Bounds for the Weighted $L^p$ Discrepancy and Tractability of Integration

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Quite recently Sloan and Woźniakowski [4] introduced a new notion of discrepancy, the so-called weighted $L^p$ discrepancy of points in the $d$-dimensional unit cube for a sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ of weights. In this paper we prove a nice formula for the weighted $L^p$ discrepancy for even $p$. We use this formula to derive an upper bound for the average weighted $L^p$ discrepancy. This bound enables us to give conditions on the sequence of weights $\gamma$ such that there exists $N$ points in $[0,1)^d$ for which the weighted $L^p$ discrepancy is uniformly bounded in $d$ and goes to zero polynomially in $N^{-1}$.

Finally we use these facts to generalize some results from Sloan and Woźniakowski [4] on (strong) QMC-tractability of integration in weighted Sobolev spaces.

Key Words: Weighted discrepancy, QMC-tractability of integration

1. INTRODUCTION AND BASIC NOTATIONS

Quite recently Sloan and Woźniakowski [4] introduced a new notion of discrepancy, the so-called weighted $L^p$ (star) discrepancy. To introduce this notion we need a sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ of weights with $1 = \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_d \geq \ldots > 0$.

At first we need some notation: let $D$ denote the index set $D = \{1, 2, \ldots, d\}$. For $u \subseteq D$ let $\gamma_u = \prod_{j \in u} \gamma_j$, $\gamma_\emptyset = 1$, $|u|$ the cardinality of $u$ and for a vector $x$ in $I^d = [0,1)^d$ let $x_u$ denote the vector from $I^{|u|} = [0,1)^{|u|}$ containing the components of $x$ whose indices are in $u$. Further let $dx_u = \prod_{j \in u} dx_j$ and let $(x_u, 1)$ be the vector $x$ from $I^d$, with all components whose indices are not in $u$ replaced by 1.

Now we can give the definition of weighted $L^p$ discrepancy.
Definition 1.1. For any $N$ points $t_1, \ldots, t_N$ in the $d$-dimensional unit cube $I^d = [0,1)^d$ and any $x = (x_1, \ldots, x_d)$ in $I^d$ let

$$\text{disc}(x) = x_1 \ldots x_d - \frac{1}{N} |\{i : t_i \in [0, x)\}|.$$ 

Then the weighted $L^p$ discrepancy $L^p_{N,\gamma}$ for $1 \leq p < \infty$ is defined as

$$L^p_{N,\gamma} = L^p_{N,\gamma}([t_n]) = \left(\frac{1}{N^d} \int_{I^d} |\text{disc}(x)|^p \, dx \right)^{1/p}.$$ 

For $p = \infty$ the weighted star discrepancy is given by

$$D^\ast_{N,\gamma} = \sup_{x \in I^d} \max_{u \in D} \gamma_u^{1/2} |\text{disc}(x_u, 1)|.$$ 

In [3] S. Joe gave an explicit formula for the weighted $L^2$ discrepancy. He proved

**Proposition 1.1.** For any $N$ points $t_1, \ldots, t_N$ in $I^d$ and any sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ of weights we have

$$(L^2_{N,\gamma})^2 = \prod_{j=1}^d \left(1 + \frac{\gamma_j}{3}\right) - \frac{2}{N} \sum_{n=1}^N \prod_{j=1}^d \left(1 + \frac{\gamma_j}{2} (1 - t_{n,j}^2)\right)$$

$$+ \frac{1}{N^2} \sum_{m=1}^N \sum_{n=1}^N \prod_{j=1}^d \left(1 + \gamma_j (1 - \max(t_{m,j}, t_{n,j}))\right),$$

where $t_{n,j}$ is the $j$-th component of the point $t_n$.

In Section 2 we generalize this formula to a formula for the weighted $L^p$ discrepancy for even $p$ (Theorem 2.1). In [4] Sloan and Woźniakowski showed that one can find points $t_1, \ldots, t_N$ in the $d$-dimensional unit cube for which the weighted $L^2$ discrepancy $L^2_{N,\gamma}$ is uniformly bounded in $d$ and goes to zero polynomially in $N^{-1}$ as long as the series $\sum_{j=1}^\infty \gamma_j < \infty$ is convergent and they anticipated that the same holds for the weighted $L^p$ discrepancy $L^p_{N,\gamma}$ as long as the series $\sum_{j=1}^\infty \gamma_j^{p/2} < \infty$ is convergent.

In Section 3 we prove a formula (Theorem 3.1) and an upper bound (Corollary 3.1) for the average weighted $L^p$ discrepancy $L^p_{N,\gamma}$ for even $p$. 
Then we use this upper bound to prove the above conjecture of Sloan and Woźniakowski (Corollary 3.2).

Let us turn now to an application of the previous results, the QMC-tractability of integration in weighted Sobolev spaces. First we repeat some basic facts and definitions. We deal with multivariate integration

\[ I_d(f) := \int_{I^d} f(t) \, dt \]

for functions defined over the \( d \)-dimensional unit cube \( I^d = [0, 1]^d \) which belong to some normed space \( F_d \). We denote the norm in \( F_d \) by \( \| \cdot \| \). A quasi-Monte Carlo (QMC) algorithm \( Q_{N,d} \) is of the form

\[ Q_{N,d}(f) := \frac{1}{N} \sum_{n=1}^{N} f(t_n), \]

where \( t_n, 1 \leq n \leq N \), are points in \( I^d \).

We define the worst case error of the QMC algorithm \( Q_{N,d} \) by its worst performance over the unit ball of \( F_d \),

\[ e(Q_{N,d}) := \sup_{f \in F_d, \|f\| \leq 1} |I_d(f) - Q_{N,d}(f)|. \]

For \( N = 0 \) we set \( Q_{0,d}(f) := 0 \) such that

\[ e(Q_{0,d}) = \sup_{f \in F_d, \|f\| \leq 1} |I_d(f)| = \|I_d\| \]

is the initial error, i.e., the a priori error in multivariate integration without sampling the function.

We would like to reduce the initial error by a factor of \( \varepsilon \), where \( \varepsilon \in (0, 1) \). Let

\[ N_{\min}(\varepsilon, d) = \min \{ N : \exists Q_{N,d} \text{ such that } e(Q_{N,d}) \leq \varepsilon e(Q_{0,d}) \}. \]

**Definition 1.2.**

1. We say that multivariate integration in the space \( F_d \) is QMC-tractable if there exist nonnegative \( C, \alpha \) and \( \beta \) such that

\[ N_{\min}(\varepsilon, d) \leq C d^\alpha \varepsilon^{-\beta} \]

holds for all dimensions \( d = 1, 2, \ldots \) and for all \( \varepsilon \in (0, 1) \).

2. We say that multivariate integration in the space \( F_d \) is strongly QMC-tractable if the inequality above holds with \( \alpha = 0 \).
3. The minimal (infimum) $\alpha$ and $\beta$ are called the $d$-exponent and the $\varepsilon$-exponent of (strong) QMC-tractability.

In this paper we consider the space

$$F_d = W_{q}^{(1,1,\ldots,1)}(\mathbb{R}^d) = W_{q}^1(\mathbb{R}) \otimes \ldots \otimes W_{q}^1(\mathbb{R}) \quad (d \text{ times}),$$

where $W_{q}^1(\mathbb{R})$ is the Sobolev space of all absolutely continuous functions with first derivative in $L^q(\mathbb{R})$, $q \geq 1$. For details on this space see the appendix.

Now let $f \in F_d$ and $t_1, \ldots, t_N \in I_d$. Then Hlawka’s identity [2] states that

$$\int_{I_d} f(x) dx - \frac{1}{N} \sum_{i=1}^{N} f(t_i) = \sum_{\emptyset \neq u \subseteq D} (-1)^{|u|} \int_{|u|} \frac{\partial |u|}{\partial x_u} f(x_u, 1) dx_u.$$

For a proof of Hlawka’s identity see also [5].

Let $\gamma = (\gamma_1, \ldots, \gamma_d)$ be a sequence of weights with $1 = \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_d > 0$. By multiplying and dividing by $\gamma_1^{1/2}$ we obtain from Hlawka’s identity

$$\int_{I_d} f(x) dx - \frac{1}{N} \sum_{i=1}^{N} f(t_i) = \sum_{\emptyset \neq u \subseteq D} (-1)^{|u|} \int_{|u|} \gamma_u^{1/2} \frac{\partial |u|}{\partial x_u} f(x_u, 1) dx_u.$$

Finally an application of Hölder’s inequality yields

$$\left| \int_{I_d} f(x) dx - \frac{1}{N} \sum_{i=1}^{N} f(t_i) \right| \leq L_{N,\gamma}^p \cdot \|f\|_{d,\gamma,q}, \quad (1)$$

where for $1 \leq q < \infty$ $p$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, and $L_{N,\gamma}^p$ is the weighted $L^p$ discrepancy of the point set $\{t_1, \ldots, t_N\}$ with respect to the sequence $\gamma$ of weights and

$$\|f\|_{d,\gamma,q} := \left( \sum_{u \subseteq D} \gamma_u^{-1} \int_{|u|} \left| \frac{\partial |u|}{\partial x_u} f(x_u, 1) \right|^q dx_u \right)^{1/q}.$$
Remark 1. It is well known that $\| \cdot \|_{d,\gamma,q}$ is a norm on the linear space $W_q(I,\ldots,I)([0,1]^d)$ (see the appendix). In [4] Sloan and Woźniakowski showed that integration in the weighted tensor product space $W_q(I,\ldots,I)([0,1]^d)$ is QMC-tractable iff $\limsup \frac{\sum_{j=1}^d \gamma_j}{\log d} < \infty$ and strongly QMC-tractable iff $\sum_{j=1}^d \gamma_j < \infty$.

In Section 4 we generalize some of their results to the non-Hilbertian case, i.e., $q \neq 2$. We show that integration in the weighted tensor product space $W_q(I,\ldots,I)([0,1]^d)$ is QMC-tractable if $\limsup \frac{\sum_{j=1}^d \gamma_j}{\log d} < \infty$ and strongly QMC-tractable if $\sum_{j=1}^\infty \gamma_j < \infty$, where $\frac{1}{q} + \frac{1}{p} = 1$ and where $p$ is even (Theorem 4.1).

2. A FORMULA FOR THE WEIGHTED $L^p$ DISCREPANCY

In this section we prove a generalization of Joe's formula for the weighted $L^p$ discrepancy for even $p$.

Theorem 2.1. Let $p$ be an even, positive integer. For any $N$ points $t_1, \ldots, t_N$ in the $d$-dimensional unit cube $I_d$ and for any sequence of weights $\gamma = (\gamma_1, \gamma_2, \ldots)$ we have

$$(L^p_{N,\gamma})^p = \sum_{l=0}^{p} \binom{p}{l} \left( \frac{-1}{N} \right)^l \sum_{(u_1, \ldots, u_l) \in \{1, \ldots, N\}^l} \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1 - \max_{1 \leq k \leq l} \sum_{u_{k,j}}^{j=1} \frac{1}{p-l+1}}{p-l+1} \right)$$

where $t_{i,j}$ is the $j$-th component of the point $t_i$.

Proof. For even $p$, for $x = (x_1, \ldots, x_d) \in [0,1)^d$ and for $u \subseteq D$, $u \neq \emptyset$, we have

$$\text{disc}(x_u, 1)^p = \left( \prod_{j \in u} x_j - \frac{1}{N} \sum_{i=1}^{N} \prod_{j \in u} 1_{[0,x_j)}(t_{i,j}) \right)^p$$

$$= \sum_{l=0}^{p} \binom{p}{l} \prod_{j \in u} x_j^{p-l} \left( \frac{1}{N} \sum_{i=1}^{N} \prod_{j \in u} 1_{[0,x_j)}(t_{i,j}) \right)^l$$

$$= \sum_{l=0}^{p} \binom{p}{l} \prod_{j \in u} x_j^{p-l} \left( \frac{1}{N} \right)^l \sum_{(u_1, \ldots, u_l) \in \{1, \ldots, N\}^l} \prod_{k=1}^{l} \prod_{j \in u} 1_{[0,x_j)}(t_{u_k,j})$$
\[
\begin{align*}
&= \sum_{l=0}^{p} \binom{p}{l} \left( -\frac{1}{N} \right)^l \sum_{(u_1, \ldots, u_l) \in \{1, \ldots, N\}^l} \prod_{k=1}^{l} t_{u_k,j}^{p-l} 1_{[0,x_j]}(t_{u_k,j}) \\
&= \sum_{l=0}^{p} \binom{p}{l} \left( -\frac{1}{N} \right)^l \sum_{(u_1, \ldots, u_l) \in \{1, \ldots, N\}^l} \prod_{j \in u} x_j^{p-l} 1_{[0,x_j]}(\max_{1 \leq k \leq l} t_{u_k,j}).
\end{align*}
\]

Therefore we get
\[
\int_{|u|} \text{disc}((x_u, 1))^p \, dx_u = \sum_{l=0}^{p} \binom{p}{l} \left( -\frac{1}{N} \right)^l \sum_{(u_1, \ldots, u_l) \in \{1, \ldots, N\}^l} \prod_{j \in u} x_j^{p-l} 1_{[0,x_j]}(\max_{1 \leq k \leq l} t_{u_k,j}) \, dx_u.
\]

Now we evaluate the integral in the above expression. We have
\[
\int_{|u|} \prod_{j \in u} x_j^{p-l} 1_{[0,x_j]}(\max_{1 \leq k \leq l} t_{u_k,j}) \, dx_u = \prod_{j \in u} \int_0^1 x_j^{p-l} 1_{[0,x_j]}(\max_{1 \leq k \leq l} t_{u_k,j}) \, dx_j
\]
\[
= \prod_{j \in u} \int_{\text{max}_{1 \leq k \leq l} t_{u_k,j}}^1 x_j^{p-l} \, dx_j
= \prod_{j \in u} \frac{1 - \text{max}_{1 \leq k \leq l} t_{u_k,j}^{p-l+1}}{p-l+1}.
\]

Thus we get
\[
\int_{|u|} \text{disc}((x_u, 1))^p \, dx_u = \sum_{l=0}^{p} \binom{p}{l} \left( -\frac{1}{N} \right)^l \sum_{(u_1, \ldots, u_l) \in \{1, \ldots, N\}^l} \prod_{j \in u} \frac{1 - \text{max}_{1 \leq k \leq l} t_{u_k,j}^{p-l+1}}{p-l+1}.
\]

Inserting this in the definition of weighted \( L^p \) discrepancy we obtain
\[
(L_{N,\gamma}^p)^p = \sum_{u \subseteq \mathbb{D}} \gamma^p/2 \sum_{l=0}^{p/2} \binom{p/2}{l} \left( -\frac{1}{N} \right)^l \sum_{(u_1, \ldots, u_l) \in \{1, \ldots, N\}^l} \prod_{j \in u} \frac{1 - \text{max}_{1 \leq k \leq l} t_{u_k,j}^{p-l+1}}{p-l+1}
\]
\[
= \sum_{l=0}^{p} \binom{p}{l} \left( -\frac{1}{N} \right)^l \sum_{(u_1, \ldots, u_l) \in \{1, \ldots, N\}^l} \sum_{u \subseteq \mathbb{D}} \gamma^p/2 \prod_{j \in u} \frac{1 - \text{max}_{1 \leq k \leq l} t_{u_k,j}^{p-l+1}}{p-l+1}.
\]
Now we use the fact that
\[ \sum_{\substack{u \subseteq D \ni 0 \neq \emptyset \ni j \in u}} \prod_{j \in u} \gamma_j^{p/2} \frac{1 - \max_{1 \leq k \leq l_{u_k,j}} p^{l_{u_k,j} - 1}}{p - l + 1} = -1 + \prod_{j=1}^d \left( 1 + \gamma_j^{p/2} \frac{1 - \max_{1 \leq k \leq l_{u_k,j}} p^{l_{u_k,j} - 1}}{p - l + 1} \right) \]

(this can be proved by induction on \(d\), or see [3]) and that
\[ \sum_{l=0}^p \binom{p}{l} \left( -\frac{1}{N} \right)^l N^l = 0. \]

Hence we obtain
\[ (L_{N,d,p}^{\gamma})^p = \sum_{l=0}^p \binom{p}{l} \left( -\frac{1}{N} \right)^l \sum_{\substack{u_1, \ldots, u_l \in \{1, \ldots, N\}}} \prod_{j=1}^d \left( 1 + \gamma_j^{p/2} \frac{1 - \max_{1 \leq k \leq l_{u_k,j}} p^{l_{u_k,j} - 1}}{p - l + 1} \right), \]

and we are done. \(\square\)

### 3. Averaging the Weighted \(L^p\) Discrepancy

In [1] Heinrich et al. analyzed the \(L^p\)-star discrepancy for uniformly distributed points in \([0,1]^d\) by computing bounds on the average \(L^p\)-star discrepancy. In the sequel we follow their calculations to derive bounds on the average weighted \(L^p\) discrepancy. For even \(p\) we define the average weighted \(L^p\) discrepancy as

\[ \text{av}_p^{\gamma}(N,d) = \left( \int_{[0,1]^N} \left( L_{N,d}^{\gamma}(t_1, \ldots, t_N) \right)^p \, dt \right)^{1/p}, \]

where \(t = (t_1, \ldots, t_N) \in I^N\).

With Theorem 2.1 we have

\[ \text{av}_p^{\gamma}(N,d)^p = \sum_{l=0}^p \binom{p}{l} \left( -\frac{1}{N} \right)^l \int_{[0,1]^N} \prod_{j=1}^d \left( 1 + \gamma_j^{p/2} \frac{1 - \max_{1 \leq k \leq l_{u_k,j}} p^{l_{u_k,j} - 1}}{p - l + 1} \right) \, dt. \]

For given \((u_1, \ldots, u_l) \in \{1, \ldots, N\}^l\) let \(\kappa(u_1, \ldots, u_l)\) be the number of different \(u_i\)'s and denote by \(v_1, \ldots, v_\kappa\) the different \(u_i\)'s \((\kappa = \kappa(u_1, \ldots, u_l))\).
Then we have
\[
\int_{[0,1]^N} \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1 - \max_{1 \leq k \leq l} t_{v_k,j}^{p-l+1}}{p-l+1} \right) dt = \\
= \prod_{j=1}^{d} \left[ \int_0^1 \cdots \int_0^1 \left( 1 + \gamma_j^{p/2} \frac{1 - \max_{1 \leq k \leq l} t_{v_k,j}^{p-l+1}}{p-l+1} \right) dt_{v_1,j} \cdots dt_{v_k,j} \right] \\
= \prod_{j=1}^{d} \left[ 1 + \frac{\gamma_j^{p/2}}{p-l+1} \int_0^1 \cdots \int_0^1 \min_{1 \leq k \leq \kappa} (1 - t_k^{p-l+1}) dt_{v_1,j} \cdots dt_{v_k,j} \right].
\]

In [1] it is shown, that
\[
\int_0^1 \cdots \int_0^1 \min_{1 \leq k \leq \kappa} (1 - t_k^{p-l+1}) dt_{v_1,j} \cdots dt_{v_k,j} = \frac{p-l+1}{\kappa(u_1, \ldots, u_l) + p-l+1}.
\]

Thus we obtain
\[
\int_{[0,1]^N} \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1 - \max_{1 \leq k \leq l} t_{v_k,j}^{p-l+1}}{p-l+1} \right) dt = \\
= \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{\kappa(u_1, \ldots, u_l) + p-l+1} \right). 
\]

Inserting this back into the above expression we have
\[
av^p_N(N, d) = \\
= \sum_{l=0}^{p} \binom{p}{l} \left( -\frac{1}{N} \right)^l \sum_{(u_1, \ldots, u_l) \in \{1, \ldots, N\}^l} \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{\kappa(u_1, \ldots, u_l) + p-l+1} \right). 
\]

Now let \(\#(l, k, N)\) be the number of tuples \((u_1, \ldots, u_l) \in \{1, \ldots, N\}^l\) such that \(k\) different elements occur (this is the number of mappings from \(\{1, \ldots, l\}\) to \(\{1, \ldots, N\}\) such that the range has cardinality \(k\)). With this notation we obtain
\[
av^p_N(N, d) = \\
= \sum_{l=0}^{p} \binom{p}{l} \left( -\frac{1}{N} \right)^l \sum_{k=0}^{l} \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{k+p-l+1} \right) \cdot \#(l, k, N).
\]
Fortunately the numbers \( \#(l, k, N) \) are well known in combinatorics and can be expressed by Stirling numbers \( s(k, i) \) of the first kind and Stirling numbers \( S(l, k) \) of the second kind. We have

\[
\#(l, k, N) = k! \binom{N}{k} S(l, k) = \sum_{i=0}^{k} s(k, i) S(l, k) N^i
\]

(this is an easy calculation using the fact that the number of surjective mappings from \{1, \ldots, l\} to \{1, \ldots, k\} is given by \( k! S(l, k) \)). Here we use \( \#(0, 0, N) = 1 \) and \( \#(l, 0, N) = 0 \) for \( l > 0 \). This is consistent with the above formula if we use the usual definitions \( s(0, 0) = S(0, 0) = 1 \) and \( S(l, 0) = 0 \) for \( l > 0 \). Therefore we get

\[
\text{av}_p^2(N, d)^p = \sum_{l=0}^{p} \binom{p}{l} \left( -1 \right)^l \sum_{k=0}^{l} \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{k + p - l + 1} \right) \times
\]

\[
\times \sum_{i=0}^{k} s(k, i) S(l, k) N^i
\]

\[
= \sum_{l=0}^{p} \sum_{k=0}^{l} \sum_{i=0}^{k} (-1)^i \binom{p}{l} \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{k + p - l + 1} \right) \right) s(k, i) S(l, k) N^{-l+i}
\]

\[
= \sum_{l=0}^{p} \sum_{i=0}^{l} \sum_{k=i}^{l} (-1)^i \binom{p}{l} \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{k + p - l + 1} \right) \right) s(k, i) S(l, k) N^{-l+i}.
\]

Letting

\[
A(i, l) := \sum_{k=i}^{l} (-1)^i \binom{p}{l} \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{k + p - l + 1} \right) \right) s(k, i) S(l, k)
\]

we have

\[
\text{av}_p^2(N, d)^p = \sum_{l=0}^{p} \sum_{i=0}^{l} A(i, l) \cdot N^{-l+i} = \sum_{r=0}^{p} \sum_{i=0}^{p-r} A(i, i + r) \cdot N^{-r}
\]

\[
= \sum_{r=0}^{p} C(r, p, d) \cdot N^{-r},
\]
where

\[
C(r, p, d) = \sum_{i=0}^{p-r} A(i, i + r)
\]

\[
= \sum_{i=0}^{p-r} \sum_{k=i}^{i+r} (-1)^{i+r} \left( \frac{p}{i+r} \right) \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{k + p - i - r + 1} \right) \right) \times
\]

\[
\times s(k, i) S(i + r, k).
\]

Now we consider \( C(r, p, d) \). First note that \( C(p, p, d) = 0 \). Next we write \( C(r, p, d) \) in the form

\[
C(r, p, d) = (-1)^r \sum_{l=0}^{r} \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{p + 1 - r - l} \right) \cdot \beta(r, p, l)
\]

with

\[
\beta(r, p, l) = \sum_{k=l}^{p-r+l} \left( \frac{p}{r + k - l} \right) (-1)^{k-l} s(k, k - l) S(k - l + r, k).
\]

In [1, Lemma 1] it is shown that \( \beta(r, p, l) = 0 \) for \( r = 0, \ldots, p/2 - 1 \) and \( l = 0, \ldots, r \) for all even integers \( p \). Thus we have \( C(r, p, d) = 0 \) for \( r = 0, \ldots, p/2 - 1 \).

We summarize:

**Theorem 3.1.** Let \( p \) be an even integer and let \( d \in \mathbb{N} \). Then we have

\[
\text{av}_2(N, d) = \sum_{r=p/2}^{p-1} C(r, p, d) \cdot N^{-r},
\]

where \( C(r, p, d) \) is given by

\[
C(r, p, d) = \sum_{i=0}^{p-r} \sum_{k=i}^{i+r} (-1)^{i+r} \left( \frac{p}{i+r} \right) \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{k + p - i - r + 1} \right) \right) \times
\]

\[
\times s(k, i) S(i + r, k).
\]

**Remark 3.1.** It is easy to prove that for the special case \( p = 2 \) we have

\[
\text{av}_2(N, d) = \frac{1}{N^{1/2}} \cdot \left[ \prod_{j=1}^{d} \left( 1 + \frac{\gamma_j}{2} \right) - \prod_{j=1}^{d} \left( 1 + \frac{\gamma_j}{3} \right) \right]^{1/2}.
\]
Compare this formula to the subsequent Corollary 3.1.

So to obtain upper bounds on $av_{p}^\gamma(N, d)$ one needs to have upper bounds for $|C(r, p, d)|$. We continue in the following way:

$$
|C(r, p, d)| \leq \sum_{i=0}^{p-r} \binom{p}{i+r} \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{k + p - i + r + 1} \right) \right) \times |s(k, i) S(i + r, k)|
$$

$$
\leq \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{p - r + 1} \right) \sum_{i=0}^{p-r} \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{k + p - i + r + 1} \right) \right) \times |s(k, i) S(i + r, k)|.
$$

Now following the proof of [1, Theorem 5] we get

$$
|C(r, p, d)| \leq \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{p - r + 1} \right) \cdot (r + 1) (4p)^p.
$$

We use this bound to get an upper estimate on $av_{p}^\gamma(N, d)$:

$$
\sum_{r=p/2}^{p-1} |C(r, p, d)| \cdot N^{-r} \leq \sum_{r=p/2}^{p-1} \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{p - r + 1} \right) \right) (r + 1) (4p)^p \frac{1}{N^r}
$$

$$
\leq \frac{p (4p)^p}{N^{p/2}} \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{2} \right) \right) \sum_{r=p/2}^{p-1} N^{-(r-\frac{p}{2})}
$$

$$
= \frac{p (4p)^p}{N^{p/2}} \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{2} \right) \right) \sum_{r=0}^{p-1} N^{-r}
$$

$$
\leq 2 \frac{p (4p)^p}{N^{p/2}} \left( \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{2} \right) \right).
$$

Noting that $\sqrt[p]{p} \leq \sqrt{2}$ for even $p$ we get the following corollary from Theorem 3.1:

**Corollary 3.1.** Let $p$ be an even integer and let $d \in \mathbb{N}$. Then we have

$$
av_{p}^\gamma(N, d) \leq 8p \frac{1}{N^{1/2}} \cdot \left[ \prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \frac{1}{2} \right) \right]^{1/p}.
$$
From this corollary we immediately get the following result, which was already conjectured by Sloan and Woźniakowski in [4].

**Corollary 3.2.** Let $p$ be an even integer and let $d \in \mathbb{N}$.

1. If the series
   \[ \sum_{j=1}^{\infty} \frac{\gamma_j^{p/2}}{j} < \infty, \]
   then one can find points $t_1, \ldots, t_N$ in $I^d$ such that we have
   \[ L_{N,\gamma}^p(\{t_i\}) < C \cdot \frac{a}{N^{1/2}}, \]
   where $a := e^{\frac{1}{2p} \sum_{j=1}^{\infty} \gamma_j^{p/2}}$ and where $C := 8p$.

2. If
   \[ \limsup_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_j^{p/2}}{\log d} < \infty, \]
   then one can find points $t_1, \ldots, t_N$ in $I^d$ such that we have
   \[ L_{N,\gamma}^p(\{t_i\}) < C \cdot \frac{d^{\alpha}}{\sqrt{N}}, \]
   where $\alpha := \frac{1}{2p} \cdot \limsup_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_j^{p/2}}{\log d}$ and where $C := 8p$.

**Proof.** From Corollary 3.1 we know that there exist points $t_1, \ldots, t_N$ in $I^d$ for which
   \[ L_{N,\gamma}^p(\{t_i\}) \leq 8p \cdot \frac{1}{N^{1/2}} \left[ \prod_{j=1}^{d} \left( 1 + \frac{\gamma_j^{p/2}}{2} \right) \right]^{1/p} \]
   holds. Further we have
   \[ \prod_{j=1}^{d} \left( 1 + \frac{\gamma_j^{p/2}}{2} \right) = e^{\sum_{j=1}^{d} \log(1+\gamma_j^{p/2}/2)} \leq e^{\frac{1}{2} \sum_{j=1}^{d} \gamma_j^{p/2}} \]
   and part 1 follows. Since
   \[ e^{\sum_{j=1}^{d} \log(1+\gamma_j^{p/2}/2)} \leq e^{\frac{1}{2} \sum_{j=1}^{d} \gamma_j^{p/2}} = d^{\frac{1}{2} \sum_{j=1}^{d} \gamma_j^{p/2}} \]
4. CONDITIONS FOR TRACTABILITY IN \( q \)-SPACES

In this section we shall give sufficient conditions on the sequence of weights \( \gamma = (\gamma_1, \gamma_2, \ldots) \) for (strong) QMC-tractability of integration in the space \( F_{d,\gamma} = W^{(1,\ldots, 1)}_q([0, 1]^d), \ q \geq 1, \) endowed with the norm \( \| \cdot \|_{d,\gamma,q}. \)

For this purpose we prove upper bounds on \( N = N_{\min}(\varepsilon,d) \) such that

\[
e(Q_{N,d}) \leq \varepsilon e(Q_{0,d})
\]

holds. As in [4] we use reproducing kernels to derive the initial error \( e(Q_{0,d}). \)

Define

\[
K_{d,\gamma}(x, t) := \prod_{j=1}^d (1 + \gamma_j \min(1 - x_j, 1 - t_j)).
\]

Then \( K_{d,\gamma} \) is a reproducing kernel for the space \( F_{d,\gamma}, \) i.e.,

\[
f(t) = \langle f(.), K(., t) \rangle,
\]

where for every element \( f \in F_{d,\gamma} \) and every element \( f' \) in the Banach dual space of \( F_{d,\gamma}, \) \( (f, f') := f'(f) \) denotes the action of \( f' \) on \( f. \)

Here \( K(., t) \) clearly is in the dual space (see appendix) for every \( t \) and

\[
\langle f(\cdot), K(\cdot, t) \rangle = \sum_{u \subseteq D} \gamma_u^{-1} \int_{1^{|u|}} f(x) K(x, t) dx, \quad (f \in F_{d,\gamma}).
\]

If we define \( h_d(x) := \int_{I^d} K(x, t) dt \) then \( h_d \) is the representer of integration on \( F_{d,\gamma}, \) i.e.,

\[
\langle f, h_d \rangle = \langle f(\cdot), \int_{I^d} K(\cdot, t) dt \rangle = \int_{I^d} \langle f(\cdot), K(\cdot, t) \rangle dt = \int_{I^d} f(t) dt,
\]

for all \( f \in F_{d,\gamma}. \) Thus we have

\[
e(Q_{0,d}) = \| I_d \| = \sup_{f \in F_{d,\gamma}, \| f \|_{d,\gamma,q} \leq 1} \left| \int_{I^d} f(t) dt \right| = \| h_d \|_{d,\gamma,p},
\]
such that the initial error is the norm of $h_d$ in $F^*_{d, \gamma}$ (here $p$ is such that $\frac{1}{p} + \frac{1}{q} = 1$). Further we have

$$h_d(x) = \int_{\mathbb{R}^d} K_d(x, t) dt$$

$$= \int \prod_{j=1}^{d} (1 + \gamma_j \min(1 - x_j, 1 - t_j)) dt$$

$$= \prod_{j=1}^{d} \int_0^1 (1 + \gamma_j \min(1 - x_j, 1 - t_j)) dt_j$$

$$= \prod_{j=1}^{d} \left(1 + \gamma_j \min(1 - x_j, 1 - t_j)\right)$$

and

$$\int_0^1 \min(1 - x_j, 1 - t_j) dt_j = \frac{1}{2} (1 - x_j^2).$$

Putting this together we get

$$h_d(x) = \prod_{j=1}^{d} \left(1 + \gamma_j \frac{1}{2} (1 - x_j^2)\right)$$

and

$$\|h_d\|_{d, \gamma, p}^p = h_d(1)^p + \sum_{\emptyset \neq u \subseteq D} \gamma_u^{-p/2} \int_{\mathbb{R}^d} \left| \frac{\partial |u|}{\partial x_u} h_d(x_u, 1) \right|^p dx_u,$$

where $1$ is the vector in $\mathbb{R}^d$ with all components equal to 1. We evaluate the integral:

$$\frac{\partial |u|}{\partial x_u} h_d(x) = \prod_{j \in u} (-\gamma_j) (1 - x_j) \prod_{j \notin u} \left(1 + \gamma_j \frac{1}{2} (1 - x_j^2)\right),$$

$$\frac{\partial |u|}{\partial x_u} h_d(x_u, 1) = \prod_{j \in u} \gamma_j (x_j - 1).$$
and hence
\[
\int_{I[\|u\|]} \left| \frac{\partial u}{\partial x_u} h_d(x_u, 1) \right|^p d x_u = \int_{I[\|u\|]} \prod_{j \in u} |\gamma_j|^p |x_j - 1|^p d x_u
\]
\[
= \prod_{j \in u} \gamma_j^p \int_0^1 |x_j - 1|^p d x_j
\]
\[
= \prod_{j \in u} \frac{\gamma_j^p}{p+1}.
\]

Since \( h_d(1) = 1 \) we have
\[
\|h_d\|_{d, \gamma, p}^p = 1 + \sum_{\emptyset \neq u \subseteq D} \gamma_u^{-p/2} \prod_{j \in u} \frac{\gamma_j^p}{p+1} = \prod_{j=1}^d \left( 1 + \frac{\gamma_j^{p/2}}{p+1} \right),
\]
such that
\[
e(Q_0, d)^p = \prod_{j=1}^d \left( 1 + \frac{\gamma_j^{p/2}}{p+1} \right).
\]

We summarize:

**Proposition 4.1.** The initial error for integration in the weighted space \( F_{d, \gamma} = W_1^{(1, \ldots, 1)}([0, 1]^d) \) with weight sequence \( \gamma = (\gamma_1, \gamma_2, \ldots) \) and for \( q \geq 1 \) is given by
\[
e(Q_0, d) = \prod_{j=1}^d \left( 1 + \frac{\gamma_j^{p/2}}{p+1} \right)^{1/p},
\]
where \( p \) is such that \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Remark 4.1.** Observe that the initial error \( e(Q_0, d) \) is bounded as a function of \( d \) if \( \sum_{j=1}^\infty \gamma_j^{p/2} \) is finite.

Now we are ready to state our tractability results for integration in the space \( F_{d, \gamma} \).

**Theorem 4.1.** Let \( q = \frac{p}{p-1} \), where \( p \) is an even integer, and let \( d \geq 1 \). Then we have
1. If
\[ \sum_{j=1}^{\infty} \gamma_j^{p/2} < \infty, \]
then integration in the weighted space \( F_{d, \gamma} \) is strongly QMC-tractable. More detailed for \( \varepsilon \in (0, 1) \) we have \( N_{\min} \leq C \cdot \varepsilon^{-2} \) where the constant \( C \) is given by \( C = (8p \alpha)^2 \) and where \( \alpha = \frac{p}{p+1-b} \).

2. If
\[ b := \limsup_{d \to \infty} \sum_{j=1}^{d} \frac{\gamma_j^{p/2}}{\log d} < \infty, \]
then integration in the weighted space \( F_{d, \gamma} \) is QMC-tractable. More detailed for \( \varepsilon \in (0, 1) \) we have \( N_{\min} \leq C \cdot d^\alpha \cdot \varepsilon^{-2} \), where \( C = (8p)^2 \) and where \( \alpha = \frac{p-1}{p(p+1)} \cdot b \).

Remark 4.2. Note that in the case of QMC-tractability the \( d \)-exponent \( \alpha \) tends to zero as \( q \) tends to one. But unfortunately \( C \) depends on \( p \) and therefore tends to infinity in this case.

Proof. From the definition of the worst case error \( e(Q_{N,d}) \) and from the generalized Koksma-Hlawka bound (1) for the integration error in \( F_{d, \gamma} \) we get
\[ e(Q_{N,d}) \leq L_{N,\gamma}^p \leq \frac{L_{N,\gamma}^p}{\prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \right)^{1/p}} \cdot e(Q_{0,d}). \]
Due to Corollary 3.1 we can find \( N \) points \( t_1, \ldots, t_N \) in the \( d \)-dimensional unit cube \( I^d \) with weighted \( L^p \) discrepancy \( L_{N,\gamma}^p \) such that
\[ \frac{L_{N,\gamma}^p}{\prod_{j=1}^{d} \left( 1 + \gamma_j^{p/2} \right)^{1/p}} \leq \frac{8p}{N^{1/2}} \cdot \exp \left( \frac{1}{p} \sum_{j=1}^{d} \log \frac{1 + \gamma_j^{p/2} \left( 1 + \frac{\gamma_j^{p/2}}{p+1} \right)}{1 + \gamma_j^{p/2}} \right) \]
\[ = \frac{8p}{N^{1/2}} \cdot \exp \left( \frac{1}{p} \sum_{j=1}^{d} \log \frac{1 + \gamma_j^{p/2} \left( 1 + \frac{\gamma_j^{p/2}}{p+1} \right)}{1 + \gamma_j^{p/2}} \right). \]
\[ \leq \frac{8p}{N^{1/2}} \cdot \exp \left( \frac{1}{p} \sum_{j=1}^{d} \log \left( 1 + \frac{p-1}{2(p+1)} \gamma_j^{p/2} \right) \right) \]

\[ \leq \frac{8p}{N^{1/2}} \cdot \exp \left( \frac{p-1}{2p(p+1)} \sum_{j=1}^{d} \gamma_j^{p/2} \right). \]

Hence there is a point set \( \{ t_1, \ldots, t_N \} \) in \( I^d \) such that

\[ e(Q_{N,d}) \leq \frac{8p}{N^{1/2}} \cdot \exp \left( \frac{p-1}{2p(p+1)} \sum_{j=1}^{d} \gamma_j^{p/2} \right) \cdot e(Q_{0,d}). \]

Let \( \varepsilon \in (0, 1) \).

1. If \( \sum_{j=1}^{\infty} \gamma_j^{p/2} < \infty \), then we have

\[ \frac{8p}{N^{1/2}} \cdot \exp \left( \frac{p-1}{2p(p+1)} \sum_{j=1}^{d} \gamma_j^{p/2} \right) \leq \frac{8p}{N^{1/2}} \cdot a, \]

where \( a \) is as in the statement of Theorem 4.1. Now the right hand side of this inequality is smaller than \( \varepsilon \) iff \( (8pa)^{2 \cdot \varepsilon^{-2}} < N \) and the result follows.

2. It \( b := \limsup_{d \to \infty} \sum_{j=1}^{d} \frac{\gamma_j^{p/2}}{\log d} < \infty \), then we have

\[ \frac{8p}{N^{1/2}} \cdot \exp \left( \frac{p-1}{2p(p+1)} \sum_{j=1}^{d} \gamma_j^{p/2} \right) \leq \frac{8p}{N^{1/2}} \cdot d^{\gamma_j^{p/2}} = \frac{8p}{N^{1/2}} \cdot d^{\alpha/2}, \]

where \( \alpha \) is as in the statement of Theorem 4.1. Now \( \frac{8p}{N^{1/2}} \cdot d^{\alpha/2} < \varepsilon \) iff \( (8p)^{2 \cdot \varepsilon^{-2}} < N \) and the result follows.

\[ \text{APPENDIX: THE SPACE } W_q^{(1, \ldots, 1)}([0, 1]^d) \]

For the whole section let \( 1 \leq q \leq \infty \). \( D = \{ 1, \ldots, d \} \).

**Definition A.1.** \( W_q^{(1, \ldots, 1)}([0, 1]^d) \) is the space of functions

\[ f : [0, 1]^d \to \mathbb{R} \]
which are absolutely continuous on $[0,1]^d$, for which \( \frac{\partial u}{\partial x} f(x,1) \) exists for almost every \( x \in [0,1]^d \), \( u \subseteq D := \{1, \ldots, d\} \), and for which

\[
\|f\|_{d,q} := \left( \sum_{u \in D} \int \left| \frac{\partial |u|}{\partial x} u f(x,1) \right|^q dx \right)^{1/q} < \infty.
\]

**Example A.1.** For the case \( d = 1 \) we have

\[
W^1_q([0,1]) := \{ f : [0,1] \rightarrow \mathbb{R} : f \text{ absolutely continuous}, \int_0^1 |f'(x)|^q dx < \infty \}
\]

**Definition A.2.** Let \( X \) and \( Y \) be linear spaces. The algebraic tensor product \( X \otimes Y \) is the linear space of linear combinations of elementary tensors \( x \otimes y, x \in X \) and \( y \in Y \),

\[
X \otimes Y := \text{span}\{x \otimes y : x \in X, y \in Y\}
\]

with the relations ("simplification rules")

1. \( x_1 \otimes y + x_2 \otimes y = (x_1 + x_2) \otimes y \),
2. \( x \otimes y_1 + x \otimes y_2 = x \otimes (y_1 + y_2) \),
3. \( \lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y) \).

Each element \( z \in X \otimes Y \) has a representation as a sum of elementary tensors, i.e.,

\[
z = \sum_{k=1}^m x_k \otimes y_k
\]

\( x_1, \ldots, x_m \in X, y_1, \ldots, y_m \in Y \). Note that we do not need scalar factors due to simplification rule 3.

**Example A.2.** Let \( X \) be a space of functions on a set \( A \) and \( Y \) be a space of functions on a set \( B \). Then we can identify elementary tensors with the functions

\[
f \otimes g : A \times B \rightarrow \mathbb{R}
\]

\[
(a, b) \mapsto f(a)g(b) \quad (a \in A, b \in B),
\]
for \( f \in X, \ g \in Y \).
Every element in \( X \odot Y \) is of the form
\[
\sum_{k=1}^{m} f_k \otimes g_k
\]
for \( f_1, \ldots, f_m \in X, \ g_1, \ldots, g_m \in Y \), and can be identified with the function
\[
(a, b) \mapsto \sum_{k=1}^{m} f_k \otimes g_k(a, b) = \sum_{k=1}^{m} f_k(a)g_k(b)
\]
for \( a \in A, \ b \in B \).

Let \( X_1, \ldots, X_n \) be linear spaces. Then the tensor product of these spaces is defined by induction
\[
X_1 \odot \ldots \odot X_n := (X_1 \odot \ldots \odot X_{n-1}) \odot X_n.
\]

**Definition A.3.** Let \( X_1, \ldots, X_n \) be normed spaces. Then we define their \( q \)-direct product as the Cartesian product
\[
\prod_{k=1}^{n} X_k := \{(x_1, \ldots, x_n) : x_k \in X_k\}
\]
equipped with the norm
\[
\|(x_1, \ldots, x_n)\|_{\oplus q} := \left(\sum_{k=1}^{n} \|x_k\|^{q}\right)^{1/q},
\]
and we denote it by \( \bigoplus_{k=1}^{n} X_k \). Note that \( \bigoplus_{k=1}^{n} X_k \) is complete iff each \( X_k, \ 1 \leq k \leq n \) is complete.

**Proposition A.1.** The spaces \( W_q^{(1, \ldots, 1)}([0, 1]^d) \) and \( \bigoplus_{u \subseteq D} L_q([0, 1]^{|u|}) \) are isometrically isomorphic.

**Proof.** Define a linear mapping
\[
I : \bigoplus_{u \subseteq D} L_q([0, 1]^{|u|}) \longrightarrow W_q^{(1, \ldots, 1)}([0, 1]^d)
\]
by

\[ I((f_u)_{u \subseteq D}) := \sum_{u = \{k_1, \ldots, k_{|u|}\} \subseteq D} (-1)^{|u|} \int_{x_{k_1}}^{1} \cdots \int_{x_{k_{|u|}}}^{1} f_u(\xi_1, \ldots, \xi_{|u|}) d\xi_1 \cdots d\xi_{|u|}. \]

$I$ is a linear operator and $I((f_u)_{u \subseteq D})$ is absolutely continuous as a finite sum of primitive functions.

$I$ is isometric: Let $(f_u)_{u \subseteq D} \in \bigoplus_{u \subseteq D} L_q([0, 1]^{[u]})$.

\[ \|I((f_u)_{u \subseteq D})\|_{d,q}^q = \sum_{u \subseteq D} \int_{D^{[u]}} \left| \frac{\partial |u|}{\partial \mathbf{x}_u} \right| \sum_{v = \{k_1, \ldots, k_{|v|}\} \subseteq D} (-1)^{|v|} \int_{x_{k_1}}^{1} \cdots \int_{x_{k_{|v|}}}^{1} f_v(\xi_v) d\xi_v \right| d\mathbf{x}_u \]

\[ = \sum_{u \subseteq D} \int_{D^{[u]}} |f_u(\mathbf{x}_u)|^q d\mathbf{x}_u = \sum_{u \subseteq D} \|f_u\|_{q}^q = \left(\|I((f_u)_{u \subseteq D})\|_{\oplus_q}^q \right)^q. \]

It remains to show that $I$ is onto $W_q^{(1, \ldots, 1)}([0, 1]^d)$. Let $g \in W_q^{(1, \ldots, 1)}([0, 1]^d)$.

Define

\[ f_u(\mathbf{x}_u) := \frac{\partial |u|}{\partial \mathbf{x}_u} g(\mathbf{x}_u, 1). \]

Then it is easy to show that $I((f_u)_{u \subseteq D}) = g$. §

Note that for $f_1, \ldots, f_d \in W_q^1$ the function

\[ (x_1, \ldots, x_d) \mapsto f_1(x_1) \cdots f_d(x_d) \]

is in $W_q^{(1, \ldots, 1)}([0, 1]^d)$. Thus $W_q^1([0, 1]) \oplus \ldots \oplus W_q^1([0, 1])$ can be viewed as a subspace of $W_q^{(1, \ldots, 1)}([0, 1]^d)$, as in Example A.2.

**Proposition A.2.** The $d$-fold algebraic tensor product $W_q^1([0, 1]) \oplus \ldots \oplus W_q^1([0, 1])$ is dense in $W_q^{(1, \ldots, 1)}([0, 1]^d)$.

**Proof.** Let $f \in W_q^{(1, \ldots, 1)}([0, 1]^d)$. For all $u \subseteq D$ there is a function $g_u \in L_q[0, 1] \oplus \ldots \oplus L_q[0, 1]$ ($|u|$-times), such that

\[ \int_{D^{[u]}} \left| g_u(\mathbf{x}_u) - \frac{\partial |u|}{\partial \mathbf{x}_u} f(\mathbf{x}_u, 1) \right|^q d\mathbf{x}_u < \frac{\varepsilon^q}{2^d}. \]
since \( L_q[0,1] \odot \ldots \odot L_q[0,1] \) (\(|u|\)-times) is dense in \( L_q([0,1]^n] \). Define

\[
g := I ((g_u)_{u \subseteq D}) \in W_q^{(1,\ldots,1)}([0,1]^d),
\]

where \( I \) is the isomorphism from Proposition A.1. It is straightforward to show that \( g \) is a linear combination of simple tensors, i.e., \( g \) lies in \( W_q^{1}([0,1]) \odot \ldots \odot W_q^{1}([0,1]) \). Now

\[
\|g - f\|_{d,q}^q = \sum_{u \subseteq D} \int |g_u(x_u) - \frac{\partial |u|}{\partial x_u} f(x_u,1)|^q \, dx_u^u < \sum_{u \subseteq D} \epsilon^q \|u\|^{1/q} = \epsilon^q,
\]

and we are done.

**Definition A.4.** Let \( X_1, \ldots, X_d \) be normed spaces with norms \( p_1, \ldots, p_d \) respectively. A norm \( p \) on the algebraic tensor product \( X_1 \odot \ldots \odot X_d \) is called a \textit{tensor product norm}, if

\[
p(x_1 \odot \ldots \odot x_d) = p_1(x_1) \ldots p_d(x_d),
\]

for every elementary tensor \( x_1 \odot \ldots \odot x_d, x_k \in X_k, 1 \leq k \leq d \).

So what would a tensor product norm \( p \) on \( W_q^1([0,1]) \odot \ldots \odot W_q^1([0,1]) \) look like?

Let \( f_1, \ldots, f_d \in W_q^1([0,1]) \). We write \( f = f_1 \odot \ldots \odot f_d \)

\[
p(f) = \|f_1\|_{1,q} \ldots \|f_d\|_{1,q} = \left( \prod_{k=1}^{d} \left( |f_k(1)|^q + \int_0^1 \left| \frac{\partial |u|}{\partial x_k} f_k(x_k) \right|^q \, dx_k \right) \right)^{1/q}
\]

\[
= \left( \sum_{u \subseteq D} \prod_{k \not\in u} |f_k(1)|^q \left( \prod_{k \in u} \int_0^1 \left| \frac{\partial |u|}{\partial x_k} f_k(x_k) \right|^q \, dx_k \right) \right)^{1/q}
\]

\[
= \left( \sum_{u \subseteq D} \int_{f|\leq} \left| \frac{\partial |u|}{\partial x_u} f(x_u,1) \right|^q \, dx_u \right)^{1/q}
\]

\[
= \|f\|_{d,q}.
\]

So our norm \( \|\|_{d,q} \) is a tensor product norm and it indeed looks like the most natural one. (But be careful: There still may be tensor product norms
p on $W^1_q([0,1]) \odot \ldots \odot W^1_q([0,1])$ with $p(f_1 \odot \ldots \odot f_d) = \|f_1 \odot \ldots \odot f_d\|_{d,q}$ but $p \neq \|\cdot\|_{d,q}$.

**Definition A.5.** We call the completion of $W^1_q([0,1]) \odot \ldots \odot W^1_q([0,1])$ ($d$-times) with respect to the norm $\|\cdot\|_{d,q}$ the tensor product of the spaces $W^1_q([0,1]), \ldots, W^1_q([0,1])$ and we simply denote it by $W^1_q([0,1]) \odot \ldots \odot W^1_q([0,1])$.

Now we introduce the weights $\gamma_1 \geq \gamma_2 \geq \ldots \geq 0$. Obviously the function

$$p_k : W^1_q([0,1]) \to [0, \infty)$$

$$f \mapsto \left( |f(1)|^q + \gamma_k^{-1} \int_0^1 |f'(x)|^q dx \right)^{1/q},$$

$1 \leq k \leq d$, defines a norm on $W^1_q([0,1])$ which is equivalent to $\|\cdot\|_{1,q}$.

As in the unweighted case this norm can be extended in a natural way to the $d$-fold tensor product $W^1_q([0,1]) \odot \ldots \odot W^1_q([0,1]) = W_q^{(1,\ldots,1)}([0,1]^d)$.

**REFERENCES**