidence that non-trivial transitive systems containing a fixed point are Li-Yorke chaotic.

Recall that if \((X, f)\) is a transitive system with dense periodic points and \( X \) is infinite, then \((X, f)\) is chaotic in the sense of Devaney (by results of [BBC+92] the third condition in Devaney’s original definition [Dev03] is redundant). This shows that Li-Yorke chaos may be a prerequisite for complicated dynamics.

In [SS94] Schweizer and Smítal proved the equivalence of distributional chaos and positive topological entropy for maps on the unit interval, which was later generalized to topological graphs as follows (see [MO11] and references therein for more historical remarks).

**Theorem 43** Let \((X, f)\) be a dynamical system, where \( X \) is a topological graph. Then the following conditions are equivalent:

43.1. \( h_{top}(f) > 0 \),
43.2. there exists a DC3 pair for \((X, f)\),
43.3. there exists a Cantor set which is distributionally scrambled of type 1.

Unfortunately, in general, connections between distributional chaos and positive topological entropy are not as tight as in the case of Li-Yorke chaos. For example, there exist minimal systems with zero topological entropy which have an uncountable distributionally scrambled set of type 1. The first example of this kind was obtained in [LF98]. Later in [Opr09a] it was proved that full shift contains an uncountable family of minimal systems of this kind. This paper also proves that a proximal system can never have DC1 pairs. Therefore, by taking any proximal system with positive entropy, we see that positive topological entropy is not strong enough to imply distributional chaos. The first example showing this was obtained in [Pik07]. For a while some authors were hoping that positive topological entropy and some other topological condition such as strong mixing can be enough to imply distributional chaos. This was motivated by the fact that the specification property, which implies both positive topological entropy and strong mixing, is sufficient for distributional chaos [Opr07] (first attempts for proving this were made in [BSSS03]; see also [OŠ08]). Unfortunately, there exist proximal strongly mixing systems with entropy arbitrarily close to entropy of the full shift [Opr10a], so there is also no DC1 pairs in such system. But in the strongly mixing system from [Opr10a], the same as many other examples of dynamical systems with positive topological entropy without DC1 pairs, there are plenty of DC2 pairs (in fact, a Cantor subset which is distributionally scrambled of type 2). Therefore, for many years the claim that this situation is not accidental remained open. Recently Downarowicz proved in [Dow12] the following result.

**Theorem 44** If \((X, f)\) has positive topological entropy then there exists a Cantor set \( S \subset X \) which is distributionally scrambled of type 2.
Despite the fact that all the objects in Theorem 44 have purely topological definitions, the proof relies strongly on the Shannon-McMillan-Breiman Theorem and other advanced methods of ergodic theory. It is noteworthy, that results of [Dow12] were further developed in [DL12b], where the authors search for definition of chaos suitable for measurable dynamical systems (i.e. suitable for ergodic theory approach). Still, there is hope that, similar to Theorem 39 for Li-Yorke chaos, a purely topological or combinatorial proof of Theorem 44 can be developed in the future.

While in the definition of various types of chaos, we require only an uncountable scrambled set \( S \), it is desirable that \( S \) is a Cantor set. In fact, in many situations it was possible to prove the existence of a Cantor set which is scrambled, and recently it was proved in [BHS08] that if for some \( \varepsilon > 0 \) a dynamical system \((X, f)\) has an uncountable \( \varepsilon \)-scrambled set then \((X, f)\) has a Cantor \( \varepsilon \)-scrambled set. The proof in [BHS08] was obtained by a clever application of Corollary 36. Unfortunately it is not clear how to use this technique for the case of scrambled sets (i.e. when it is impossible to pick up one uniform parameter \( \varepsilon > 0 \) for all Li-Yorke pairs in a scrambled set). The main problem is that it is hard to determine if the set of all Li-Yorke pairs \( D \) contained in a perfect set \( A \times A \) is \( G_\delta \), when it is dense, that is when \( D = A \times A \). In other words, it is not easy to decide if Corollary 36 can be applied in this case or not. Therefore, the following question, which we repeat after [BHS08], still remains open.

**Problem 45** When \((X, f)\) has an uncountable scrambled set, does it also have a Cantor scrambled set?

Still, there are some examples of a scrambled set \( S \), when there is no a uniform parameter \( \varepsilon \) (i.e. \( S \) does not contain an uncountable \( \varepsilon \)-scrambled set for any \( \varepsilon > 0 \)) but the answer to Problem 45 is positive (and Cantor scrambled set is contained in \( S \)). Particular examples of such situation are strong Li-Yorke pairs, that is pairs \( (x, y) \) which are Li-Yorke and recurrent for \((X^2, f^2)\) at the same time. Namely, the set of all strong Li-Yorke pairs is a \( G_\delta \) subset of \( X^2 \), which again allows us to work with Corollary 36. Authors of [AAG08] call a dynamical system \((X, f)\) semi-distal if it does not contain strong Li-Yorke pairs, and prove that systems with positive topological entropy are never semi-distal (see [AAG08] for further properties of semi-distal systems). Note that semi-distal dynamical systems form a wider class than almost distal systems discussed earlier.

The topic of topological chaos is quite large presently, so it is impossible to discuss all important developments of recent years. The reader is encouraged to refer to [Bla09] for a more extensively introductory of this topic (see also the paper [BHS08]). Here we only mention a few possible directions for further reading. In this section we presented many results on size of scrambled sets in terms of their placement in the space (e.g. its density). The other question on the size of scrambled sets, expressed in terms of Lebesgue measure, were considered in the case of unit interval or the unit cube (e.g. in [BH87, BJL99, Mis85]). Later, the same questions were addressed for the case of distributionally scrambled sets in [OŠ08, Šte07] (see references in [OŠ08] for a more complete history of this research). Possibility of
extension of these results to other types of compact manifolds (e.g. two dimensional surfaces) is not known.

One of the motivations for introducing distributional chaos was its equivalence to positive topological entropy in the setting of interval maps. But it is not the only possibility. Another extension of Li-Yorke pairs are so-called $\omega$-chaotic pairs introduced by S. H. Li in [Li93]. This definition strongly relies on the structure of $\omega$-limit sets in the system, and so there is almost no hope for application of Theorem 36 in the case of this type of pairs. In fact, not much is known about this type of pairs, beyond maybe a few results, while in some cases it is still possible to construct a dense Mycielski scrambled set consisting only of $\omega$-chaotic pairs [SS03]. Another equivalent condition for positive topological entropy on the interval is obtained when we additionally assume that a scrambled set is invariant for some iteration (e.g. see [Du05] or [Opr09b]). Systems with dense Mycielski invariant scrambled sets are kind of intermediate step between Li-Yorke chaotic and completely scrambled systems.

In 1987 it was proved by Gedeon [Ged87] that a dynamical system on the interval can never have a residual scrambled set, and later this was extended to all topological graphs [BHS08]. From the other point of view, some dendroids can admit completely scrambled flows [HY01], while on some other dendroids such homeomorphism can never exist [Nag11]. The topological characterization of spaces (or even dendroids) admitting completely scrambled flows is still far from complete.

Recently Moothathu initiated in [Moo11] studies on Li-Yorke pairs which are syndetically proximal pairs at the same time. Later, extensive studies on this type of Li-Yorke pairs were undertaken in [MO12]. While both papers [MO12, Moo11] provide some insight into the structure of systems with syndetically proximal Li-Yorke pairs, many questions remain open (e.g. see questions stated in [Moo11]).

### 1.6 Weakly mixing pairs, tuples and sets

Weak mixing of a dynamical system, defined as transitivity of $(X^2, f^2)$ is a global property which was an object of study for many years. It strongly contradicts distality, in the sense that distal minimal systems are always disjoint from weakly mixing ones [Fur67]. Additionally, mixing is usually connected with some kind of non-predictability in the dynamics, in particular as it was pointed out in previous section, weak mixing always imply Li-Yorke chaos. By the definition, weak mixing of a dynamical system is much stronger than transitivity but, in some sense, weaker than positive topological entropy as will be explained in this section. As we said before, the classical definition of weak mixing is a global property. However, it is also possible to define local versions of weak mixing, which to some extent mimic features of the global definition. In recent years there were various attempts to provide a “good” version of “local” weak mixing. In this section we will present most successful approaches to this problem.
Let us first present the concept of weakly mixing pairs. In [Pet70] Petersen was using the condition that \((X, f)\) is not weakly mixing if and only if there are nonempty open sets \(U, V \subset X\) such that there is no \(n \in \mathbb{N}\) with \(U \cap f^{-n}(U) \neq \emptyset\) and \(U \cap f^{-n}(V) \neq \emptyset\) (compare with a list of conditions to weak mixing in [Ban99]). Motivated by this property, Huang et al. introduced the concept of weakly mixing pairs in [HLSY03], which was next generalized to tuples in [MS07].

Given \(n \geq 2\), we say that \((x_1, \cdots, x_n) \in X^n \setminus A_n(X)\) is a **weakly mixing \(n\)-tuple** if \(\bigcap_{i=1}^{n} N(U_i, U_i) \neq \emptyset\) for any open neighborhoods \(U_1, \cdots, U_n\) of points \(x_1, \cdots, x_n\), respectively, where \(N(U, V) = \{k \in \mathbb{Z}_+ : U \cap f^{-k}V \neq \emptyset\}\). Denote by \(WM_n(X, f)\) the set of all weakly mixing \(n\)-tuples of \((X, f)\).

Note that the definition of weakly mixing tuple is not symmetric (we look only on transfer times from the set \(U_1\) but not from others), therefore it is not surprising that even the relation \(WM_2(X, f)\) does not have to be symmetric, as shown in [HLSY03].

Strictly speaking, there exists a transitive system \((X, f)\) and points \(x, y \in X\) such that \((x, y) \in WM_2(X, f)\) but \((y, x) \notin WM_2(X, f)\).

The following result of [HLSY03] shows that there are strong relations between weakly mixing pairs and weak mixing of \((X, f)\), which can also be deduced from a more general analysis on aspects of topological properties defined in terms of directed graphs [Ban99].

**Theorem 46** \((X, f)\) is weakly mixing if and only if \(WM_2(X, f) = X^2 \setminus A_2(X)\).

The regionally proximal pairs and weakly mixing pairs are related together by the following result proved in [HLSY03] (see [MS07] for generalization of this result to tuples).

**Proposition 47** If \((X, f)\) is a flow then \(WM_2(X, f) \subset RP(f) \setminus A_2(X)\). If additionally \((X, f)\) is minimal then \(WM_2(X, f) = RP(f) \setminus A_2(X)\).

It was also shown in [HLSY03] that \(WM_2(X, f) \subsetneq RP(f) \setminus A_2(X)\) for some flows that are not minimal.

Long time ago, Furstenberg proved in [Fur67] that \((X^2, f^{(2)})\) is transitive if and only if \((X^n, f^{(n)})\) is transitive for all \(n \geq 2\). In other words, if \((X, f)\) is weakly mixing, then for any nonempty open sets \(U_1, \cdots, U_n, V_1, \cdots, V_n\) there is \(k > 0\) such that \(f^k(U_i) \cap V_i \neq \emptyset\) for every \(i = 1, \cdots, n\).

Motivated by this property and an equivalent condition for weak mixing from [XY91]. Blanchard and Huang introduced in [BH08] the notion of weakly mixing set (see Theorem 48 below). We say that a closed and nonempty set \(A \subset X\) is weakly mixing if there exists a Mycielski set \(B \subset A\) dense in \(A\) (i.e. \(\overline{B} = A\)) such that for any \(E \subset B\) and any continuous mapping \(g : E \rightarrow A\) there exists a sequence \(\{n_1 < n_2 < \cdots\} \subset \mathbb{N}\) satisfying \(\lim_{n \to \infty} f^n(x) = g(x)\) for every \(x \in E\).

It was proved in [XY91] that a system \((X, f)\) is weakly mixing if and only if the whole space \(X\) is a weakly mixing set according to the above definition. Blanchard and Huang proved in [BH08] that analogous condition is true for subsets, that is:

**Theorem 48** Let \((X, f)\) be a dynamical system and let \(A \subset X\) a closed set with at least two points. Then \(A\) is a weakly mixing set if and only if for any choice of open...
sets $U_1, \cdots, U_n, V_1, \cdots, V_n$ intersecting $A$ (i.e. $U_i \cap A \neq \emptyset$ and $V_i \cap A \neq \emptyset$ for every $i = 1, \cdots, n$) there exists $m > 0$ such that $f^m(V_i \cap A) \cap U_i \neq \emptyset$ for each $i = 1, \cdots, n$.

When we have Theorem 48 at hand, it is easier to understand the real motivation behind the criterion for chaos in Theorem 42.

Note that by the definition every weakly mixing set must be perfect. Moreover, for any dynamical system the set of all weakly mixing sets forms a $G_δ$ subset of $2^\mathbb{X}$. In a system with positive entropy many entropy sets are weakly mixing and in particular the following result holds [BH08]. It provides another motivation for introducing the notion of weakly mixing set.

**Theorem 49** Every system with positive entropy contains a weakly mixing set.

Obviously, systems with zero entropy may also contain weakly mixing sets, e.g. weakly mixing systems with zero topological entropy and $\#X > 1$. Probably the first weakly mixing system with dense periodic points and entropy zero was described in [Wei71]. In [BH08] the authors constructed an example of a transitive dynamical system on an infinite space with dense periodic points but without weakly mixing sets. Clearly, by Theorem 49 this system must also have zero topological entropy. In [BH08] there was also constructed a transitive completely scrambled system which contains no weakly mixing subsets. Recall that a system is completely scrambled if the whole space is a scrambled set.

For a better understanding of the dynamics over sets, in [OZ11] the authors introduced the following definitions, inspired by Theorem 48. Let $A \subset X$ be nonempty and $n \geq 2$. We say that $A$ is a transitive set of $(X, f)$ if for each pair of open subsets $(U, V)$ of $X$ intersecting $A$ there exists $m \in \mathbb{N}$ such that $f^m(V \cap A) \cap U \neq \emptyset$ and weakly mixing of order $n$ if $A^n$ is a transitive set of $(X^n, f^n)$. In other words, $A$ is weakly mixing of order $n$ if for any open subsets $U_1, \cdots, U_n, V_1, \cdots, V_n$ intersecting $A$ there exists an integer $k > 0$ such that $f^k(V_i \cap A) \cap U_i \neq \emptyset$ for each $i = 1, \cdots, n$.

Note that $A$ is a weakly mixing set if and only if $\#A > 1$ and $A$ is closed and weakly mixing of all orders, that is, $A$ is weakly mixing of order $m$ for each $m \geq 2$. We emphasize here, that in contrast to [BH08], in definitions from [OZ11] the authors neither assume that $A$ is closed nor that it has at least two points.

From the definitions we immediately see that $(X, f)$ is transitive if and only if $X$ is transitive and $(X, f)$ is weakly mixing if and only if $X$ is weakly mixing of order 2. In the latter case $X$ is a weakly mixing set of order $n$ for every $n \geq 2$ by Furstenberg’s characterization of weak mixing mentioned earlier. It is easy to see that the image of a transitive set or a weakly mixing set (of order $n$) by a factor map remains transitive or weakly mixing (of order $n$), respectively. Similarly, if $A$ is a transitive set (weakly mixing, etc.) then so is its closure $\overline{A}$.

Note that entropy sets were defined by the condition that any tuple from this set is an entropy tuple. This does not work for weakly mixing sets and tuples. First of all WM$_n(X, f)$ need not be a symmetric relation, but this is not the main problem. The main difficulty is that even if $A^2 \setminus T_2(A) \subset$ WM$_2(X, f)$ it may happen that there exist open sets $U_1$ and $U_2$ intersecting $A$ such that $f^m(U_1 \cap A) \cap U_2 = \emptyset$ for every $m \geq 0$ (e.g. see [OZ12a]). There is another difference between weakly mixing sets
and entropy sets. It can be proved that any closed non-trivial weakly mixing set of order 2 is perfect [OZ11], while entropy sets are hereditary, i.e. each non-trivial subset of an entropy set is again an entropy set. Despite of many differences, there are strong connections between positive entropy systems and weakly mixing sets, that is, the entropy of a dynamical system equals exactly the supremum of entropy capacity over all closed weakly mixing entropy sets [OZ12b]. The above results show that in contrast to (global) weak mixing of transformations, the local approach to weak mixing may reflect a wider variety of dynamics.

The definition of a weakly mixing set suggests that the dynamics over this set should be very similar to global weak mixing (i.e. weak mixing of \( (X, f) \)), however this first impression is a little bit misleading. While a weakly mixing set of order \( n + 1 \) is weakly mixing of order \( n \), the converse implication cannot be guaranteed in general. The special case of \( n = 2 \) was proved in [OZ11] for an abstract dynamical system, where the authors were using residual properties of some special group of homeomorphisms. It was motivated by a technique of construction of skew products developed by Glasner and Weiss in [GW79]. Recently, by completely different tools, we were able to construct in [OZ12a] the following examples.

**Theorem 50** For every \( n \geq 2 \) there exists a minimal subshift with zero topological entropy which contains a perfect weakly mixing set of order \( n \) but all weakly mixing sets of order \( n + 1 \) are trivial, that is, are singletons.

In dimension one this difference is not that much visible. In fact, we have the following full characterization of weakly mixing sets for one dimensional systems [OZ11, OZ12a].

**Theorem 51** Assume that \( (X, f) \) is a dynamical system acting on a topological graph. Then the following statements are equivalent:

51.1. \( h_{\text{top}}(f) > 0 \),
51.2. \( (X, f) \) contains a weakly mixing set,
51.3. \( (X, f) \) contains a non-trivial weakly mixing set of order 2,
51.4. there exists a perfect set \( A \subset X \) with \( A^3 \Delta A_3(A) \subset \text{WM}_3(X, f) \).

The following result shows that these relations are even more tight (see [OZ11] and [OZ12c]).

**Proposition 52** Assume that \( (X, f) \) is a dynamical system acting on a topological graph and \( A \) is a closed non-trivial weakly mixing set of order 2. Then for every \( \epsilon > 0 \) there exists a weakly mixing set \( D \) such that \( S_\Delta(A, D) < \epsilon \).

In fact, in dimension one there are many relations between positive topological entropy and triples of points. Another example, similar to Condition 51.4 are sequence entropy triples considered in [Tan11, TYZ10].

Theorem 51 suggests that sometimes we may expect an interesting relationship between weakly mixing pairs, tuples and sets. A trivial observation is that for each \( n \geq 2 \) if a nonempty set \( A \subset X \) is weakly mixing of order \( n \) then \( A^n \Delta A_n(A) \subset \text{WM}_n(X, f) \). In practice, the notion of weakly mixing sets of order 2 is
much stronger than the notion of weakly mixing tuples, and the above trivial relation may be the only positive relation between them. Simply, an example from [OZ12a] shows that there exists a flow $(X, f)$ containing a non-trivial path-connected (and so uncountable) closed invariant subset $A \subset X$ such that $A^m \setminus A_m(A) \subset WM_n(X, f)$ for all $m \geq 2$ while all weakly mixing sets of order 2 for $(X, f)$ are trivial, that is are singletons. When dynamics is minimal we can guarantee stronger relations between weakly mixing sets and tuples (see [OZ12a]).

**Proposition 53** Let $(X, f)$ be a minimal flow and $A$ a non-trivial weakly mixing set of order 2. Then $A^n \setminus A_n(A) \subset WM_n(X, f)$ for every $n \geq 2$.

The next result shows connections between weakly mixing sets and proximality. While we need weak mixing of order $n$ to ensure proximal $n$-tuples within $A$, it is interesting that for regionally proximal $n$-tuples $RP_n(f)$ weak mixing of order 2 is enough [OZ11, OZ12a]. The regional proximal relation $RP(f)$ can be generalized easily to regionally proximal $n$-tuples $RP_n(f)$ by putting:

$$RP_n(f) = \bigcap \left\{ \bigcup_{m \in \mathbb{Z}_+} (f^m)^{-m}(U) : U \text{ is a neighborhood of } A_n(X) \text{ in } X^n \right\}.$$

**Proposition 54** Let $K$ be a nonempty closed weakly mixing set of order 2. Then $K^n \subset RP_n(f)$ for each $n \geq 2$.

**Proposition 55** Fix any integer $n \geq 2$ and let $K \subset X$ be a closed non-trivial weakly mixing set of order $n$. Then there exist $\delta > 0$ and a Mycielski set $A$ such that if $x_1, \ldots, x_n \in A$ are pairwise distinct then

55.1. $\liminf_{m \to \infty} \max_{1 \leq i, j \leq n} d(f^m x_i, f^m x_j) = 0$,

55.2. $\limsup_{m \to \infty} \min_{1 \leq i < j \leq n} d(f^m x_i, f^m x_j) \geq \delta$,

55.3. $\liminf_{m \to \infty} \max_{1 \leq i \leq n} d(f^m x_i, x_i) = 0$.

Observe that the above set $A$ is scrambled, therefore the existence of a non-trivial weakly mixing set of order 2 is a sufficient condition for Li-Yorke chaos.

As a natural modification of topological entropy which can distinguish between systems with topological entropy zero, the concept of topological sequence entropy was introduced by Goodman [Goo74] as another measure of complexity of systems. Namely, systems with zero topological entropy can have positive sequence entropy, and similar to topological entropy, systems with different values of sequence entropy (along the same sequence) cannot be conjugate. Then sequence entropy is a more sensitive tool than topological entropy in some cases. Here we will not present a detailed definition of sequence entropy. The reader not familiar with this notion is referred to [Goo74] or [Wal82]. It can be proved that the existence of a weakly mixing set implies infinite sequence entropy [OZ12a], so systems with finite sequence entropy do not admit such sets. However it is also possible to provide an example (see [OZ12a]) of systems with infinite sequence entropy but without non-trivial (i.e. other than singletons) weakly mixing sets of order 2. In Theorem 49 we saw...
that weakly mixing sets (of all orders) are strongly related to positive topological entropy. The following result from [OZ12a] shows a similar relationship between sequence entropy and weakly mixing sets of order 3.

**Theorem 56** Every minimal flow containing a non-trivial weakly mixing set of order 3 has positive topological sequence entropy.

We repeat here after [OZ12a] the following natural question.

**Problem 57** Does every minimal flow have positive topological sequence entropy when it contains a non-trivial weakly mixing set of order 2?

It is very easy to construct a minimal system with zero sequence entropy, since any isometry has sequence entropy zero, so every irrational rotation of the unit circle serves as an example. The following question appeared first in [HLSY03] and later has been proposed several times in other references.

**Problem 58** Does there exist a transitive but not minimal dynamical system with zero topological sequence entropy?

The above question is closely related with Problem 57. Namely, it was demonstrated in [OZ12a] that any flow $(X, f)$ with zero topological sequence entropy and a non-trivial weakly mixing set of order 2 (if such a system exists) can be transformed into a non-minimal transitive system with zero topological sequence entropy.

To finish this section let us only mention a few topics that may be of some interest for further reading. The concept of weakly mixing pairs and tuples is closely related to the study of topological sequence entropy tuples, complexity tuples and sensitivity by numerous results established in [HLSY03, MS07, SYZ08, Zha07]. While we do not explore these notions here, the reader is referred to [GY09] by Glasner and Ye (and references therein) for further reading.

In §1.4 we have mentioned the paper [Kin90] by King. It motivates the following general problem. What are necessary and sufficient conditions on a dynamical system $(X, f)$ under which dynamical system $(X^n, f \times f^2 \times \cdots \times f^n)$ is transitive or minimal, for every $n \geq 1$. For example if $(X, f)$ is minimal then $(X^n, f \times f^2 \times \cdots \times f^n)$ is transitive for each $n \in \mathbb{N}$ if and only if $(X, f)$ is weakly mixing. It was first proved by Glasner [Gla00] for flows but is also valid for surjections (e.g. see [KO11]). While definitely weak mixing is a necessary condition for minimality of $(X^n, f \times f^2 \times \cdots \times f^n)$ for all $n \in \mathbb{N}$, it is still not sufficient (see [KO11] and references therein). In particular, the question when $(X^n, f \times f^2 \times \cdots \times f^n)$ is minimal for every $n \in \mathbb{N}$ is still open.

In the case of actions of topological groups minimality, transitivity, weak mixing of order $n$ (so-called $n$-fold mixing) can be defined similarly to the case of a flow in terms of density of orbits. The classical result [Fur67, Proposition II.3] by Furstenberg asserts that if the acting group is Abelian then 2-fold weak mixing implies weak mixing of all orders. Observe that although [Fur67, Proposition II.3] just considers the case of the integer group actions, its proof works for all Abelian group actions. This result is no longer true for the case of more general groups. For example Weiss constructed in [Wei00] a minimal weakly mixing group action $(X, G)$ such that the
group action \((X^3, G^{(3)})\) is not transitive. Another basic result which successfully transfers to Abelian group actions, is the theorem saying that a minimal dynamical system is not weakly mixing exactly when it admits a non-trivial equicontinuous factor (see [Pet70]). Again for groups which are not Abelian it cannot be guaranteed anymore. A counterexample was constructed long time ago by McMahon in [McM76]. By [FG78] and [McM80] implication that a minimal weakly mixing dynamical system is weakly mixing of all orders is valid for amenable group actions. For a more extensive exploring of a weakly mixing group action the reader is encouraged to consult [Gla05] (see also related parts of [Gla76]). In general, local aspects of weak mixing of actions of groups are yet to be understood.

1.7 Product recurrence

In this section we will compare recurrence properties of a given point in pair with arbitrary recurrent points from some specified class of dynamical systems. Such studies were initiated by Furstenberg in his book [Fur81]. In recent years the original definition from [Fur81] was widely extended. Good motivation was provided by the fact that recurrence in pairs may be strongly related to another interested topic of disjointness with a specified class of systems. Let us start with defining product recurrence using the most recent terminology which we introduce following [DSY10].

First, let us make some preparations. Recall that a (Furstenberg) family \(\mathcal{F}\) is a collection of subsets of \(\mathbb{Z}^+\) which is upwards hereditary, that is, \(F_1 \in \mathcal{F}\) and \(F_1 \subset F_2 \Rightarrow F_2 \in \mathcal{F}\).

We say that a set \(A \subset \mathbb{Z}^+\) is thick if for every \(n > 0\) there is an \(i \in \mathbb{N}\) such that \(\{i, i + 1, \cdots, i + n\} \subset A\). It is direct to see that each thick subset intersects all syndetic subsets. Denote by \(\mathcal{F}_{\text{inf}}, \mathcal{F}_{\text{t}},\) and \(\mathcal{F}_{\text{s}}\) the family of all infinite subsets, thick subsets and syndetic subsets of \(\mathbb{Z}^+\), respectively. We denote by \(\mathcal{F}_{\text{ps}}\) the family of all piecewise syndetic sets, that is sets which can be obtained as the intersection of a thick set and a syndetic set. We denote by \(\mathcal{F}_{\text{pubd}}\) the family of sets with positive upper Banach density, that is sets \(F \subset \mathbb{Z}^+\) such that

\[
\limsup_{n-m \to \infty} \frac{\#(F \cap \{m, m + 1, \cdots, n\})}{n - m + 1} > 0.
\]

Now, let \((X, f)\) be a dynamical system and let \(\mathcal{F} \subset \mathbb{Z}^+\) be a family. Recall, that a point \(x \in X\) is \(\mathcal{F}\)-recurrent if \(N(x, U) \in \mathcal{F}\) for any open neighborhood \(U\) of \(x\), where \(N(x, U) = \{n \in \mathbb{Z}^+ : f^n(x) \in U\}\). Note that \(x\) is recurrent if and only if it is \(\mathcal{F}_{\text{inf}}\)-recurrent. For an interesting exposition on recurrence properties expressed in terms of families the reader is referred to the book [Aki97] by Akin. We say that \(x \in X\) is \(\mathcal{F}\)-product recurrent (\(\mathcal{F}\)-PR for short) if for any dynamical system \((Y, g)\) and any \(\mathcal{F}\)-recurrent point \(y \in Y\) the pair \((x, y)\) is recurrent for the product system \((X \times Y, f \times g)\).
Clearly if $F_1 \subset F_2$ and $x$ is $F_1$-recurrent then it is also $F_2$-recurrent. Similarly, if $x$ is $F_2$-PR then it is also $F_1$-PR. In particular we have the following implications

$$F_{\inf} - PR \implies F_{\pub} - PR \implies F_{ps} - PR \implies F_s - PR.$$  

The study of product recurrence was initiated by Furstenberg who fully characterized $F_{\inf}-PR$ as follows (see [Fur81, Theorem 9.11]).

**Theorem 59** The following conditions are equivalent:

59.1. $x$ is distal,

59.2. $x$ is $F_{\inf}-PR$,

59.3. for any $(Y, g)$ and any minimal point $y \in Y$ the pair $(x, y)$ is a minimal point of the product system $(X \times Y, f \times g)$.

It is worth mentioning that the above theorem has many interesting extensions. For example Auslander and Furstenberg were able to express in [AF94] relations between distality and product recurrence in terms of algebraic properties of idempotents in its Ellis semigroup, extending that way Theorem 59 to more general semigroup actions. These ideas were extended even further in [EEN01], where among other interesting results, the authors proved the equivalence of distality and product recurrence for actions of infinite groups (see [EEN01, Corollary 5.36]).

As we mentioned earlier, every distal point is minimal. The same can be proved about any $F_{ps}$-PR point [DSY10]. Furthermore, it was recently announced that $F_{ps}$-PR point is necessarily distal [OZ12c]. We can express this fact in the following way.

**Theorem 60** The properties of $F_{\inf}$-PR, $F_{\pub}$-PR and $F_{ps}$-PR are equivalent, i.e. if a point $x$ satisfies one of these properties then it automatically satisfies all of them.

In view of the above result, it is natural to ask if it is possible to include $F_s$-PR to the list of conditions equivalent to distality. This question was stated explicitly by Auslander and Furstenberg in [AF94]. The negative answer was first given in [HO08] by constructing a transitive non-minimal system with a point $x$ with dense orbit which additionally is $F_s$-PR (note that in this case $x$ is not distal, since is not minimal). In fact the authors of [HO08] proved that if mixing in the system is sufficiently strong, then every point with a dense orbit is $F_s$-PR. After publication of [HO08] a few less restrictive necessary conditions on $F_s$-PR were discovered. Unfortunately, we are still missing a complete characterization of such systems. The following fact (with surprisingly easy proof) was obtained independently in [DSY10] and [Opr10b].

**Theorem 61** If $(X, f)$ is disjoint from every minimal dynamical system and $x \in Tran(X, f)$ then $x$ is $F_s$-PR.

The above theorem immediately gives us numerous examples of systems with $F_s$-PR points which are not minimal points (therefore not distal). Simply, it is enough to consider any transitive non-minimal system $(X, f)$ which is disjoint from...
every minimal system, and pick up a transitive point \( x \in \text{Tran}(X, f) \), then by Theorem 61 one has that \( x \) is \( \mathcal{F}_s \)-PR but not minimal. First class of such systems was described by Furstenberg in [Fur67], where he proved that any weakly mixing system with dense periodic points is disjoint from every minimal dynamical system. This fact has at least two generalizations (see [DSY10, HY05, Opr10b]).

**Theorem 62** Let \((X, f)\) be a weakly mixing system and additionally assume that at least one of the following conditions is satisfied:

62.1. the set of all distal points is a dense subset of \( X \),

62.2. \((X, f)\) has dense small periodic sets, i.e. for every nonempty open set \( U \) there is \( x \in X \) and \( k > 0 \) such that \( \text{Orb}^+(x, f^k) \subset U \).

Then \((X, f)\) is disjoint from every minimal dynamical system.

Unfortunately, disjointness with minimal systems is very far from a complete characterization of \( \mathcal{F}_s \)-PR. For example, we have the following necessary condition [Opr10b]:

**Theorem 63** If \( A \) is a weakly mixing set and for any open set \( U \) intersecting \( A \) there are \( x \in A \cap U \) and \( k > 0 \) such that \( \text{Orb}^+(x, f^k) \subset U \), then there is a residual subset of \( A \) consisting of points which are \( \mathcal{F}_s \)-PR but not \( \mathcal{F}_{inf} \)-PR.

But now, if we take the Cartesian product of any non-trivial weakly mixing system having dense periodic points (e.g. on the unit interval) with an adding machine then clearly all assumptions of the above theorem are satisfied, while the constructed system surely is not disjoint from an adding machine.

So far, different types of product recurrence were obtained by putting restrictions on the set of possible return times. But there are other possibilities. In [DSY10] the authors introduced the notion of \( \mathcal{F}_s \)-PR\(_0\). Simply, we say that \( x \in X \) is \( \mathcal{F}_s \)-PR\(_0\) if for any dynamical system \((Y, g)\) with zero topological entropy and any \( \mathcal{F} \)-recurrent point \( y \in Y \) the pair \((x, y)\) is recurrent in the product system \((X \times Y, f \times g)\). Again, the following implications follow just by the definition:

\[
\mathcal{F}_{inf} \text{-PR}_0 \implies \mathcal{F}_{publ} \text{-PR}_0 \implies \mathcal{F}_{ps} \text{-PR}_0 \implies \mathcal{F}_s \text{-PR}_0.
\]

Additionally, almost without any modification in the proof of Theorem 61, it is possible to show that any transitive point in a system disjoint from all minimal zero entropy systems is \( \mathcal{F}_s \) – \( \mathcal{F}_{inf} \)-PR\(_0\). Having results of [HPY07] at hand, the authors of [DSY10] were able to provide a short and elegant proof of the following fact.

**Theorem 64** A point \( x \in X \) is distal if and only if it is \( \mathcal{F}_{inf} \)-PR\(_0\).

It is also possible to prove the following nice property of \( \mathcal{F}_{ps} \)-PR\(_0\) points.

**Theorem 65** If \( x \) is \( \mathcal{F}_{ps} \)-PR\(_0\) then it is a minimal point.

As we saw before, there are \( \mathcal{F}_s \)-PR points which are not minimal, and so as a direct corollary of Theorem 65 we see that \( \mathcal{F}_s \) – PR \( \nRightarrow \mathcal{F}_{ps} \text{-PR}_0 \).
The above facts have highlighted some analogues between $\mathcal{F}_{\text{-PR}}$ and $\mathcal{F} - \text{PR}_0$; however, as we will see there are also many differences between these two notions. First, it was proved in [HPY07] that if $(X, f)$ is a minimal flow such that any of its invariant measures is a $K$-measure, then it is disjoint from any transitive zero entropy system with fully supported measure (in particular, every minimal zero entropy system belongs to this class). If we take any point $y$ which is $\mathcal{F}_{\text{pubd}}$-recurrent then its orbit closure is a transitive system with a fully supported measure, therefore the above mentioned result implies that if $(X, f)$ is a strictly ergodic flow with its unique invariant measure being a $K$-measure, then every point $x \in X$ is $\mathcal{F}_{\text{pubd}}$-$\text{PR}_0$. But obviously every such system has positive entropy, therefore contains proper asymptotic pairs. In particular, there is a point $x \in X$ which is not distal. Even more, if $x \neq y$ are asymptotic then definitely $(x, y)$ is not recurrent. This shows that $\mathcal{F}_{\text{pubd}} - \text{PR}_0 \not\iff \mathcal{F} - \text{PR}$, since there are many strictly ergodic flows with an invariant $K$-measure (a direct construction of such systems was obtained first by Grillenberger [Gri73] but in fact, by Jewett-Krieger Theorem, every measurable Kolmogorov system gives rise to such a system [Gla03]).

Summing up, we have revealed almost all relations between considered notions of product recurrence as shown by the following Figure 1.1.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (inf) at (0,0) {\mathcal{F}_{\text{inf}} - \text{PR}}; 
  \node (pubd) at (1.5,0) {\mathcal{F}_{\text{pubd}} - \text{PR}}; 
  \node (ps) at (3,0) {\mathcal{F}_{\text{ps}} - \text{PR}}; 
  \node (s) at (4.5,0) {\mathcal{F}_{\text{s}} - \text{PR}}; 
  \node (inf0) at (0,-1.5) {\mathcal{F}_{\text{inf}} - \text{PR}_0}; 
  \node (pubd0) at (1.5,-1.5) {\mathcal{F}_{\text{pubd}} - \text{PR}_0}; 
  \node (ps0) at (3,-1.5) {\mathcal{F}_{\text{ps}} - \text{PR}_0}; 
  \node (s0) at (4.5,-1.5) {\mathcal{F}_{\text{s}} - \text{PR}_0}; 

  \draw[->] (inf) -- (pubd); 
  \draw[<-] (pubd) -- (inf); 
  \draw[->] (pubd) -- (ps); 
  \draw[<-] (ps) -- (pubd); 
  \draw[<-] (inf) -- (ps); 
  \draw[<->] (pubd) -- (s); 
  \draw[->] (inf) -- (s); 
  \draw[<-] (inf0) -- (pubd0); 
  \draw[<-] (pubd0) -- (inf0); 
  \draw[->] (pubd0) -- (ps0); 
  \draw[->] (ps0) -- (pubd0); 
  \draw[->] (inf0) -- (ps0); 

  \node at (1.5,-0.75) {not}; 
  \node at (3,-0.75) {not}; 
  \node at (4.5,-0.75) {not}; 
\end{tikzpicture}
\caption{Product recurrence and product recurrence with zero entropy systems}
\end{figure}

However there are at least two open questions, which haven’t been answered so far. We repeat them from [HO08] and [DSY10], respectively.

**Problem 66** Does there exist a minimal point $x$ which is $\mathcal{F}_{\text{s}}$-$\text{PR}$ but not distal?

**Problem 67** Does $\mathcal{F}_{\text{ps}} - \text{PR}_0$ imply $\mathcal{F}_{\text{pubd}} - \text{PR}_0$?

As we have seen, there are many relations between product recurrence of points and results on disjointness between some classes of dynamical systems. While we do not state open problems on topological disjointness, many of them are similar in spirit to open problems on product recurrence. The reader is referred to [DSY10] and references therein for more details.

**Acknowledgements**

We gratefully acknowledge receiving many helpful comments and important suggestions from Professors: Ethan Akin, Joe Auslander, Henk Bruin, Tomasz Dow-
The first author was supported by the Marie Curie European Reintegration Grant of the European Commission under grant agreement no. PERG08-GA-2010-272297. The second author was supported by Foundation for the Authors of National Excellent Doctoral Dissertation of China, grant no. 201018.

References


1 Topological aspects of dynamics of pairs, tuples and sets


1 Topological aspects of dynamics of pairs, tuples and sets


1. **Topological aspects of dynamics of pairs, tuples and sets**


Index

<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_n$</td>
<td>4</td>
</tr>
<tr>
<td>$\text{Asy}(f)$</td>
<td>17</td>
</tr>
<tr>
<td>$\mathcal{F}_{pr}$</td>
<td>33</td>
</tr>
<tr>
<td>$\mathcal{F}_{-PR}$</td>
<td>35</td>
</tr>
<tr>
<td>$\mathcal{F}_{-PR}$</td>
<td>33</td>
</tr>
<tr>
<td>$\mathcal{F}_{-recurrent}$</td>
<td>33</td>
</tr>
<tr>
<td>$\mathcal{F}_r$</td>
<td>33</td>
</tr>
<tr>
<td>$\mathcal{F}_{ps}$</td>
<td>33</td>
</tr>
<tr>
<td>$\mathcal{F}_{pub}$</td>
<td>33</td>
</tr>
<tr>
<td>$\text{Prox}(f)$</td>
<td>14</td>
</tr>
<tr>
<td>$\text{RP}(f)$</td>
<td>15</td>
</tr>
<tr>
<td>$\text{Rec}(X, f)$</td>
<td>3</td>
</tr>
<tr>
<td>$\text{SynProx}(f)$</td>
<td>14</td>
</tr>
<tr>
<td>$\text{Tran}(X, f)$</td>
<td>3</td>
</tr>
<tr>
<td>$\text{WM}_n(X, f)$</td>
<td>28</td>
</tr>
<tr>
<td>$h_{top}(f)$</td>
<td>6</td>
</tr>
<tr>
<td>$\mathcal{S}_n$</td>
<td>3</td>
</tr>
<tr>
<td>$f^{(m)}$</td>
<td>3</td>
</tr>
<tr>
<td>asymptotic</td>
<td>17</td>
</tr>
<tr>
<td>c-expansive</td>
<td>18</td>
</tr>
<tr>
<td>c.p.e.</td>
<td>7</td>
</tr>
<tr>
<td>DC1 (DC2, DC3)</td>
<td>22</td>
</tr>
<tr>
<td>diagonal flow</td>
<td>7</td>
</tr>
<tr>
<td>disjoint dynamical systems</td>
<td>4</td>
</tr>
<tr>
<td>distal</td>
<td>14</td>
</tr>
<tr>
<td>distributionally chaotic</td>
<td>22</td>
</tr>
<tr>
<td>entropy pair</td>
<td>6</td>
</tr>
<tr>
<td>entropy set</td>
<td>9</td>
</tr>
<tr>
<td>entropy tuple</td>
<td>9</td>
</tr>
<tr>
<td>equicontinuous</td>
<td>4</td>
</tr>
<tr>
<td>expansive</td>
<td>17</td>
</tr>
<tr>
<td>factor map</td>
<td>4</td>
</tr>
<tr>
<td>flow</td>
<td>2</td>
</tr>
<tr>
<td>Furstenberg family</td>
<td>33</td>
</tr>
<tr>
<td>ICER</td>
<td>4</td>
</tr>
<tr>
<td>IE-tuple</td>
<td>11</td>
</tr>
<tr>
<td>Li-Yorke chaotic</td>
<td>22</td>
</tr>
<tr>
<td>Li-Yorke pair</td>
<td>22</td>
</tr>
<tr>
<td>mildly mixing</td>
<td>4</td>
</tr>
<tr>
<td>minimal point</td>
<td>4</td>
</tr>
<tr>
<td>minimal system</td>
<td>3</td>
</tr>
<tr>
<td>Mycielski set</td>
<td>3</td>
</tr>
<tr>
<td>natural extension</td>
<td>3</td>
</tr>
<tr>
<td>orbit</td>
<td>3</td>
</tr>
<tr>
<td>positive density</td>
<td>9</td>
</tr>
<tr>
<td>positively expansive</td>
<td>17</td>
</tr>
<tr>
<td>proximal</td>
<td>14</td>
</tr>
<tr>
<td>recurrent</td>
<td>3</td>
</tr>
<tr>
<td>regionally proximal</td>
<td>15</td>
</tr>
<tr>
<td>strictly ergodic</td>
<td>5</td>
</tr>
<tr>
<td>strongly mixing</td>
<td>4</td>
</tr>
<tr>
<td>support</td>
<td>5</td>
</tr>
<tr>
<td>syndetically proximal</td>
<td>14</td>
</tr>
<tr>
<td>topological $K$</td>
<td>10</td>
</tr>
<tr>
<td>topological dynamical system</td>
<td>2</td>
</tr>
<tr>
<td>topological entropy</td>
<td>6</td>
</tr>
<tr>
<td>topological graph</td>
<td>3</td>
</tr>
<tr>
<td>transitive</td>
<td>3</td>
</tr>
<tr>
<td>transitive set</td>
<td>29</td>
</tr>
<tr>
<td>u.p.e.</td>
<td>7</td>
</tr>
<tr>
<td>uniquely ergodic</td>
<td>5</td>
</tr>
<tr>
<td>weakly mixing set</td>
<td>28</td>
</tr>
<tr>
<td>weakly mixing set of order $n$</td>
<td>29</td>
</tr>
<tr>
<td>weakly mixing system</td>
<td>3</td>
</tr>
<tr>
<td>weakly mixing tuple</td>
<td>28</td>
</tr>
</tbody>
</table>