Identification of 2D Roesser models by using linear fractional transformations

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Abstract—In this paper, the problem of identifying a 2D linear time-invariant Roesser model is tackled. Based on the strong relation between the linear fractional representation and the nD Roesser model, a gradient-based optimization algorithm is suggested to estimate the state-space matrices of a standard Roesser model in the black-box as well as the gray-box model identification frameworks. Contrary to the developments available in the literature, no specific restriction (to the 2D causal, recursive and separable-in-denominator (CRSD) state-space models) is required by the non-linear programming technique developed in this article. The efficiency of this method is illustrated through two simulation examples: a CRSD state-space model and a 2D Roesser model of a co-current flow heat exchanger.

I. INTRODUCTION

Many efficient methods [1], [2] and software tools [3], [4] are now available to estimate a 1D state-space representation of a linear time-invariant (LTI) multivariable system from different types of input-output data sets in a consistent way. According to the authors’ knowledge, this is far from being the case when nD LTI state-space models are concerned. Nevertheless, over the last three decades, there has been a strong interest in studying nD systems through the development of specific representations (Roesser [5], Attassi [6], Fornasini-Marchesini [7] models) for which dedicated realization theories [8], elementary operation approach [9] or oriented control methods [10] have been introduced. Unfortunately, identifying efficiently an nD LTI state-space model is still a challenging problem. This difference of treatment of 1D and nD model identification is probably linked to the fact that many standard results available for 1D models have no counterpart for nD models [11].

nD LTI state-space models appear in many different fields (image processing [5], iterative learning control [12], filter design [13], discretization of partial differential equations [14]) because of their flexibility in modeling multi-dimensional systems. Thus, getting reliable nD (state-space) models is essential when such applications are concerned. In this paper, we focus on nD LTI Roesser models because of their generic nature [9]. Solutions for the identification of 2D state-space representations of a LTI multi-dimensional systems are available in the literature. Most of them focus on 2D causal, recursive and separable-in-denominator (CRSD) state-space models in Roesser form [15], [16], [17]. This type of state-space representation indeed corresponds to a transfer function with a denominator which can be factored into two polynomials, each being a function of only one shifted operator. Thus, a CRSD Roesser model can be tackled as two linked 1D state-space models. In the aforementioned references, several subspace-based algorithms, inspired by their 1D counterparts, are modified to estimate discrete-time 2D CRSD Roesser models from deterministic and/or stochastic inputs consistently.

In this paper, standard (i.e., without the CRSD restriction) Roesser models are handled. For the sake of conciseness, a specific attention is paid to 2D LTI state-space models. Notice however that the following developments can be extended to any nD Roesser model, n ≥ 3, straightforwardly. As shown in the following section, the algorithm described in this paper relies on the strong link between the Roesser model and the linear fractional representation (LFR) [18]. By noticing that a Roesser model can be described with a linear fractional transformation (LFT) [18], an extension of algorithms dedicated to LFRs and available, e.g., in [19], [20], is suggested to identify standard 2D Roesser models. More precisely, an extension of the non-linear programming algorithm introduced in [20] to 2D Roesser model identification is introduced in the sequel by considering, in a similar way, a black-box or a gray-box model identification framework.

The paper is organized as follows. Section II introduces the notations and states the addressed problem. Section III focuses on the strong link between a 2D Roesser model and the standard linear fractional representation. This is indeed the basis of the following developments. Section IV is dedicated to the gradient-based optimization algorithm developed to estimate the state-space matrices of the Roesser model to identify. Two simulation examples are introduced in Section V to illustrate the efficiency of this technique. Section VI concludes this contribution.

II. PROBLEM FORMULATION AND NOTATIONS

Herein, we consider LTI 2D systems assumed to be modeled by a 2D Roesser representation [5] defined as

\[ x_1(k_1 + 1, k_2) = A_1 x_1(k_1, k_2) + A_2 x_2(k_1, k_2) + B_1 u(k_1, k_2) \] (1a)

\[ x_2(k_1, k_2 + 1) = A_3 x_1(k_1, k_2) + A_4 x_2(k_1, k_2) + B_2 u(k_1, k_2) \] (1b)
\[ y(k_1, k_2) = C_1 x_1(k_1, k_2) + C_2 x_2(k_1, k_2) + D u(k_1, k_2) \quad (1c) \]
in the discrete-time framework where, at the \((k_1, k_2)\)th localization of a finite spatial domain \( \mathbb{D} = \{(k_1, k_2), 0 \leq k_1 \leq N, 0 \leq k_2 \leq M\} \), \( x_1(k_1, k_2) \in \mathbb{R}^{n_1} \) and \( x_2(k_1, k_2) \in \mathbb{R}^{n_2} \) are the local components of the state respectively, \( y(k_1, k_2) \in \mathbb{R}^{n_y} \) is the output vector, \( u(k_1, k_2) \in \mathbb{R}^{n_u} \) is the input vector, \( A_k, \ k \in \{1, \cdots, 4\}, \ B_i, \ i \in \{1, 2\}, \ C_i, \ i \in \{1, 2\} \) and \( D \) are constant matrices of appropriate dimensions.

In this paper, contrary to the cases studied, e.g., in [16], [21], \( A_2 \) or \( A_3 \) are not required to be equal to a zero matrix. Now, our identification problem can be formulated as follows

**Problem 1:** From \( \{u(k_1, k_2), y(k_1, k_2)\}, \ k_1 \in \{0, \cdots, N\}, \ k_2 \in \{0, \cdots, M\} \), estimate the unknown parameters of the state-space matrices \( A_i, \ i \in \{1, \cdots, 4\}, \ B_i, \ i \in \{1, 2\}, \ C_i, \ i \in \{1, 2\} \) and \( D \).

In the following, when it is necessary, these unknown parameters, gathered into a parameter vector \( \theta \in \mathbb{R}^{n_{\theta}} \), will be explicitly introduced into the state-space matrices composing the state-space form (1) as follows: \( A_k(\theta), \ k \in \{1, \cdots, 4\}, \ B_i(\theta), \ i \in \{1, 2\}, \ C_j(\theta), \ j \in \{1, 2\} \) and \( D(\theta) \). It is interesting to notice that such a problem formulation can be stated in an equivalent way when black-box or gray-box (i.e., physically-structured) state-space representations are handled. More precisely, when black-box models are considered, the state-space matrices involved in Eq. (1) are fully-parameterized and \( \theta \in \mathbb{R}^{n_{\theta}} \) (unknown (physical) parameters, the position of which in \( A_i, B_i, C_i \) and \( D \) is known a priori. As shown in Section V, the gradient-based algorithm introduced in Section IV can indeed be used for black-box or gray-box model in a similar manner. As far as the measurement noise acting on the system is concerned, it is assumed that the outputs of the real system can be disturbed by 2D zero-mean white or colored Gaussian noises. The output-error framework introduced hereafter can indeed deal with such noise characteristics straightforwardly.

### III. 2D Roesser Model and Linear Fractional Representation

As explain, for instance, in [18, Chapter 10], the linear fractional representation can be used to link the input-output representation and the state-space form of a model in a smart way. Such a property is used in the sequel. In order to describe this link correctly, let us start by recalling important definitions and notations (see [18, Chapter 10] for details).

**Definition 1:** Let \( M \) be a real matrix partitioned as

\[ M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{R}^{(p_1+p_2) \times (q_1+q_2)}. \quad (2) \]

Then, the (upper) linear fractional transformation of \( M \) with respect to \( \Delta \in \mathbb{R}^{n \times p} \) is given by (see Fig. 1 for an illustration)

\[ \mathcal{F}(M, \Delta) = M_{22} + M_{21} (I_{p_1 \times p_1} - M_{11} \Delta)^{-1} M_{12} \quad (3) \]

provided that the inverse of \( (I_{p_1 \times p_1} - M_{11} \Delta) \) exists.

![Fig. 1. LFR (or LFT description) as a matrix \( M \) with a partial feedback connection \( \Delta \).](image)

It is interesting to notice that, if the matrix \( M \) is composed of the state-space matrices \( (A, B, C, D) \) of an \( n_x \) order 1D state-space representation and \( \Delta = z^{-1} I_{n_x \times n_x} \), then the corresponding 1D transfer matrix \( G(z) \) satisfies

\[ G(z) = C(z I_{n_x \times n_x} - A)^{-1} B + D = \mathcal{F} \left( \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}, z^{-1} I_{n_x \times n_x} \right) \quad (4) \]

where \( z \) denotes the forward shift operator. Similarly, the transfer matrix of the 2D Roesser model (1) satisfies\(^1\)

\[ G(\theta, z_1, z_2) = C(\theta) \Delta (I_{n_x \times n_x} - \Delta A(\theta))^{-1} B(\theta) + D(\theta) = \mathcal{F} \left( \begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix}, \Delta \right) \quad (5) \]

where, now, \( n_x = n_{x_1} + n_{x_2}, \ z_i, \ i \in \{1, 2\}, \) are forward shift operator and

\[ A(\theta) = \begin{bmatrix} A_1(\theta) & A_2(\theta) \\ A_3(\theta) & A_4(\theta) \end{bmatrix}, \ B(\theta) = \begin{bmatrix} B_1(\theta) \\ B_2(\theta) \end{bmatrix} \quad (6) \]

\[ C(\theta) = \begin{bmatrix} C_1(\theta) \\ C_2(\theta) \end{bmatrix} \]

and

\[ \Delta = \begin{bmatrix} z_1^{-1} I_{n_{x_1} \times n_{x_1}} & 0_{n_{x_1} \times n_{x_2}} \\ 0_{n_{x_2} \times n_{x_1}} & z_2^{-1} I_{n_{x_2} \times n_{x_2}} \end{bmatrix}. \quad (7) \]

Notice that this relation can be extended to any \( nD \) Roesser model, \( n \geq 3 \).

Interesting from a notation viewpoint because of its compactness, this relation between the Roesser model and its LFR counterpart allows the user to discard the complicated interconnection\(^2\) between the state variables \( x_1 \) and \( x_2 \) by focusing on an input-output representation of the model without destroying the state-space model parameterization involved in the standard Roesser model.

Having access to this reformulation of the Roesser model with an LFR, the next step of the procedure deals with the estimation \( A(\theta), B(\theta), C(\theta) \) and \( D(\theta) \), more precisely the parameter vector \( \theta \), from the available data-sets.

\(^1\)The inverse of \( (I_{n_{x_1} + n_{x_2}} \times (n_{x_1} + n_{x_2}) - \Delta \Delta) \) exists when the transfer matrix of the underlying system is defined and the system is stable which is assumed in this paper.

\(^2\)This interconnection is the main reason why the authors of [16], [17] have focused on the CRSD description.
IV. LFR PARAMETER ESTIMATION OF A ROESSER MODEL

The LFT framework for model identification has been recently studied with attention in [19], [20], [22]. These results are used in the following to solve Problem 1 through the LFT description (5)-(7).

A. Identification algorithm

A standard way to estimate the parameter vector $\theta$ consists in minimizing the cost function

$$
\frac{1}{N+1} \frac{1}{M+1} \sum_{k_1=0}^{N} \sum_{k_2=0}^{M} \|y(k_1, k_2) - \gamma(k_1, k_2|\theta)\|_2^2
$$

$$
= \frac{1}{(N+1)(M+1)} \sum_{k_1=0}^{N} \sum_{k_2=0}^{M} \|\epsilon(k_1, k_2|\theta)\|_2^2
$$

(8)

where $\gamma(k_1, k_2|\theta)$, $k_1 \in \{0, \cdots, N\}$, $k_2 \in \{0, \cdots, M\}$, denotes the model output, $y(k_1, k_2)$, $k_1 \in \{0, \cdots, N\}$, $k_2 \in \{0, \cdots, M\}$, the system output measurements and $\epsilon(k_1, k_2|\theta)$, $k_1 \in \{0, \cdots, N\}$, $k_2 \in \{0, \cdots, M\}$, the corresponding output error.

Such a non-convex criterion can be minimized by using a gradient-based optimization algorithm dedicated to non-linear least-squares problem [23]. A Levenberg-Marquardt algorithm is a standard technique to optimize this non-linear least-squares cost function. This type of technique requires the calculation of the associated gradients and Hessians. As shown in [19], the LFR structure (5)-(7) can be really useful for such computations. A direct extension of the results available in [19] indeed leads to the following proposition.

Proposition 1: The parameter vector $\theta_i$ can be updated as follows

$$
\theta_{i+1} = \theta_i - (\alpha I + H(\theta_i))^{-1}g(\theta_i)
$$

(9)

where $\alpha$ stands for a user-defined positive real number, $H(\theta_i)$ is (an approximation of) the Hessian and $g(\theta_i)$ denotes the gradient at $\theta_i$. These vectors and matrices are respectively defined as follows

$$
g(\theta) = \frac{2}{(N+1)(M+1)} \psi^T_{NM}(\theta) e_{NM}(\theta)
$$

(10)

where

$$
e_{NM}(\theta) = \left[ e^T(0,0|\theta) \quad e^T(1,0|\theta) \quad \cdots \quad e^T(N,0|\theta) \quad e^T(0,1|\theta) \quad \cdots \quad e^T(N,M|\theta) \right]^T
$$

(11)

and

$$
\psi_{NM}(\theta) = \frac{\partial e_{NM}(\theta)}{\partial \theta}
$$

while the Hessian approximation satisfies

$$
H(\theta) \approx \frac{2}{(N+1)(M+1)} \psi^T_{NM}(\theta) \psi_{NM}(\theta).
$$

(12)

The explicit computation of the sensitivity functions

$$
\frac{\partial e_{NM}(\theta)}{\partial \theta^i} \quad \text{involved in the former equations can be obtained by simulating } \frac{\partial e_{NM}(\theta)}{\partial \theta^i}, \text{ i.e.,}
$$

$$
\frac{\partial e(k_1, k_2|\theta)}{\partial \theta^i} = F \left( \begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} \right) \Delta
$$

× \left[ \xi(k_1, k_2|\theta) \right]

(13)

with $n_x = n_{x_1} + n_{x_2}$ and

$$
\left[ \xi(k_1, k_2|\theta) \right] = F \left( \begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} \right) \Delta \left[ u(k_1, k_2) \right].
$$

B. Discussion

As noticed first in [19], then thoroughly studied in [24], the data driven local coordinates (DDLC) parameterization can be used to improve the convergence of the algorithm introduced in Proposition 1 when fully-parameterized black-box LFR models are considered. Indeed, when fully-parameterized state-space LFR models are handled, identifiability [1] problems appear because the estimated state matrices are known up to a similarity transformation. First considered as an issue, this non-uniqueness can be used as an interesting feature to constrain the optimization algorithm. More precisely, by noticing that the set of similar systems (indistinguishable from the available data-sets) forms a manifold in the parameter space, the DDLC approach consists in determining the tangent plane of the aforementioned manifold, then updating the parameter vector $\theta$ along the orthogonal complement of this manifold. Details about the implementation of such a technique are available in [19]. Because of the strong link between LFR and 2D Roesser model, such a numerically-improved parameterization can be used as well for any fully-parameterized black-box 2D Roesser model.

As any iterative optimization technique, the efficiency of the method used in this paper, more precisely the consistency of the final estimated parameter vector, highly depends on its initialization because of the existence of local minima. In this paper, a simple approach is applied where random values are used as initial parameters and several trials are carried out until final small cost function values are reached. In order to circumvent such a naive technique, the authors of this contribution are currently studying an alternative approach based on a combination of a new subspace-based identification technique able to yield a reliable fully-parameterized 2D Roesser model and an extension of the null-space-based technique developed in [25] to LFRs.

V. SIMULATION EXAMPLES

In this section, the algorithm described in Proposition 1 is tested with simulated data generated respectively by two different numerical examples: a 2D CRS model and a co-current flow heat exchanger. The first simulation example is borrowed from [17]. This CRS model state-space model is used to perform a fair comparison with an existing subspace-based technique dedicated to the identification of CRS state-space models. Notice however that our algorithm can tackle fully-parameterized 2D state-space representations without any
modification. The second one, a 2D Roesser model of a heat
exchanger, is used to show that the method suggested in this
paper can also be applied to estimate physical parameters of a
real system, i.e., in the gray-box model framework.

A. A 2D CRSD model

In this subsection, our algorithm (called “LFR” algorithm
in the following figures) is compared with the 2D N4SID al-
gorithm introduced in [17]. In this afore-cited paper, a CRSD
state-space model is considered. This model is characterized
by the following state matrices (see Eq. (1) for a definition)

\[
A_1 = \begin{bmatrix}
-0.2589 & -0.4977 & -0.5811 \\
-0.4526 & 0.2943 & -0.0616 \\
-0.0736 & 0.0822 & -0.0859
\end{bmatrix},
A_2 = \begin{bmatrix}
-0.3967 & 0.1232 \\
-0.0459 & 0.2855 \\
-0.0396 & -0.0614
\end{bmatrix},
A_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
A_4 = \begin{bmatrix}
-0.1175 & -0.1809 \\
0.1257 & 0.3264 \\
-0.0614 & -0.0859
\end{bmatrix},
B_1 = \begin{bmatrix}
0.1006 & 0.1273 & -0.2313
\end{bmatrix}^T,
B_2 = \begin{bmatrix}
-1.0636 & -0.1246
\end{bmatrix}^T,
C_1 = \begin{bmatrix}
1.4578 & 1.7139 & 0.6824
\end{bmatrix},
C_2 = \begin{bmatrix}
0.3152 & 2.5705
\end{bmatrix}
\]

and \(D = -0.2638\). As suggested in [17], the input signal
is chosen as an \(N\)-by-\(M\) matrix containing pseudo-random
values drawn from the standard normal distribution with
\(N = M = 50\). For this comparison, the tuning parameters
of the 2D N4SID algorithm are equal to those available in
[17], i.e., \(n_h = 3, n_v = 2, j = 32\) and \(i = 10\).

As a gradient-based optimization algorithm, it is
well-known that the technique introduced in Proposition 1 can
be very sensitive to the user-defined initial parameter vector.
Because of the lack of physical insights for the black-box
framework considered in this subsection, it has been decided
to fix \(\theta_{init} = 0.6\theta_{real}\) and to postpone the sensitivy study
(w.r.t. the initialization) to Sub-Section V-B where prior
physical information can be introduced in an easier manner.

Because the 2D CRSD model studied herein is not
identifiable, focusing on the state matrices entries as a
performance index is not possible. It is necessary to introduce
specific invariants of the 2D system in order to compare
the performance of both identification techniques. In the 1D
framework, the poles and the zeros of the estimated models
are often used as reliable invariants of the system dynamics
because the pole-zero map is an accurate representation of
the dynamical behavior of the system. In the 2D framework,
these standard tools are not usable because a 2D LTI model
can have an infinite number of poles and zeros [11]. Thus,
in this study, the eigenvalues of the estimated \(A\) matrices
are used as a reliable performance index. For the sake of
conciseness, only two eigenvalues are plotted in Fig. 2 and
Fig. 3 respectively. Notice that similar conclusions can be
drawn for the others. The box-plots available in Fig. 2-3
are obtained after carrying out a Monte Carlo simulation of
size 100 where the tuning stochastic element is the output
measurement noise generated as a white Gaussian noise
characterized by a signal-to-noise ratio equal to 40 \(dB\) and
30 \(dB\) respectively. As shown in Fig. 2-3, the algorithm
introduced in this paper outperforms the 2D N4SID algorithm,
especially when the signal-to-noise ratio is equal to 30 \(dB\).
This difference can be explained by the fact that, currently,
the 2D N4SID algorithm developed in [17] does not contain
any instrumental variable to reduce the bias induced by the
disturbances acting on the system.

B. A co-current flow heat exchanger

Heat exchangers are widely used in industry such as
in power plants, chemical plants, petroleum refineries, for
space heating, refrigeration, and air conditioning to name
a few. Initially used only to cool and to heat large scale
industrial processes, heat exchangers are now more and more
implemented in industry in order to recover lost energy
from any hot product or effluent by heating a different
stream in the process. Due to the increasing exploitation
of heat exchangers in different industrial frameworks, it is
necessary to have access to reliable and relevant models of
such processes. Depending on the use, many types of heat
exchangers can be found: cross-flow, rotary, parallel-flow,
counter-flow and co-current flow [26]. In this sub-section,
a specific attention is paid to the identification of the physical
parameters of a co-current flow heat exchanger where the hot
and the cold fluids run in the same direction through parallel
pipes separated by a solid plate preventing the mixing of the
fluids.

The dynamics of such a system can be described by using
the following set of first order partial differential equations
The unknown parameters to estimate are the cold and hot fluid mass flow rates \( m_c \) and \( m_h \), the cold and hot specific heats \( c_c \) and \( c_h \), the cold and hot thickness \( d_c \) and \( d_h \) related to specific physical parameters of the system. Indeed, these unknown quantities are linked to the width \( W \) of the heat exchanger, the cold and hot convection coefficient \( h_c \) and \( h_h \) resp., the cold and hot mass flow rates \( m_c \) and \( m_h \) resp., the cold and hot fluid densities \( \rho_c \) and \( \rho_h \) resp., the cold and hot specific heats \( c_c \) and \( c_h \) resp. as follows:

\[
\begin{align*}
\nu_h &= \frac{m_h}{\rho_h d_h W}, \\
\alpha &= -\beta = -\frac{h_c h_h}{\rho_c c_h d_h (h_c + h_h)}, \\
\nu_c &= \frac{m_c}{\rho_c d_c W}, \\
\eta &= -\delta = -\frac{h_c h_h}{\rho_c c_c d_c (h_c + h_h)}.
\end{align*}
\]

Simulating as well as identifying the parameters of such partial differential equations is far from being easy. As far as the simulation is concerned, various numerical methods are available to deal with advective-diffusive problems, both finite element methods as, e.g., in [28], or also finite difference methods as, e.g., in [29]. Concerning the identification, recent solutions (see, e.g., [30] and the references therein) have been suggested. In the current contribution, it has been chosen to follow the basic idea initially suggested in [31].

This technique aims at transforming the partial differential equations governing the behavior of the system to identify into two-dimensional linear equations by introducing specific derivative approximations. In this paper, in order to get a 2D Roesser model of the heat exchanger, we use the forward and backward difference quotients for \( \frac{\partial T}{\partial x} \) and \( \frac{\partial T}{\partial t} \) respectively. Such approximations lead to the following set of discrete-time linear equations:

\[
\begin{align*}
T_h(k_1,k_2+1) - T_h(k_1,k_2) &= \alpha T_h(k_1,k_2) - \beta T_c(k_1,k_2), \\
-\beta T_c(k_1,k_2) + \nu_h T_h(k_1,k_2) - T_h(k_1-1,k_2) &= 0, \\
T_c(k_1,k_2+1) - T_c(k_1,k_2) &= \eta T_c(k_1,k_2) - \delta T_h(k_1,k_2), \\
-\delta T_h(k_1,k_2) + \nu_c T_c(k_1,k_2) - T_c(k_1-1,k_2) &= 0.
\end{align*}
\]

After straightforward manipulations, we find the following 2D Roesser discrete-time model:

\[
\begin{bmatrix}
T_1(k_1,k_2) \\
T_2(k_1,k_2) \\
T_3(k_1,k_2) \\
T_4(k_1,k_2)
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\frac{\beta \Delta t}{\Delta x} & 0 & 0 & 0 \\
\frac{\alpha \Delta t}{\Delta x} & \frac{\beta \Delta t}{\Delta x} & 0 & 0 \\
0 & \frac{\eta \Delta t}{\Delta x} & \frac{\beta \Delta t}{\Delta x} + \eta \Delta t & 0
\end{bmatrix}
\begin{bmatrix}
T_1(k_1,k_2) \\
T_2(k_1,k_2) \\
T_3(k_1,k_2) \\
T_4(k_1,k_2)
\end{bmatrix}
\tag{16}
\]

and

\[
\begin{bmatrix}
T_1(k_1-1,k_2) \\
T_2(k_1-1,k_2) \\
T_3(k_1-1,k_2) \\
T_4(k_1-1,k_2)
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\frac{\beta \Delta t}{\Delta x} & 0 & 0 & 0 \\
\frac{\alpha \Delta t}{\Delta x} & \frac{\beta \Delta t}{\Delta x} & 0 & 0 \\
0 & \frac{\eta \Delta t}{\Delta x} & \frac{\beta \Delta t}{\Delta x} + \eta \Delta t & 0
\end{bmatrix}
\begin{bmatrix}
T_1(k_1-1,k_2) \\
T_2(k_1-1,k_2) \\
T_3(k_1-1,k_2) \\
T_4(k_1-1,k_2)
\end{bmatrix}
\tag{18}
\]

defined for \( k_1 > 1 \). In the following, the initial/boundary conditions for the Roesser model are assumed to be known, i.e., \( T_1(0,k_2) = T_2(0,k_2) = T_3(k_1,0) = T_4(k_1,0) \) are known and fixed a priori. Contrary to a standard Roesser model, we don’t have access to the input and output vectors for any values of \( k_1 \) and \( k_2 \), \( k_1 \in \{1, \ldots, N\} \) and \( k_2 \in \{1, \ldots, M\} \) respectively. Indeed, on a real heat exchanger, the sensors and actuators are located at the outlet (\( x = W \)) and inlet (\( x = 0 \)) of the system respectively. Thus, in the following, we assume that the input signal is

\[
u(k_1 = 0, k_2) = \begin{bmatrix} T_3(0,k_2) \\ T_4(0,k_2) \end{bmatrix}, \quad k_2 \in \{1, \ldots, M\}
\tag{19}
\]

and the output signal is

\[
y(k_1 = N, k_2) = \begin{bmatrix} T_3(N,k_2) \\ T_4(N,k_2) \end{bmatrix}, \quad k_2 \in \{1, \ldots, M\}
\tag{20}
\]

Thus, the involved cost function is reduced to

\[
J_{MN} = \frac{1}{M+1} \sum_{k_2=0}^{M} \|y(N,k_2) - y(N,k_2|\theta)\|_2^2
\tag{21}
\]

but its optimization still requires the simulation of 2D Roesser models. This feature justifies the use of the algorithm introduced in Proposition 1.
ratio equal to 10 dB, are added up to the noise-free outputs plotted in Fig. 4 and the initial parameter vector is generated by considering the following formula
\[ \theta_{i,real} = \theta_{i,real}(r_i - 0.5) \] (22)
where, for \( i = \{1, \cdots, 4\} \), \( \theta_{i,real} \) is the real value of the sought parameter and \( r_i \) is a random number uniformly distributed on the open interval \((0, 1)\). The resulting estimated parameters are plotted in Fig. 5. These box-plots prove that, for this simulation example, the optimization suggested in this paper is quite robust to the initialization as well as a low signal-to-noise ratio.

![Box-plot of the estimated parameter values for the heat exchanger.](image)

**VI. CONCLUSION**

In this paper, a gradient-based algorithm is introduced in order to estimate the state matrices of a 2D Roesser model consistently. This algorithm is obtained thanks to the close relation between the Roesser model representation and the linear fractional transformation description. Contrary to the solutions available in the literature until now, no restriction to a causal, recursive, separable-in-denominator model is required. This algorithm has been tested with simulation data. A specific attention has been paid to the estimation of physical parameters of a co-current flow heat exchanger. These numerical results have proved the efficiency of the developed non-linear programming algorithm.

**REFERENCES**