$L^p$-OPTIMAL BOUNDARY CONTROL FOR THE WAVE EQUATION∗

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Abstract. We study problems of boundary controllability with minimal $L^p$-norm ($p \in [2, \infty]$) for the one-dimensional wave equation, where the state is controlled at both boundaries through Dirichlet or Neumann conditions. The problem is to reach a given terminal state and velocity in a given finite time, while minimizing the $L^p$-norm of the controls. We give necessary and sufficient conditions for the solvability of this problem. We show as follows how this infinite-dimensional optimization problem can be transformed into a problem which is much simpler: The feasible set of the transformed problem is described by a finite number of simple pointwise equality constraints for the control function in the Dirichlet case while, in the Neumann case, an additional integral equality constraint appears. We provide explicit complete solutions of the problems for all $p \in [2, \infty]$ in the Dirichlet case and solutions for some typical examples in the Neumann case.

Key words. optimal control, boundary control, wave equation, analytic solution, distributed parameter systems, robust optimization, controllability, state constraints, sensitivity, test examples

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1. Introduction. In this paper, we discuss two-sided Dirichlet or Neumann controls for the one-dimensional wave equation for $p$ between 2 and $\infty$. We consider the problem of exact control; that is, starting from the zero position we want to reach a given terminal state in a given finite time. Our aim is to find control functions with minimal $L^p$-norm that steer the system to the target. For certain typical cases, we present explicit representations of such optimal control functions in terms of the given target state.

It is well known that, in the $L^2$-case, the optimal control functions can be characterized as the $L^2$-norm minimal solutions of a trigonometric moment problem, which has been analyzed in depth (see [4], [19], [15]). For the $L^p$-case ($p > 2$) there are only a few publications on the subject (see [2], [16], [12], [11], [14], [10]), and even the question of existence of solutions, which is equivalent to the question of $L^p$-controllability, has not been solved completely.

In the present paper, we give a complete analysis of this problem for the boundary control of the one-dimensional wave equation. The problem can be reduced to the case of the minimal time interval, where controllability is possible. This allows an answer to be given to the question of solvability of the problem of $L^p$-controllability in terms of the properties of the target states. We use the control function for the minimal time interval to transform the infinite-dimensional problem into a problem, which is much simpler because it has only a finite number of simple pointwise equality constraints with an additional integral equality constraint (see (3.29) below) in the Neumann case. The transformation is based upon the representation of the state as a trigonometric series and on the corresponding description of the feasible set by

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a sequence of moment equations on the control time interval that is transformed into a sequence of moment equations on a shorter time interval, namely, the interval corresponding to the time that a characteristic curve needs to travel from one end of the string to the other.

In the Dirichlet case, solutions of our problem of optimal control exist for all time intervals which are at least as long as the time that waves need to travel from one boundary of the system to the other, provided that the functions that describe the target are sufficiently regular. The required regularity is that the initial state and the primitive of the initial velocity are both in the space $L^p(0, L)$. The optimal controls are given explicitly in terms of these functions for all $p \in [2, \infty)$. In general, the $L^\infty$-norm minimal control is not determined uniquely. Hence, in general there is a set of $L^\infty$-norm minimal controls that contains more than one element. This convex set contains a unique element with minimal $L^2$-norm. In Theorem 2.2, we give this element explicitly.

For Neumann boundary controls, the situation is more complicated. On the time interval that allows a characteristic curve to travel from one boundary of the system interval to the other, in general, controllability is possible up to a constant only, in the sense that instead of the desired target a state can be reached that differs from the desired target by an appropriate constant. If this time interval is enlarged by an arbitrarily small time, $L^p$-controllability is possible on the elongated time interval if and only if the target functions are sufficiently regular. In this case, the required regularity is that the initial velocity and the derivative of the initial state are both in the space $L^p(0, L)$. We transform the optimal control problem to a simpler problem, where the feasible set is defined by a finite number of constraints. For some cases, we give explicit expressions for the solution of the optimal control problem in terms of the given target state.

The relation between the $L^p$ regularity of the controls and the data in the Neumann and Dirichlet cases is consistent with what is known in the classical $L^2$ theory. The form of the optimal solutions depends on the relation between the length of the time interval and the time that the waves need to travel from one boundary of the system to the other. The structure of the optimal control functions is, in general, quite complicated. The optimal controls usually do not show bang-bang behavior; this is also true in the $L^\infty$-case. This may be surprising, since the optimal solutions of discretized problems often are of bang-bang type. The difference between the structure of the controls that solve the PDE-constrained optimization problem and controls that are solutions of discretized problems has been the subject of recent research; see [9], [21], and the references therein.

This paper is also a contribution to robust optimal control: We have found controls whose optimality is robust with respect to perturbations of the objective function. For symmetric targets in the case of Dirichlet boundary controls and for antisymmetric target states in the case of Neumann boundary controls, the controls that are optimal for $L^p$ ($p > 2$) are exactly the controls that are optimal with respect to the $L^2$-norm and sufficiently regular to be contained in the space $L^p$.

Our results are also interesting from the point of view of sensitivity analysis (see [3]), since they allow us to analyze how the solutions of the optimization problems depend on $p$, the target state, and the time interval.

The explicit solutions of problems of optimal control that we present provide valuable test examples for numerical methods.

In the linear case, $L^p$-boundary controls have been considered in Krabs and
Leugering [16], where $W^{1,p}$ and also $L^p$-Dirichlet controls for $p \in [2, \infty]$ are applied at one of the boundary points. Optimal control problems for first-order-in-time equations with distributed $L^p$-controls have been considered in Fabre, Puel, and Zuazua [5]. See also Glowinski and Lions [7], [8].

Our work is related to existing controllability results for nonlinear problems; see [6] for approximate controllability results for semilinear parabolic equations with $L^p$-interior or boundary controls and [20] for controllability results for a semilinear wave equation for a class of nonlinearities that grow superlinearly at infinity.

We hope that our results will prove helpful for future analysis of optimal control problems with nonlinear systems.

This paper is organized as follows. Section 2 considers the problem of Dirichlet boundary control. We define the exact optimal boundary control problem, give an exact controllability result (Theorem 2.1), and state Theorem 2.2, where boundary controls that solve the optimal control problem are given in terms of the target state. This result allows a sensitivity analysis for the optimal control problem that is the subject of the next section. Then examples for the solutions presented in Theorem 2.2 are given. For the proof of the results, a series representation of weak solutions of the initial value problem is given and the moment problem describing the successful controls is defined. Then the minimal time interval, where controllability holds, is studied (see Lemma 2.6). Longer time intervals are studied section 2.10. The corresponding moment equations are transformed to moment equations on the minimal time interval. This transformation allows us to prove Theorem 2.1. Also by transformation to a problem on the minimal time interval, Theorem 2.2 is proved. At the end of section 2, the cases of symmetric and antisymmetric targets are discussed.

In section 3, we consider the problem of Neumann boundary control. The optimal control problem is defined and an exact controllability result (Theorem 3.1) is given. A series representation of the weak solution of the initial value problem and the moment problem that describes the successful controls is stated. Exact controllability up to a constant is proved for the minimal time interval (see Lemma 3.2). For larger time intervals, exact controllability is proved. Theorem 3.4 on the solutions of the optimization problem for a certain range of control times is stated. Finally, the cases of symmetric and antisymmetric targets are considered.

2. Dirichlet boundary control. In this section we present a complete solution of the problem of $L^p$-norm minimal Dirichlet boundary control of our system.

2.1. The initial-value problem. Let an interval $[0, L]$, a time interval $[0, T]$, and a wave speed $c > 0$ be given. We consider the initial-value problem for the wave equation

\begin{equation}
(2.1) \quad y_{tt}(x, t) = c^2 y_{xx}(x, t), \quad (x, t) \in [0, L] \times [0, T],
\end{equation}

subject to the initial conditions

\begin{equation}
(2.2) \quad y(x, 0) = 0, \quad y_t(x, 0) = 0, \quad x \in [0, L],
\end{equation}

and the Dirichlet boundary conditions

\begin{equation}
(2.3) \quad y(0, t) = f_1(t), \quad y(L, t) = f_2(t), \quad t \in [0, T].
\end{equation}

For the description of the desired target state, we add the following end conditions:

\begin{equation}
(2.4) \quad y(x, T) = y_0(x), \quad y_t(x, T) = y_1(x), \quad x \in [0, L].
\end{equation}
The function \( y_0 \) is in the Hilbert space \( L^2(0, L) \) of square integrable functions on the interval \((0, L)\), and the function \( y_1 \) is integrable such that \( Y_1(x) = \int_0^x y_1(z) \, dz \) is in \( L^2(0, L) \), so \( y_1 \) is contained in the corresponding Sobolev space \( W^{-1/2}(0, L) \).

### 2.2. The optimization problem.

For a fixed time \( T > 0 \) and a given value of \( p \in [2, \infty) \), we consider the following optimization problem:

\[
C(p) : \inf \| f_1 \|_{p,(0,T)}^p + \| f_2 \|_{p,(0,T)}^p \quad \text{s.t.} \quad f_1, f_2 \in L^p(0, T)
\]

and the solution \( y \) of the initial boundary-value problem (2.1)–(2.3) satisfies the end conditions (2.4).

In the case \( p = \infty \), our optimization problem is the following:

\[
C(\infty) : \inf \max\{\| f_1 \|_{\infty,(0,T)}, \| f_2 \|_{\infty,(0,T)}\} \quad \text{s.t.} \quad f_1, f_2 \in L^\infty(0, T)
\]

and the solution \( y \) of (2.1)–(2.3) satisfies the end conditions (2.4).

Here, for \( p \in [2, \infty) \), we use the norm

\[
\| f \|_{p,(0,T)} = \left( \int_0^T |f(t)|^p \, dt \right)^{1/p}
\]

and for \( p = \infty \) we use the norm \( \| f \|_{\infty,(0,T)} = \text{ess sup}\{ |f(t)| : t \in (0, T) \} \).

### 2.3. Exact controllability.

**Theorem 2.1.** Let \( p \in [2, \infty] \) and \( T \geq L/c \) be given. The initial boundary-value problem (2.1)–(2.3) has a weak solution that satisfies the end conditions (2.4) with \( f_1, f_2 \in L^p(0, T) \) if and only if the target states \( y_0, y_1 \) satisfy the following conditions: \( y_0 \in L^p(0, L) \) and \( Y_1 \in L^p(0, L) \), where \( Y_1(x) = \int_0^x y_1(z) \, dz \), that is, \( y_1 \in W^{-1/p}(0, L) \). This implies that problem \( C(p) \) is solvable if and only if \( y_0 \) and \( Y_1 \) are in \( L^p(0, L) \).

A standard method for proving an exact controllability result of this type is to reduce the exact controllability problem to a moment problem and to prove the solvability of the moment problem using Ingham's classical inequalities (see [13]) or its generalizations (see [18], [16]). For the cases \( p = 2 \) and \( p = \infty \), Ingham's results provide an alternative proof of Theorem 2.1. In this paper, we use a different approach: We give a solution of the moment problem explicitly (see section 2.9).

We expect that Theorem 2.1 holds for all \( p \geq 1 \), but for the case \( 1 \leq p < 2 \) a different method of proof should be used.

In the next section we will state our main result, which gives the solution of problem \( C(p) \) in terms of two functions that depend on the target states.

### 2.4. Solution of the optimal control problem.

In this section we present optimal control functions that solve problem \( C(p) \). For \( p < \infty \), the solution is unique, and for \( p = \infty \), we present one element on the set of \( L^\infty \)-norm minimal controls, namely, the element of this convex set with minimal \( L^2 \)-norm.

**Theorem 2.2.** Let \( p \in [2, \infty] \) and a time \( T \geq L/c \) be given. Choose a natural number \( k \) such that \( kL/c \leq T < (k+1)L/c \). Define the function

\[
Y_1(x) = \int_0^x y_1(t) \, dt.
\]

Assume that \( y_0, Y_1 \in L^p(0, L) \) and define the functions \( g_1, g_2 \) in \( L^p(0, L/c) \) by

\[
g_1(t) = y_0(ct)/2 - (1/(2c)) Y_1(ct), \quad g_2(t) = y_0(L - ct)/2 + (1/(2c)) Y_1(L - ct).
\]
If \( p < \infty \), let \( \hat{r} \) denote the real number that minimizes the function
\[
h_p(r) = \int_0^{T-kL/c} \frac{1}{(k+1)^{-p}+r} \left[ |g_1(t) + r|^p + |g_2(t) - r|^p \right] dt
\]
while, if \( p = \infty \), let \( \hat{r} \) be the real number that minimizes
\[
h_\infty(r) = \max \left[ \|(g_1(t) + r)/(k+1)\|_{\infty,(0,T-kL/c)}, \|(g_2(t) - r)/(k+1)\|_{\infty,(0,T-kL/c)} \right].
\]
For \( j \in \{0, \ldots, k\} \), define the intervals
\[
I_j^1 = [jL/c, T - kL/c + jL/c],
\]
and for \( j \in \{0, \ldots, k-1\} \), define the intervals
\[
I_j^2 = [T - kL/c + jL/c, (j+1)L/c].
\]
For natural numbers \( j \) and \( n \) let \( g_b(n+j) = g_1 \) if \( n+j \) is odd and \( g_b(n+j) = g_2 \) if \( n+j \) is even. Then a solution of problem \( C(p) \) is given by the pair of control functions \((f_1, f_2)\) defined as follows:
\[
(2.5) \quad f_n(T-t) = \frac{(-1)^j g_b(n+j)(t - jL/c) - (-1)^n \hat{r}}{k+1}
\]
for \( n \in \{1,2\}, j \in \{0,\ldots,k\}, t \in I_j^1 \), and
\[
(2.6) \quad f_n(T-t) = \frac{(-1)^j g_b(n+j)(t - jL/c) - (-1)^n \hat{r}}{k}
\]
for \( n \in \{1,2\}, j \in \{0,\ldots,k-1\}, t \in I_j^2 \).

If \( p < \infty \), this is the unique solution of problem \( C(p) \). If \( p = \infty \), this is a solution of \( C(\infty) \), namely, the element of the set of solutions that has the smallest \( L^2 \)-norm.

For certain interesting target states, the value of \( \hat{r} \) can be computed explicitly. If \( y_0 \) and \( y_1 \) are symmetric, \( \hat{r} = Y_1(L)/(4c) \). In particular, in this case the value of \( \hat{r} \) is independent of \( p \). This implies that the solution is also independent of \( p \). Hence for a symmetric target state with \( y_0, Y_1 \in L^p(0,L) \), our optimal control that solves problem \( C(p) \) also solves problem \( C(q) \) for all \( q \in [2, p] \).

Later we will characterize the feasible controls \((f_1, f_2)\) that steer the system to the desired target state as the solutions of a trigonometric moment problem. To do this, we need a series representation of the solution of the initial boundary-value problem that we obtain from the weak form of the problem.

Then we show that the set of successful controls can be described by a set of two equations for each \( t \in [0, L/c] \). This leads to a family of optimization problems, with parameter \( t \in [0, L/c] \), whose solutions are coupled by a constant \( r \). The solutions of these optimization problems yield the values of the optimal controls up to the constant \( r \), and thus we obtain a parametric family of successful controls with parameter \( r \). Inserting the elements of this family into the objective function yields the values \( h_p(r) \). The optimal control is the element of this family of controls for which the value \( h_p(r) \) is minimal.
2.5. Sensitivity analysis for the optimal control problem. The explicit solutions that we have obtained allow a detailed study of their sensitivity with respect to data perturbations, which is useful for obtaining some idea about what might hold in the general case of optimal control problems with hyperbolic PDEs. Here we study only the continuity of the solutions of $C(p)$ as functions of the parameter $p$.

**Lemma 2.3.** The number $\hat{r}(p)$ that minimizes the function $h_p$ depends continuously on $p$. In fact, if $y_0$ and $Y_1$ are in $L^q(0,L)$ for some $q \in [2,\infty]$, we have

\[ \lim_{p \to q} \hat{r}(p) = \hat{r}(q) \text{ and for } p_1 < q \text{ we have } \lim_{p_2 \to p_1} \hat{r}(p_2) = \hat{r}(p_1). \]

**Proof.** Case 1: If $h_q(\hat{r}(q)) = 0$, we have $\hat{r}(p) = \hat{r}(q)$ for all $p < q$. Case 2: Assume that $q < \infty$ and $h_q(\hat{r}(q)) > 0$. Then for all $p \in [2,q]$, $h_p(\hat{r}(p)) > 0$ and $h''_p(\hat{r}(p)) > 0$. Consider the function $F: [2,q] \times R \to R$, $F(p,r) = h'_p(r)$. Then for all $p \in [2,q]$, $F(p,\hat{r}(p)) = 0$ and $\partial_r F(p,\hat{r}(p)) = h''_p(\hat{r}(p)) > 0$. Hence the implicit function theorem implies that the function $\hat{r}$ is continuously differentiable on $[2,q]$. Case 3: $q = \infty$. Let $f_1(r)$, $f_2(r)$ denote the control functions defined in Theorem 2.2 that correspond to $r \in R$. Then for all $p \in [2,\infty)$ we have $h_p(r) = \|f_1(r)\|_{p,0,T}^p + \|f_2(r)\|_{p,0,T}^p$ and $h_\infty(r) = \max\{\|f_1(r)\|_{\infty,0,T}^\infty + \|f_2(r)\|_{\infty,0,T}^\infty\}$. Thus for all $r$, $\lim_{p \to 0} h_p(r)^{1/p} = h_\infty(r)$. Moreover, the triangle inequality for the $p$-norm implies that for all $p \in [2,\infty)$, $r_1, r_2 \in R$, we have

\begin{equation}
|h_p(r_1)^{1/p} - h_p(r_2)^{1/p}| \leq (\|f_1(r_1) - f_1(r_2)\|_{p,0,T}^p + \|f_2(r_1) - f_2(r_2)\|_{p,0,T}^p)^{1/p}.
\end{equation}

For all $p \in [2,\infty)$ we have $h_p(\hat{r}(p))^{1/p} \leq h_p(\hat{r}(\infty))^{1/p}$ and hence the set $\{\hat{r}(p), p \in [2,\infty]\}$ is also bounded. Suppose that a sequence $(p_k)$ converging to $\infty$ with $p_k \in [2,\infty)$ for all $k$ is given and $\lim_k \hat{r}(p_k) = r_*$. Using (2.7) it can be shown that $h_\infty(r_*) = \lim_{p \to \infty} h_p(r_*)^{1/p} \leq \limsup_{p \to \infty} h_p(\hat{r}(p))^{1/p} = h_\infty(\hat{r}(\infty))$. Thus $h_\infty(r_*) \leq h_\infty(\hat{r}(\infty))$. Since $\hat{r}(\infty)$ is the minimizer of $h_\infty$, this implies that $h_\infty(r_*) = h_\infty(\hat{r}(\infty))$, and since the minimizer of $h_\infty$ is determined uniquely, this implies that $r_* = \hat{r}(\infty)$, and the assertion follows. □

Lemma 2.3 and Theorem 2.2 imply the following proposition.

**Proposition 2.4.** Let $p \in [2,\infty]$ be given. Assume that $y_0$ and $Y_1$ are in $L^p(0,L)$. Consider a sequence $(q_k)_k$ ($q_k \leq p$) that converges to $q_0 \leq p$. Then for the solutions $(f_{1,k}, f_{2,k})^T$ of the optimization problems $C(q_k)$ presented in Theorem 2.2, we have

\[ \lim_{k \to \infty} \|f_{1,k} - f_{1,0}\|_{p,0,T} + \|f_{2,k} - f_{2,0}\|_{p,0,T} = 0, \]

where $(f_{1,0}, f_{2,0})$ is the solution of $C(q_0)$ presented in Theorem 2.2.

2.6. Examples. For our examples, let $L = 1$, $c = 1$, and $T = 3.25$, and hence $k = 3$.

2.6.1. Example 1. Let $y_0(x) = x - L/2$, $y_1(x) = 1$. For $p = \infty$, the optimal $\hat{r}$ is $5/28$ and we have $h_\infty(5/28) = 1/7$. Figure 2.1(a) shows the optimal controls. The thick lines show $f_1$ and the dotted line shows $f_2$. A plot of the corresponding optimal state $y$ in the interior of the rectangle $[0,L] \times [0,T]$ is shown in Figure 2.1(b). Here the optimal state is piecewise linear and the optimal velocity is piecewise constant on areas that are bounded by characteristic curves in the interior of the rectangle $[0,L] \times [0,T]$. 
2.6.2. Example 2. The desired state is $y_0(x) = x - L/2, y_1(x) = 0$. For $p = \infty$, the optimal $\hat{r}$ equals $-1/28$. For $p = 2$, the optimal $\hat{r}$ is $-1/80$. Figure 2.2(a) shows the optimal controls for $p = \infty$. The thick lines show $f_1$, and the dotted line shows $f_2$. Figure 2.2(b) is a plot of the corresponding state $y$ in the interior of the rectangle $[0, L] \times [0, T]$. Due to the form of the target, the optimal state is piecewise linear and the optimal velocity is piecewise constant.

2.6.3. Example 3. In [21], [6] it is pointed out that the optimal boundary controls can also be determined as boundary traces of solutions of adjoint optimal control problems. This example illustrates that for $p \in [2, \infty)$, this approach yields the same controls as Theorem 2.1.

Consider the following optimal control problem:

\[(A) \quad \inf \|v\|_{L^2(0, 2)} \text{ s.t. } y_{tt}(x, t) = y_{xx}(x, t), \quad y(0, t) = 0, \quad y(1, t) = v(t),\]

$y(x, 0) = y^0(x), \quad y_t(x, 0) = 0, \quad y(x, T) = y_t(x, T) = 0, \quad (x, t) \in (0, 1) \times (0, 2)$.

Let $y^0 \in L^2(0, 1)$ be continued to the interval $(-1, 1)$ as the antisymmetric function $y^0_0$ that is, $y^0_0(x) = y^0(x)$ for $x \in (0, 1)$, $y^0_0(x) = -y^0(-x)$ for $x \in (-1, 0)$. Then problem (A) has the same solutions as problem C(2) with $c = 1, L = T = 2, y_0(x) = y_a^0(x + 1), y_1(x) = 0$ in the sense that the optimal controls satisfy $f_2(t) = v(T - t) - f_1(t)$. The adjoint optimization problem presented in [21] is

\[\inf \frac{1}{2} \int_0^1 |y^0(x)|^2 dt + \int_0^1 y^0(x)u^1(x) - y^1(x)u^0(x) dx \text{ s.t. } u_{tt}(x, t) = u_{xx}(x, t),\]

$u(0, t) = u(1, t) = 0, \quad u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad (x, t) \in (0, 1) \times (0, 2)$.

The solution of this adjoint optimization problem is $u^0(x) = \text{const}, \quad u^1(x) = -y^0(x)/2$, which yields the optimal control $v(t) = u_x(1, t) = y^0(1 - t)/2$ for $t \in (0, 1)$,

$v(t) = u_x(1, t) = -y^0(t - 1)/2$ for $t \in (1, 2)$.

Theorem 2.2 yields exactly the same solutions. With $k = 1, g_1(t) = y^0_a(t - 1)/2, \quad g_2(t) = y^0_a(t - 1)/2$, we have $\hat{r} = 0$, and hence for $t \in (0, 1) = I_a^0$ we have $f_2(T - t) = g_2(t)$ and for $t \in (1, 2) = I_a^2$ we have $f_2(T - t) = -g_1(t)$.

2.6.4. Example 4. For $p = \infty$, consider the following optimal control problem:

\[(B) \quad \inf \|v\|_{L^\infty(0, 2)} \text{ s.t. } y_{tt}(x, t) = y_{xx}(x, t), \quad y(0, t) = 0, \quad y(1, t) = v(t),\]

$y(x, 0) = y^0(x), \quad y_t(x, 0) = 0, \quad y(x, T) = y_t(x, T) = 0, \quad (x, t) \in (0, 1) \times (0, 2)$.

Let $y^0 \in L^2(0, 1)$ be continued to the interval $(-1, 1)$ as the antisymmetric function $y^0_0$. Then problem (B) has the same solutions as problem C(\infty) with $c = 1, L = T = 2, y_0(x) = y^0_a(x + 1), y_1(x) = 0$ in the sense that the optimal controls satisfy $f_2(t) = v(T - t) - f_1(t)$.

Following [6], for a given solution $u$ of the adjoint optimization problem

\[\inf \frac{1}{2} \left( \int_0^1 |u_x(1, t)| dt \right)^2 + \int_0^1 y^0(x)u^1(x) - y^1(x)u^0(x) dx \text{ s.t. } u_{tt}(x, t) = u_{xx}(x, t),\]

$u(0, t) = u(1, t) = 0, \quad u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad (x, t) \in (0, 1) \times (0, 2)$ a solution $v$ of problem (B) is quasi bang-bang in the sense that $v(t) \in \text{sign}(u_x(1, t))$.\]
Fig. 2.1.

(a) The optimal controls

(b) The optimal state
Fig. 2.2.
Assume that $y^0$ attains its infinity norm on a set of measure greater than zero. Then the necessary optimality conditions for the adjoint optimization problem imply that there is a constant $c_0 > 0$ such that for $t \in (0, 1)$, $u_x(1, t) = y^0(1 - t)c_0$ if $y^0(1 - t) \in \{\|y^0\|_{L^\infty(0, 1)}, -\|y^0\|_{L^\infty(0, 1)}\}$, $u_x(1, t) = 0$ otherwise, and for $t \in (1, 2)$, $u_x(1, t) = -y^0(t - 1)c_0$ if $y^0(t - 1) \in \{\|y^0\|_{L^\infty(0, 1)}, -\|y^0\|_{L^\infty(0, 1)}\}$, $u_x(1, t) = 0$ otherwise.

Theorem 2.2 yields exactly the same solutions $(f_1, f_2)$ as in Example 3. The solution $v(t) = f_2(2 - t)$ satisfies the quasi-bang-bang characterization given above, which is a restriction only on the set where $y^0$ attains its infinity norm. Note that $v(t)$ satisfies at the same time the quasi-bang-bang characterization and is given as the boundary trace of the optimal solution of the adjoint optimization problem for the $L^2$-case considered in Example 3.

2.7. Weak solutions of the initial-value problem. A general description of the weak form of the initial boundary-value problems with Dirichlet or Neumann boundary conditions can be found in [17]. The solution $y$ of our initial boundary-value problem (2.1)–(2.3) has the series representation

$$y(x, t) = \sum_{j=1}^{\infty} \frac{(2c/L)}{\sqrt{J}} \int_{0}^{t} [f_1(s) - (-1)^j f_2(s)] \sin((c\pi j/L)(t - s)) \, ds \sin((j\pi/L)x),$$

and for the time-derivative $y_t$ we have

$$y_t(x, t) = \sum_{j=1}^{\infty} \frac{2c^2 j \pi}{L^2} \int_{0}^{t} [f_1(s) - (-1)^j f_2(s)] \cos((c\pi j/L)(t - s)) \, ds \sin((j\pi/L)x).$$

2.8. End conditions and a trigonometric moment problem. For $j \in \mathbb{N}$, define the function $\varphi_j(x) = (\sqrt{2}/\sqrt{L}) \sin(j\pi x/L)$ and the numbers

$$y_0^j = \int_{0}^{L} y_0(x) \varphi_j(x) \, dx, \quad y_1^j = \int_{0}^{L} y_1(x) \varphi_j(x) \, dx.$$ 

Inserting the series representations of the solution $y$ and its time derivative $y_t$ into the end conditions (2.4) yields the trigonometric moment equations

$$\int_{0}^{T} (\sqrt{2}/\sqrt{L}) [f_1(s) - (-1)^j f_2(s)] \sin((c\pi j/L)(T - s)) \, ds = y_0^j,$$

$$\int_{0}^{T} (\sqrt{2}c^2 \pi j/L^{3/2}) [f_1(s) - (-1)^j f_2(s)] \cos((c\pi j/L)(T - s)) \, ds = y_1^j$$ 

for $j \in \mathbb{N}$. Hence, we have described the set of feasible controls as the solution set of a trigonometric moment problem. This approach to controllability via moment problems is well established (see, for example, [19], [1]).

2.9. The minimal time interval with controllability. In this section we study controllability on the time interval with $T = L/c$. Since this is the time that a characteristic curve starting at one end of the system needs to reach the other end, it is clear that this is the minimal time interval, where controllability for general target states $y_0 \in L^2(0, L)$, $y_1 \in W^{-1}_{2}$ can possibly hold.

Definition 2.5. A function $f \in L^2(0, L)$ is symmetric with respect to the midpoint $L/2$ if $f(L/2 - x) = f(L/2 + x)$ for all $x \in (0, L/2)$. The function $f$ is
antisymmetric on the interval $[0, L]$ with respect to the midpoint $L/2$ if $f(L/2 - x) = -f(L/2 + x)$ for all $x \in (0, L/2)$.

Remark 2.1. Each function $f \in L^2(0, L)$ can be written as a sum $f = f^{\text{even}} + f^{\text{odd}}$ with a symmetric function $f^{\text{even}} \in L^2(0, L)$ and an antisymmetric function $f^{\text{odd}} \in L^2(0, L)$. The functions $f^{\text{even}}$ and $f^{\text{odd}}$ are determined uniquely. Moreover, $f \in L^p(0, L)$ if and only if $f^{\text{even}}$ and $f^{\text{odd}}$ are in $L^p(0, L)$. In fact, we have

$$f^{\text{even}}(x) = (f(x) + f(L - x))/2, \quad f^{\text{odd}}(x) = (f(x) - f(L - x))/2.$$  

Note that $f^{\text{even}}(x) - f^{\text{odd}}(x) = f(L - x)$.

For given control functions $f_1$ and $f_2$ we introduce the sum

$$S(t) = (f_1(T - t) + f_2(T - t))/2$$

and the difference

$$D(t) = (f_1(T - t) - f_2(T - t))/2.$$  

The trigonometric moment equations (2.8), (2.9) are equivalent to two moment problems for the functions $S$ and $D$. We start with the moment problem for $D$:

$$\int_0^{Tc} D(t/c)(\sqrt{2}/\sqrt{L})\sin(2\pi jt/L) dt = y_0^2j/2,$$

$$\int_0^{Tc} D(t/c)(\sqrt{2}/\sqrt{L})\cos(2\pi jt/L) dt = L y_1^2j/(4\pi j).$$

This means that we know all the Fourier coefficients of the function $D(\cdot/c)$ except the coefficient that corresponds to the constant function. Hence there exists a real number $r$ such that for all $x \in [0, L]$ we have

$$D\left(\frac{x}{c}\right) = r + \sum_{j=1}^{\infty} \frac{y_0^2j}{2} \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi j x}{L}\right) - \frac{1}{2} \sum_{j=1}^{\infty} \frac{y_1^2j}{2\pi j} \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi j x}{L}\right).$$

We define the symmetric function

$$Y_1^{\text{even}}(x) = \sum_{j=1}^{\infty} -y_1^2j(L/(2\pi j))\sqrt{(2/L)} \cos((2\pi j/L)x)$$

and the antisymmetric functions

$$y_1^{\text{odd}} = \sum_{j=1}^{\infty} (y_1^2j)\sqrt{(2/L)} \sin(2\pi j x/L), \quad y_0^{\text{odd}} = \sum_{j=1}^{\infty} (y_0^2j)\sqrt{(2/L)} \sin(2\pi j x/L).$$

We have $(Y_1^{\text{even}})'(x) = y_1^{\text{odd}}$, and for the function $D$ for all $x \in [0, L/c]$, we have

$$D(x) = r + y_0^{\text{odd}}(cx)/2 - (1/(2c)) \ Y_1^{\text{even}}(cx).$$

Now we consider the moment problem for the function $S$:

$$\int_0^{Tc} S(t/c)(\sqrt{2}/\sqrt{L})\sin((2j - 1)\pi t/L) dt = y_0^{2j-1}/2,$$

$$\int_0^{Tc} S(t/c)(\sqrt{2}/\sqrt{L})\cos((2j - 1)\pi t/L) dt = Ly_1^{2j-1}/(2(2j - 1)\pi c).$$
We define the symmetric functions
\[ y_0^{\text{even}}(x) = \sum_{j=1}^{\infty} (y_0^{2j-1}) \sqrt{(2/L)} \sin((2j-1)\pi x/L), \]
\[ y_1^{\text{even}}(x) = \sum_{j=1}^{\infty} (y_1^{2j-1}) \sqrt{(2/L)} \sin((2j-1)\pi x/L) \]
and the antisymmetric function
\[ Y_1^{\text{odd}}(x) = \sum_{j=1}^{\infty} (-y_1^{2j-1}) \sqrt{(2/L)} \cos((2j-1)\pi x/L). \]

Then we have \( (Y_1^{\text{odd}})'(x) = y_1^{\text{even}}(x) \). For the function \( S \) for all \( x \in [0,L/c] \), we have
\[ S(x) = y_0^{\text{even}}(cx)/2 - (1/(2c)) Y_1^{\text{odd}}(cx). \]

Thus, for the control functions \( f_1, f_2 \) that steer the system to the target state in the time \( T = L/c \), we have
\[ f_1(T-t) = S(t) + D(t) = r + y_0(ct)/2 - (1/(2c)) Y_1(ct), \]
\[ f_2(T-t) = S(t) - D(t) = -r + y_0(L-ct)/2 + (1/(2c)) Y_1(L-ct), \]
where \( r \) is a real number.

This representation of the functions \( f_1 \) and \( f_2 \) implies that if \( y_0 \) and \( Y_1 \) are in the space \( L^p(0,L) \), then the functions \( f_1 \) and \( f_2 \) are in the space \( L^p(0,L/c) \). On the other hand, if \( f_1 \) and \( f_2 \) are in the space \( L^p(0,L/c) \), then \( D \) and \( S \) also are in the space \( L^p(0,L/c) \), which implies that \( y_0^{\text{odd}}, y_1^{\text{even}}, y_0^{\text{even}}, Y_1^{\text{odd}} \) are in \( L^p(0,L) \). This, in turn, is equivalent to the statement that \( y_0 \) and \( Y_1 \) are in \( L^p(0,L) \). Thus, we have the following.

**Lemma 2.6.** Let \( p \in [2, \infty] \) and \( T = L/c \). If the control functions \( f_1 \) \( f_2 \) are in \( L^p(0,T) \), then the state \( y_1(T), y_1(T) \) that the system has reached at time \( T \) has the following regularity: \( y_1(T) \) is in \( L^p(0,L) \) and \( h(x) = \int_0^x y_1(z,T) dz \) is also in \( L^p(0,L) \).

For a given target state \( (y_0, y_1) \) with \( y_0, y_1 \in L^p(0,L) \), there exist control functions \( f_1 \) \( f_2 \) in \( L^p(0,T) \) that steer the system to this target; moreover, these controls are uniquely determined up to the constant \( r \) in (2.18), (2.19).

The uniqueness follows from the fact that the moment problem for \( S \) has a unique solution in \( L^2(0,T) \) and the moment problem for \( D \) determines \( D \) up to a constant.

**2.10. Controllability on larger time intervals.** In this section we show how the question of controllability for a time interval \( [0,T] \) with \( T > L/c \) can be reduced to the question for the minimal time interval \( [0, L/c] \) that was considered in the last section. This reduction depends upon the fact that all the trigonometric functions that appear in the moment equations have similar periodicity properties.

**2.10.1. Transformation of the moment equations.** Assume that \( T \geq L/c \). Choose the natural number \( k \) such that \( kL/c < T < (k+1)L/c \). Let the function \( \varphi(s) \) be an element of the set \( \{ \sin((c\pi j)/L)s), \cos((c\pi j)/L)s) \) \} with \( j \in \mathbb{N}, j \text{ odd} \). Then we have \( \varphi(s + L/c) = -\varphi(s) \), and for all functions \( v \in L^2(0,T) \), the following equation
is valid:
\[
\int_0^T v(s) \varphi(s) \, ds = \int_0^{T-kL/c} \left[ \sum_{j=0}^k (-1)^j v(s + jL/c) \right] \varphi(s) \, ds \\
+ \int_{T-kL/c}^{L/c} \left[ \sum_{j=0}^{k-1} (-1)^j v(s + jL/c) \right] \varphi(s) \, ds.
\]

Define the function
\[
\hat{\vartheta}(t) = \sum_{j=0}^k (-1)^j v(t + jL/c) \text{ for } t \in (0, T-kL/c),
\]
\[
\hat{\vartheta}(t) = \sum_{j=0}^{k-1} (-1)^j v(t + jL/c) \text{ for } t \in (T-kL/c, L/c).
\]

Then
\[
\int_0^{L/c} \hat{\vartheta}(s) \varphi(s) \, ds = \int_0^T v(s) \varphi(s) \, ds.
\]

As in the last section, let the functions \(S\) and \(D\) be defined by (2.10) and (2.11). Then for a function \(S\) that satisfies the moment equations (2.15), (2.16) for all \(j \in \mathbb{N}\), the corresponding function \(\hat{\vartheta}\) must satisfy these moment equations with integrals on the interval \((0, L/c)\). In Lemma 2.6, we have stated that the moment equations (2.15), (2.16) with \(T = L/c\) determine a unique solution \(\hat{S}\), which is given by (2.17).

For a function \(D\) that satisfies the moment equations (2.12), (2.13), the corresponding function \(\hat{\vartheta}\) is defined as in (2.20), (2.21) but the numbers \((-1)^j\) are replaced by 1 and must satisfy (2.12), (2.13) on the interval \((0, L/c)\). These moment equations determine \(\hat{D}\), which is given by (2.14) up to a constant.

In what follows, let \(\hat{D}\) be defined by the equation
\[
\hat{D}(x) = y_0^{\text{odd}}(cx)/2 - (1/(2c)) Y_1^{\text{even}}(cx)
\]
and \(\hat{S}\) by
\[
\hat{S}(x) = y_0^{\text{even}}(cx)/2 - (1/(2c)) Y_1^{\text{odd}}(cx).
\]

So we see that we can describe the feasible controls, that is, the controls that steer the system to the target, by the following equations (with \(\Delta = T - kL/c\)):
\[
\hat{S}(t) = \sum_{j=0}^k (-1)^j S(t + jL/c), \quad \hat{D}(t) + \hat{\varrho} = \sum_{j=0}^k D(t + jL/c), t \in (0, \Delta),
\]
\[
\hat{S}(t) = \sum_{j=0}^{k-1} (-1)^j S(t + jL/c), \quad \hat{D}(t) + \hat{\varrho} = \sum_{j=0}^{k-1} D(t + jL/c), t \in (\Delta, L/c),
\]

where \(\hat{\varrho}\) can be any real number.

This means that we have reduced our problem of optimal control to an optimization problem with four affine linear pointwise equality constraints.
2.10.2. **Proof of Theorem 2.1.** Now we proceed to the proof of Theorem 2.1. It is clear that \( f_1 \) and \( f_2 \) are in \( L^p(0, T) \) if and only if \( S \) and \( D \) are in \( L^p(0, T) \).

If \( y_0 \) and \( Y_1 \) are in \( L^p(0, L) \), Lemma 2.6 implies that we can find \( \hat{S} \) and \( \hat{D} \) in \( L^p(0, L/c) \) that satisfy the moment equations (2.15), (2.16) and (2.12), (2.13), respectively. Then we can find functions \( S \) and \( D \) in \( L^p(0, T) \) such that (2.24)–(2.25) hold, for example, with \( \hat{r} = 0 \) and the definitions \( D(t) = \hat{D}(t) \) and \( S(t) = \hat{S}(t) \) for \( t \in (0, L/c) \) and \( 0 = D(t) = S(t) \) for \( t \geq L/c \). In this way, we obtain feasible controls \( f_1 \) and \( f_2 \) in \( L^p(0, T) \).

Now we prove the converse. Let \( f_1 \) and \( f_2 \) in the space \( L^p(0, T) \) be given. Then the corresponding functions \( \hat{D} \) and \( \hat{S} \) defined by (2.24)–(2.25) with \( \hat{r} = 0 \) are in the space \( L^p(0, L/c) \). The corresponding controls on the time interval \( (0, L/c) \) reach the same target as \( f_1 \) and \( f_2 \) on the time interval \( (0, T) \) since they solve the corresponding moment problem on the shorter time interval \( (0, L/c) \). Hence Lemma 2.6 implies that the corresponding target state has the desired regularity, namely, \( y_0 \) and \( Y_1 \) are in \( L^p(0, L) \). So the proof of Theorem 2.1 is complete.

2.11. **Transformation of the optimization problem.** In section 2.10.1 we showed that the set of controls \( f_1, f_2 \) for which the corresponding state \( y \) satisfies the end conditions (2.4) can be described by (2.24)–(2.25), with \( S \) defined by (2.10), \( D \) defined by (2.11), \( \hat{S} \) given by (2.23), and \( \hat{D} \) as in (2.22).

Thus for \( p < \infty \), problem \( C(p) \) can be transformed into the following form:

\[
\inf \| f_1 \|_{p,(0,T)}^p + \| f_2 \|_{p,(0,T)}^p \quad \text{s.t.} \quad f_1, f_2 \in L^p[0, T] \quad \text{and} \quad S \text{ defined by (2.10)}
\]

and \( D \) defined by (2.11) satisfy the constraints (2.24)–(2.25) for some \( \hat{r} \in R \) with \( \hat{S} \) given by (2.23) and \( \hat{D} \) given by (2.22). For \( p = \infty \), \( C(p) \) is equivalent to the corresponding problem with objective function

\[
\max\{\| f_1 \|_{\infty,(0,T)}, \| f_2 \|_{\infty,(0,T)}\}.
\]

2.11.1. **Proof of Theorem 2.2.** In this section, we use the transformed form of problem \( C(p) \) that was given in the last section to prove Theorem 2.2.

First we consider the case \( p < \infty \). Define the function

\[
J(f_1 , f_2) = \| f_1 \|_{p,(0,T)}^p + \| f_2 \|_{p,(0,T)}^p,
\]

which is the objective function of problem \( C(p) \) for \( p < \infty \). We have the representation

\[(2.26) \quad J(f_1 , f_2) = \int_0^{T-kL/c} \sum_{j=0}^k |f_1(T - t - jL/c)|^p + |f_2(T - t - jL/c)|^p \, dt \]

\[+ \int_{T-kL/c}^{L/c} \sum_{j=0}^{k-1} |f_1(T - t - jL/c)|^p + |f_2(T - t - jL/c)|^p \, dt.\]

Since \( f_1(T - t) = S(t) + D(t) \) and \( f_2(T - t) = S(t) - D(t) \), the constraints (2.24) imply (for a natural number \( n \), let \( b(n) = 1 \) if \( j \) is odd and \( b(n) = 2 \) if \( j \) is even)

\[(2.27) \quad \sum_{j=0}^k (-1)^j f_{b(j+1)}(T - t - jL/c) = \hat{S}(t) + \hat{D}(t) + \hat{r},\]

\[(2.28) \quad \sum_{j=0}^k (-1)^j f_{b(j)}(T - t - jL/c) = \hat{S}(t) - \hat{D}(t) - \hat{r} \]
for all $t \in (0, T - kL/c)$, and the constraints (2.25) imply

$$
\sum_{j=0}^{k-1} (-1)^j f_{b(j+1)}(T - t - jL/c) = \hat{S}(t) + \hat{D}(t) + \hat{r},
$$

and

$$
\sum_{j=0}^{k-1} (-1)^j f_{b(j)}(T - t - jL/c) = \hat{S}(t) - \hat{D}(t) - \hat{r}
$$

for all $t \in (T - kL/c, L/c)$.

In our optimization problem, each point of the time interval $[0, T]$ corresponds to two equality constraints. The objective function $J$ is given by an integral over the time interval $[0, T]$, where for each $t \in [0, T]$ the integrand is the sum of two terms, each of which depends only on function values that appear in exactly one of the constraints. The idea of our proof is that we can minimize the objective function $J$ subject to the two pointwise constraints by minimizing for each point in time both parts of the integrand separately subject to the corresponding equality constraint.

The solutions of the resulting parametric family of optimization problems are given in the following lemma.

**Lemma 2.7.** Let $p \geq 2$, a natural number $d$, and a real number $g$ be given. Consider the optimization problem

$$
H(p, d, g) : \min_{(f_0, \ldots, f_d) \in \mathbb{R}^{d+1}} \sum_{j=0}^{d} |f_j|^p \text{ s.t. } \sum_{j=0}^{d} (-1)^j f_j = g.
$$

The unique solution of $H(p, d, g)$ has the components $f_j = (-1)^j g/(d+1)$ and the optimal value is $|g|^p/(d+1)^{p-1}$.

**Proof.** $H(p, d, g)$ is a convex optimization problem with a strictly convex objective function, and hence it has at most one solution. The point with the components $(-1)^j g/(d+1)$ is feasible and satisfies the necessary optimality conditions, and hence it is the unique solution of $H(p, d, g)$. \hfill \Box

Let the number $\hat{r}$ be given. Representation (2.26) of the objective function $J$ shows that in order to minimize $J$ subject to our pointwise constraints, it suffices to choose the values of our control functions $f_1, f_2$ as follows: For $t \in [0, T - kL/c]$, let $f_j = f_{b(j+1)}(T - t - jL/c) (j \in \{0, \ldots, k\})$ be such that they solve problem $H(p, k, \hat{S}(t) + \hat{D}(t) + \hat{r})$, that is,

$$
f_{b(j+1)}(T - t - jL/c) = (-1)^j (\hat{S}(t) + \hat{D}(t) + \hat{r})/(k + 1),
$$

and let $f_j = f_{b(j)}(T - t - jL/c)$ be the solution of problem $H(p, k, \hat{S}(t) - \hat{D}(t) - \hat{r})$, that is,

$$
f_{b(j)}(T - t - jL/c) = (-1)^j (\hat{S}(t) - \hat{D}(t) - \hat{r})/(k + 1).
$$

Similarly, for $t \in [T - kL/c, L/c]$, let $f_j = f_{b(j+1)}(T - t - jL/c) (j \in \{0, \ldots, k-1\})$ be such that they solve problem $H(p, k - 1, \hat{S}(t) + \hat{D}(t) + \hat{r})$, that is,

$$
f_{b(j+1)}(T - t - jL/c) = (-1)^j (\hat{S}(t) + \hat{D}(t) + \hat{r})/k,
$$

and let $f_j = f_{b(j)}(T - t - jL/c)$ be such that they solve problem $H(p, k - 1, \hat{S}(t) - \hat{D}(t) - \hat{r})$, that is,

$$
f_{b(j)}(T - t - jL/c) = (-1)^j (\hat{S}(t) - \hat{D}(t) - \hat{r})/k.
By Lemma 2.7, this yields the following value of the objective function:
\[
J(f_1, f_2) = \int_0^{T-kL/c} \left(1/(k+1)^{p-1}\right) \left|\hat{S}(t) + \hat{D}(t) + \hat{r}\right|^p dt + \int_{T-kL/c}^{L/c} \left|\hat{S}(t) + \hat{D}(t) + \hat{r}\right|^p dt.
\]
Since this value still depends on our choice of the real number \(\hat{r}\), we define this value as \(h_p(\hat{r})\). Now the problem remains to find the value of \(\hat{r}\) for which the corresponding value of the objective function is minimal. Since the function \(h_p\) is strictly convex and differentiable, the equation \(h'_p(\hat{r}) = 0\) uniquely determines the optimal value of \(\hat{r}\).

Remember that due to (2.18) and (2.19), we can compute \(\hat{S} + \hat{D}\) and \(\hat{S} - \hat{D}\) from the given functions \(y_0\) and \(Y_1\), so the optimal value of \(\hat{r}\) can be determined.

Now we come to the case \(p = \infty\). Also in this case, we can transform our problem \(C(\infty)\) into a problem with the four simple pointwise equality constraints (2.27)–(2.30):
\[
\inf \max\{\|f_1\|_{\infty,(0,T)}, \|f_2\|_{\infty,(0,T)}\} \text{ s.t. } f_1, f_2 \in L^\infty(0,T)
\]
and there is a real number \(\hat{r}\) such that \(f_1, f_2\) satisfy (2.27)–(2.30).

In order to solve this problem, we look for solutions at each \(t \in (0, T-kL/c)\) of the problems to minimize
\[
\max\{|f_{b(j)}(T-t-jL/c)|, j \in \{0, \ldots, k\}\} \text{ s.t. (2.27) is satisfied},
\]
\[
\max\{|f_{b(j)}(T-t-jL/c)|, j \in \{0, \ldots, k\}\} \text{ s.t. (2.28) is satisfied},
\]
and for each \(t \in (T-kL/c, L/c)\) for solutions of the analogous problems with (2.29), (2.30), respectively. We present the solutions of the resulting parametric family of optimization problems in the following.

**Lemma 2.8.** Let a natural number \(d\) and a real number \(g\) be given. Consider the optimization problem
\[
H(d, g) := \min_{(f_0, \ldots, f_d) \in \mathbb{R}^{d+1}} \max\{\|f_j\|, j \in \{0, \ldots, d\}\} \text{ s.t. } \sum_{j=0}^{d} (-1)^j f_j = g.
\]
The unique solution of \(H(d, g)\) has the components \(f_j = (-1)^j g/(d+1)\) and the optimal value is \(|g|/(d+1)\).

**Proof.** The point with the components \((-1)^j g/(d+1)\) is feasible. Hence the optimal value of \(H(d, g)\) is \(\leq |g|/(d+1)\). Suppose that there exists a point \((h_0, \ldots, h_d)\) with \(\sum_{j=0}^{d} (-1)^j h_j = g\) and \(\max|h_j| < |g|/(d+1)\). Then \(\sum_{j=0}^{d} (-1)^j h_j < \sum_{j=0}^{d} |g|/(d+1)\) = \(|g|\), a contradiction. So the optimal value of \(H(d, g)\) is \(|g|/(d+1)\). Using a similar contradiction argument we see that for every solution \((h_0, \ldots, h_d)\) of \(H(d, g)\) we have \(|h_j| = |g|/(d+1)\) for all \(j\). Inserting this condition into the equation \(\sum_{j=0}^{d} (-1)^j h_j = g\) yields \(\sum_{j=0}^{d} (-1)^j \text{sign} h_j = (d+1) \text{sign} g\), and hence for all \(j\) we have \(\text{sign} h_j = (-1)^j \text{sign} g\), and the assertion follows.

In analogy to the case \(p < \infty\), Lemma 2.8 yields the desired solutions. In order to obtain an optimal control in this case, we choose \(\hat{r}\) such that it minimizes the function \(h_\infty\) defined as
\[
h_\infty(r) = \max\left\{\|\hat{S}(t) + \hat{D}(t) + r\|/(k+1)\|_{\infty,(0,T-kL/c)}, \|\hat{S}(t) - \hat{D}(t) - r\|/(k+1)\|_{\infty,(0,T-kL/c)}, \|\hat{S}(t) + \hat{D}(t) + r\|/(k+1)\|_{\infty,(T-kL/c,L/c)}, \|\hat{S}(t) - \hat{D}(t) - r\|/k\|_{\infty,(T-kL/c,L/c)}\right\}.
\]
This determines the value of \( \hat{r} \) uniquely. However, in the \( L^\infty \)-case, the optimal control is in general not uniquely determined. For the given value of \( \hat{r} \), our construction above yields the solution of (2.27)–(2.30) with minimal \( L^2 \)-norm, which is in fact the same as our solution for the case \( p = \infty \), so we have constructed the solution of problem \( C(\infty) \) with minimal \( L^2 \)-norm.

**2.11.2. Symmetric targets.** In this subsection, we assume that for all even \( j \in \mathbb{N} \) we have

\[
\int_0^L y_0(x) \varphi_j(x) \, dx = \int_0^L y_1(x) \varphi_j(x) \, dx = 0.
\]

This means that the functions \( y_0 \) and \( y_1 \) are even on the interval \([0,L]\) with respect to the midpoint \( L/2 \). This implies that \( Y_1 \) is antisymmetric. Thus, (2.22) implies that \( \hat{D} = 0 \). We have \( h'(0) = 0 \), and hence, in this case, the number \( \hat{r} = 0 \) is the optimal choice. On the time interval \((0,L/c)\) this yields the control functions \( f_1(T-t) = y_0(ct)/2 - (1/(2c))Y_1(ct) = f_2(T-t) \). Also on larger time intervals \((0,T)\), for the optimal controls we have \( f_1 = f_2 \), since \( g_1 = g_2 \).

**2.11.3. Antisymmetric targets.** In this subsection, we assume that for all odd \( j \in \mathbb{N} \) (2.31) holds. This means that the functions \( y_0 \) and \( y_1 \) are antisymmetric on the interval \([0,L]\) with respect to the midpoint \( L/2 \). Then \( Y_1 \) is symmetric, and in the statement of Theorem 2.2 we have \( g_1(t) = -g_2(t) \). Therefore, for the optimal controls we have \( f_1 = -f_2 \).

**3. Neumann boundary control.** In this section we study the problem in which the system is controlled by Neumann boundary conditions.

**3.1. The initial-value problem.** Let a wave speed \( c > 0 \) be given. We consider the initial-value problem with the wave equation

\[
y_{tt}(x,t) = c^2y_{xx}(x,t), \quad (x,t) \in [0,L] \times [0,T],
\]

subject to the initial conditions

\[
y(x,0) = 0, \quad y_t(x,0) = 0, \quad x \in [0,L],
\]

and the Neumann boundary conditions

\[
y_x(0,t) = -f_1(t), \quad y_x(L,t) = f_2(t), \quad t \in [0,T].
\]

The desired target state is given in the following end conditions:

\[
y(x,T) = y_0(x), \quad y_t(x,T) = y_1(x), \quad x \in [0,L].
\]

The functions \( y_0, y_1 \) are in the space \( L^2(0,L) \).

**3.2. The optimization problem.** For a fixed time \( T > 0 \) and a given value of \( p \in [2,\infty) \), we consider the following optimization problem:

\[
C(p) : \inf \|f_1\|_{p,[0,T]}^p + \|f_2\|_{p,[0,T]}^p \quad \text{s.t.} \quad f_1, f_2 \in L^p[0,T]
\]

and the solution \( y \) of the initial boundary-value problem (3.1)–(3.3) satisfies the end conditions (3.4).

In the case \( p = \infty \), the objective function is \( \max \{ \|f_1\|_{\infty,[0,T]}, \|f_2\|_{\infty,[0,T]} \} \).
3.3. Exact controllability.

Theorem 3.1. Let $p \in [2, \infty]$ and $T > L/c$ be given. The initial boundary-value problem (3.1)–(3.3) has a weak solution satisfying the end conditions (3.4) with $f_1, f_2 \in L^p(0, T)$ if and only if the target states $y_0, y_1$ satisfy the following conditions: $y_1 \in L^p(0, L)$ and $y_0 \in L^p(0, L)$ is such that the derivative $y_0'$ in the sense of distributions is in the space $L^p(0, L)$, that is, $y_0 \in W^1_p(0, L)$. This implies that the optimization problem $C(p)$ has a solution if and only if $y_0'$ and $y_1$ are in $L^p(0, L)$.

3.4. Weak solution of the initial-value problem. The solution $y$ of the initial boundary-value problem (3.1)–(3.3) has the series representation

$$y(x, t) = (c^2/L) \int_0^t [f_1(s) + f_2(s)](t - s) \, ds$$

$$+ \sum_{j=1}^{\infty} \left(\frac{2}{(cj\pi)}\right) \int_0^t [f_1(s) + (-1)^j f_2(s)] \sin \left(\left((cj\pi/L)(t - s)\right)\right) \cos \left((j\pi/L)x\right) \, ds$$

and for the time derivative $y_t$ we obtain the series

$$y_t(x, t) = (c^2/L) \int_0^t [f_1(s) + f_2(s)] \, ds$$

$$+ \sum_{j=1}^{\infty} \left(\frac{2}{(cj\pi)}\right) \int_0^t [f_1(s) + (-1)^j f_2(s)] \cos \left(\left((cj\pi/L)(t - s)\right)\right) \cos \left((j\pi/L)x\right).$$

3.5. End conditions and a trigonometric moment problem. For $j \in \mathbb{N}$, define the functions

$$\varphi_0(x) = 1/\sqrt{L}, \; \varphi_j(x) = (\sqrt[2]{2}/\sqrt{L}) \cos(j\pi x/L),$$

and for $j \in \mathbb{N} \cup \{0\}$, define the numbers

$$y_0^j = \int_0^L y_0(x) \varphi_j(x) \, dx, \; y_1^j = \int_0^L y_1(x) \varphi_j(x) \, dx.$$

Inserting the series representation of the solution $y$ and its time derivative $y_t$ into the end conditions (3.4) yields the moment equations

$$\int_0^T (c^2/\sqrt{L})(f_1(T - s) + f_2(T - s)) \, ds = y_0^0,$$

$$\int_0^T \left(\sqrt[2]{2}/\sqrt{L}/(c\pi j)\right)(f_1(T - s) + (-1)^j f_2(T - s)) \sin \left(\left((c\pi j/L)s\right)\right) \, ds = y_0^j, \quad j \in \mathbb{N}.$$

$$\int_0^T (c^2/\sqrt{L}) \int_0^T f_1(T - s) + f_2(T - s) \, ds = y_1^0,$$

$$\int_0^T \left(\sqrt[2]{2}/\sqrt{L}\right)(f_1(T - s) + (-1)^j f_2(T - s)) \cos \left(\left((c\pi j/L)s\right)\right) \, ds = y_1^j, \quad j \in \mathbb{N}.$$
3.6. The minimal time interval with controllability up to a constant.  
In this section we study controllability on the time interval with $T = L/c$, which is the minimal time interval, where controllability for all target states $y_0, y_1$ in $L^2(0, L)$ can be possible.

For given control functions $f_1$ and $f_2$ we introduce the sum $S$ as in (2.10) and the difference $D$ as in (2.11). The trigonometric moment equations (3.5)–(3.8) are equivalent to two moment problems for the functions $S$ and $D$.

The moment problem for the difference function $D$ is

$$\int_0^{Tc} (1/c)D(x/c)\sqrt{(2/L)}\cos((2j-1)\pi x/L)\,dx = y_1^{2j-1}/2,$$

and the symmetric function

$$Y_0^{\text{even}}(x) = -\sum_{j=1}^{\infty} y_1^{2j-1}((2j-1)\pi/L)\sqrt{(2/L)}\sin(((2j-1)\pi/L)x).$$

Then $Y_0^{\text{even}}(x) = (y_0^{\text{odd}})'(x)$ and for all $t \in [0, L/c]$ we have

$$(3.11) \quad D(t) = (c/2)y_1^{\text{odd}}(ct) - (c^2/2)(y_0^{\text{odd}})'(ct).$$

Consider the moment problem for the function $S$:

$$(3.12) \quad \int_0^{Tc} (1/\sqrt{L})S(x/c)\,dx = y_0^0/2,$$

$$(3.13) \quad \int_0^{Tc} S(x/c)(\sqrt{2/L})\sin((2\pi j/L) x)\,dx = c^2(2\pi j/L)y_0^{2j}/2,$$

$$(3.14) \quad \int_0^{Tc} (1/\sqrt{L})S(x/c)\,dx = y_0^0/(2c),$$

$$(3.15) \quad \int_0^{Tc} S(x/c)(\sqrt{2/L})\cos((2\pi j/L) x)\,dx = cy_1^{2j}/2, \quad j \in \mathbb{N}.$$ 

This system of moment equations is in fact overdetermined. If we omit (3.12), the remaining system determines a unique solution.

We define the symmetric functions

$$y_0^{\text{even}}(x) = \sum_{j=1}^{\infty} (y_0^{2j})\sqrt{(2/L)}\cos(2j\pi x/L) + y_0^0/\sqrt{L},$$

$$y_1^{\text{even}}(x) = \sum_{j=1}^{\infty} (y_1^{2j})\sqrt{(2/L)}\cos(2j\pi x/L) + y_1^0/(c^2\sqrt{L})$$
and the antisymmetric function

\[ Y_0^{\text{odd}}(x) = \sum_{j=1}^{\infty} -y_0^{2j}(2j\pi/L)\sqrt{(2/L)} \sin((2j\pi/L)x). \]

Note that in the definition of \( y_1^{\text{even}} \) in the constant term the wave speed \( c^2 \) appears. We have \((y_0^{\text{even}})'(x) = Y_0^{\text{odd}}(x)\). For the function \( S \), we have for all \( t \in [0, L/c] \)

\[ S(t) = (c/2)y_1^{\text{even}}(ct) - (c^2/2)(y_0^{\text{even}})'(ct) \]

since all the Fourier coefficients of \( S \) are determined by (3.13)–(3.15).

Thus the unique solution of the moment problem (3.6)–(3.8) with \( T = L/c \) is

\[ f_1(T-t) = S(t) + D(t) = (c/2)[y_1(ct) + ((1/c^2) - 1)(y_1^2/\sqrt{L})] - (c^2/2)y_0'(ct), \]

\[ f_2(T-t) = S(t) - D(t) = (c/2)y_1(L-ct) - (c^2/2)y_0'(L-ct). \]

With these control functions at time \( T = L/c \) the system reaches a target state of the form \( y_0 + c_0, y_1 \) since in the representation of the state \( y(\cdot, T) \) as a series of the functions \( \varphi_j (j \in \mathbb{N} \cup \{0\}) \), all the coefficients for \( j \neq 0 \) are determined by (3.6).

To find control functions that satisfy the first moment equation (3.5) with \( c_0 = 0 \), in general we need a longer time interval. (However, for antisymmetric targets it is possible; see section 3.8.2.) Thus we see that controllability to all target states in \((y_0, y_1)\) with \( y_0' \) and \( y_1 \) in \( L^2(0, L) \) is not possible; we have only the following result.

**Lemma 3.2.** Let \( p \in [2, \infty] \) and \( T = L/c \). If the control functions \( f_1 \) and \( f_2 \) are in \( L^p(0, T) \), then the state \( y(\cdot, T), y_1(\cdot, T) \) that the system has reached at time \( T \) has the following regularity: \( \partial_x y(\cdot, T) \) and \( y_1(\cdot, T) \) are in \( L^p(0, L) \).

For a given target state \((y_0, y_1)\) with \( y_0', y_1 \in L^p(0, L) \), there exist control functions \( f_1 \) and \( f_2 \) in \( L^p(0, T) \) that steer the system to a state of the form \((y_0 + c_0, y_1)\) with a real constant \( c_0 \); moreover, these controls are uniquely determined.

The states that can be reached at the time \( T = L/c \) are exactly the states \( y_0, y_1 \) with \( y_0', y_1 \in L^p(0, L) \) for which the controls \( f_1, f_2 \) given in (3.17), (3.18) satisfy (3.5). In this case, \( f_1, f_2 \) given in (3.17), (3.18) are the unique solution of \( C(p) \).

**3.7. Controllability on larger time intervals.** In this section we show how the question of controllability for a time interval \((0, T)\) with \( T > L/c \) can be solved by transformation of the moment equations to moment equations on the interval \((0, L/c)\). This reduction depends on the fact that all the trigonometric functions that appear in the moment equations have the same periodicity properties.

**3.7.1. Transformation of the moment equations.** Assume that \( T > L/c \). Choose the natural number \( k \) such that \( kL/c < T < (k+1)L/c \).

Let the function \( \varphi(s) \) be an element of the set \( \{1, \sin((c\pi2j/L)s), \cos((c\pi2j/L)s) \text{ with } j \in \mathbb{N}\} \). Then we have \( \varphi(s + L/c) = \varphi(s) \), and for all functions \( v \in L^2(0, T) \), the following equation is valid:

\[ \int_0^T v(s)\varphi(s) \, ds = \int_0^{T-kL/c} \left[ \sum_{j=0}^{k} v(s + jL/c) \right] \varphi(s) \, ds + \int_{T-kL/c}^{T-L/c} \left[ \sum_{j=0}^{k-1} v(s + jL/c) \right] \varphi(s) \, ds. \]
Define the function

\[
\hat{v}(t) = \sum_{j=0}^{k} v(t + jL/c) \quad \text{for } t \in (0, T - kL/c),
\]

(3.20)

\[
\hat{v}(t) = \sum_{j=0}^{k-1} v(t + jL/c) \quad \text{for } t \in (T - kL/c, L/c).
\]

(3.21)

Then

\[
\int_{0}^{L/c} \hat{v}(s)\varphi(s) \, ds = \int_{0}^{T} v(s)\varphi(s) \, ds.
\]

Again, let the functions \( S \) and \( D \) be defined by (2.10) and (2.11). Then for a function \( S \) that satisfies the moment equations (3.13)–(3.15) for all \( j \in \mathbb{N} \), the corresponding function \( \hat{v} \) must satisfy these moment equations with integrals on the interval \((0, L/c)\). In section 3.6 we have stated that these moment equations with \( T = L/c \) determine a unique solution \( \hat{S} \), which is given by (3.16).

For a function \( D \) that satisfies (3.9), (3.10) for all \( j \in \mathbb{N} \), the corresponding function \( \hat{v} \) is defined as in (2.20)–(2.21). Since \( \hat{v} \) satisfies (2.20)–(2.21) on the interval \((0, L/c)\), as stated in section 3.6, it is determined uniquely and is given by (3.11). In what follows we call it \( \hat{D} \). Thus we have

\[
\hat{S}(t) = (c/2)y_{1}^{0}(ct) - (c^{2}/2) (y_{0}^{0})'(ct),
\]

\[
\hat{D}(t) = (c/2)y_{1}^{0}(ct) - (c^{2}/2) (y_{0}^{0})'(ct).
\]

Let \( \Delta = T - kL/c \). The set of feasible controls, that is, the controls that steer the system to the target, can be described by the equations

\[
\hat{S}(t) = \sum_{j=0}^{k} S(t + jL/c), \quad \hat{D}(t) = \sum_{j=0}^{k-1} (-1)^{j} D(t + jL/c), \quad t \in (0, \Delta),
\]

(3.22)

\[
\hat{S}(t) = \sum_{j=0}^{k-1} S(t + jL/c), \quad \hat{D}(t) = \sum_{j=0}^{k-1} (-1)^{j} D(t + jL/c), \quad t \in (\Delta, L/c),
\]

(3.23)

and the moment equation (3.12).

This means that we have reduced our problem of optimal control to an optimization problem with a finite number of simple pointwise equality constraints and one integral constraint.

In terms of \( f_{1} \) and \( f_{2} \), the constraints (3.22)–(3.23) can be written as

\[
\hat{S}(t) + \hat{D}(t) = \sum_{j=0}^{k} f_{b(j+1)}(t + jL/c), \quad t \in (0, \Delta),
\]

(3.24)

\[
\hat{S}(t) - \hat{D}(t) = \sum_{j=0}^{k} f_{b(j)}(t + jL/c), \quad t \in (0, \Delta),
\]

(3.25)

\[
\hat{S}(t) + \hat{D}(t) = \sum_{j=0}^{k-1} f_{b(j+1)}(t + jL/c), \quad t \in (\Delta, L/c),
\]

(3.26)

\[
\hat{S}(t) - \hat{D}(t) = \sum_{j=0}^{k-1} f_{b(j)}(t + jL/c), \quad t \in (\Delta, L/c),
\]

(3.27)
and \( f_1 + f_2 \) must satisfy (3.5). To transform (3.5), we use the equation

\[
\int_0^T tS(t) \, dt
\]

\[
= \int_0^{T - kL/c} \sum_{j=0}^k (t + jL/c)S(t + jL/c) \, dt + \int_{T - kL/c}^{L/c} \sum_{j=0}^{k-1} (t + jL/c)S(t + jL/c) \, dt
\]

\[
= \int_0^{L/c} t\tilde{S}(t) \, dt + \int_0^{L/c} \mathcal{S}(s) \, ds,
\]

with

\begin{align*}
(3.28) \quad \mathcal{S}(t) &= \sum_{j=0}^k j(L/c)S(t + jL/c) \text{ for } t \in [0, T - kL/c], \\
&= \sum_{j=0}^{k-1} j(L/c)v(t + jL/c) \text{ for } t \in [T - kL/c, L/c].
\end{align*}

Since the function \( \tilde{S} \) is known, we can replace the moment equation (3.5) by

\[
(3.29) \quad \int_0^{L/c} \mathcal{S}(t) \, dt = \sqrt{L} y_0^0/(2c^2) - \int_0^{L/c} t\tilde{S}(t) \, dt =: R_0.
\]

The description of the feasible controls by the equality constraints (3.24)–(3.29) allows us to prove Theorem 3.1.

### 3.7.2. Proof of Theorem 3.1

Now we come to the proof of Theorem 3.1. We use the fact that \( f_1 \) and \( f_2 \) are in \( L^p(0,T) \) if and only if \( S \) and \( D \) are in \( L^p(0,T) \).

Let \( f_1 \) and \( f_2 \) in the space \( L^p(0,T) \) be given such that at time \( T \), the system has reached the state \( y(T) = y_0, y(t,T) = y_1 \). The corresponding functions \( D \) and \( \tilde{S} \) defined by (3.22)–(3.23) are in the space \( L^p(0,L/c) \). The corresponding controls on the time interval \((0,L/c)\) reach at time \( L/c \) a state of the form \( y_0 + c_0, y_1 \) since they solve the moment problem (3.13)–(3.15). Lemma 3.2 implies that \( y_1 \) and \( y_0' \) are in \( L^p(0,L) \).

Now we show the converse. If \( y_0' \) and \( y_1 \) are in \( L^p(0,L) \), Lemma 3.2 implies that we can find \( \tilde{S} \in L^p(0,L/c) \) that satisfies the moment equations (3.13)–(3.15) and \( \tilde{D} \in L^p(0,L/c) \) that satisfies (3.9), (3.10). Then we can find functions \( f_1 \) and \( f_2 \) in \( L^p(0,T) \) such that (3.24)–(3.29) hold, for example, if \( T - kL/c > 0 \) with the definition

\[
\begin{align*}
f_1(t + L/c) &= f_2(t + L/c) = R_0 c/(L(T - kL/c)), \\
f_1(t) &= \tilde{S}(t) + \tilde{D}(t) - f_2(t + L/c), \\
f_2(t) &= \tilde{S}(t) - \tilde{D}(t) - f_1(t + L/c)
\end{align*}
\]

for \( t \in (0,T - kL/c) \) and \( f_1(t) = \tilde{S}(t) + \tilde{D}(t), f_2(t) = \tilde{S}(t) - \tilde{D}(t) \) for \( t \in (T - kL/c, L/c) \) and \( f_1(t) = f_2(t) = 0 \) otherwise. Then the constraints (3.24)–(3.29) hold, and thus we have found a successful control in the space \( L^p(0,L) \). So we have proved Theorem 3.1.

### 3.8. Solution of the optimization problem \( C(p) \)

In this section we consider the case \( L/c < T < 2L/c \), that is, \( k = 1 \). For \( p = \infty \), this case has also been considered in [10], but here we provide a solution for \( p < \infty \). We work with the transformed
form of problem $C(p)$, where the feasible set $F(p)$ is described by pointwise equality constraints and one integral constraint:

$$F(p) = \{ f_1, f_2 \in L^p(0, T) : (3.24)−(3.27) \text{ hold}, \overline{S} \text{ defined by (3.28) satisfies (3.29)} \}.$$ 

In our case $k = 1$, (3.26), (3.27) reduce to the equations

$$(3.30) \quad f_1(t) = \hat{S}(t) + \hat{D}(t), \quad f_2(t) = \hat{S}(t) - \hat{D}(t), \quad t \in (T - L/c, L/c).$$

This means that the values of all feasible controls, and thus also of the optimal control, are prescribed on the interval $(T - L/c, L/c)$. This fact has important consequences for the structure of the optimal controls: We see that the functions $f_1$ and $f_2$ can have any form, so in general there is no reason why they should have a bang-bang, bang-off, or a similar structure. The optimization only takes place on the two intervals $(0, T - L/c)$ and $(L/c, T)$. The constraints (3.24), (3.25) can be written as

$$(3.31) \quad f_2(t + L/c) = \hat{S}(t) + \hat{D}(t) - f_1(t), \quad f_1(t + L/c) = \hat{S}(t) - \hat{D}(t) - f_2(t)$$

for all $t \in (0, T - L/c)$. On the middle interval $(T - L/c, L/c)$ we have $\overline{S}(t) = 0$ and on $(0, T - L/c)$ we have $\overline{S}(t) = (L/c)\overline{S}(t + L/c)$, so (3.29) becomes

$$\int_0^{T-L/c} (L/c)(f_1(t + L/c) + f_2(t + L/c))/2 \, dt = R_0.$$

We insert (3.31) and obtain the constraint

$$\int_0^{T-L/c} (L/c) \left[ \hat{S}(t) - (f_1(t) + f_2(t))/2 \right] \, dt = R_0,$$

and hence for $p < \infty$, problem $C(p)$ reduces to the problem of minimizing

$$\|f_1\|^p_{p,(0,\Delta)} + \|f_2\|^p_{p,(0,\Delta)} + \|\hat{S} + \hat{D} - f_1\|^p_{p,(0,\Delta)} + \|\hat{S} - \hat{D} - f_2\|^p_{p,(0,\Delta)}$$

s.t. $f_1, f_2 \in L^p(0, \Delta), \int_0^\Delta (f_1(t) + f_2(t))/2 \, dt$

$$= \int_0^\Delta \hat{S}(t) \, dt - (c/L)R_0 \text{ (with } \Delta = T - L/c).$$

We set $G = \hat{S} + \hat{D}, \quad H = \hat{S} - \hat{D}, \text{ and } C_0 = \int_0^\Delta \hat{S}(t) \, dt - (c/L)R_0$. Then we can write the above optimization problem in the form

$$\min_{f_1, f_2} \|f_1\|^p_{p,(0,\Delta)} + \|f_2\|^p_{p,(0,\Delta)} + \|G - f_1\|^p_{p,(0,\Delta)} + \|H - f_2\|^p_{p,(0,\Delta)}$$

s.t. $f_1, f_2 \in L^p(0, \Delta), \int_0^\Delta (f_1(t) + f_2(t))/2 \, dt = C_0$. The corresponding necessary optimality condition states that there exists a Lagrange multiplier $\lambda \in R$ such that for all $t \in (0, \Delta)$ the following equations hold:

$$|f_1(t)|^{p-1}\text{sign}(f_1(t)) + |f_1(t) - G(t)|^{p-1}\text{sign}(f_1(t) - G(t)) = \lambda,$$

$$|f_2(t)|^{p-1}\text{sign}(f_2(t)) + |f_2(t) - H(t)|^{p-1}\text{sign}(f_2(t) - H(t)) = \lambda.$$

For the solution of the optimality system, we use the following lemma.
Lemma 3.3. Let \( p \in [2, \infty) \). For a real number \( a \), define the function
\[
h_a(x) = |x|^{p-1}\text{sign}(x) + |x-a|^{p-1}\text{sign}(x-a).
\]
Then \( h_a \) is strictly increasing and \( \lim_{x \to \infty} h_a(x) = \infty \), \( \lim_{x \to -\infty} h_a(x) = -\infty \). So the inverse function \( \psi_a = h_a^{-1} \) exists and is strictly increasing with \( \lim_{\lambda \to \infty} \psi_a(\lambda) = \infty \), \( \lim_{\lambda \to -\infty} \psi_a(\lambda) = -\infty \), and for all \( \lambda \in \mathbb{R} \) the equation \( h_a(x) = \lambda \) has the unique solution \( x = \psi_a(\lambda) \). For fixed \( \lambda \), the function \( a \mapsto \psi_a(\lambda) \) is continuous. If \( p = 2 \), we have \( \psi_a(\lambda) = (\lambda + a)/2 \).

Proof. Consider the function \( g(x) = |x|^{p-1}\text{sign}(x) \). Then \( g \) is strictly increasing and \( \lim_{x \to \infty} g(x) = \infty \), \( \lim_{x \to -\infty} g(x) = -\infty \). Since \( h_a(x) = g(x) + g(x-a) \), the assertions for \( h_a \) follow, except the continuity of the map \( a \mapsto \psi_a(\lambda) \).

Define \( F : \mathbb{R}^2 \to \mathbb{R} \), \( F(a,x) = h_a(x) \). Then \( F \) is continuously differentiable and \( F_x(a,x) = g'(x) + g'(x-a) > 0 \) for \((a,x) \neq (0,0)\). We have \( F(a,\psi_a(\lambda)) = \lambda \), and hence the implicit function theorem implies the continuity of the map \( a \mapsto \psi_a(\lambda) \) for \((a,\lambda) \neq (0,0)\). Since \( h_a(a) = g(a) \) and \( h_a(0) = -g(a) \) we have \( \psi_a(0) \in (-|a|,|a|) \), so the continuity for \( a = \lambda = 0 \) also follows.

Hence for \( p < \infty \), the solution of problem \( C(p) \) can be characterized in the following form.

Theorem 3.4. Assume that \( L/c < T < 2L/c, p \in [2, \infty) \), and that \( y_1 \) and \( y'_0 \) are in \( L^p(0,L) \). Let \( G = \dot{S} + \dot{D} \), \( H = \dot{S} - \dot{D} \), and \( C_0 = \int_0^{T-L/c} \dot{S}(t) \, dt - (c/L)R_0 \). Let \( \lambda \in \mathbb{R} \) be the uniquely determined solution of the equation
\[
\int_0^{T-L/c} \psi_G(t)(\lambda) + \psi_H(t)(\lambda) \, dt = 2C_0.
\]

Then the unique solution of problem \( C(p) \) is given by
\[
f_1(t) = \psi_G(t)(\lambda), \quad f_2(t) = \psi_H(t)(\lambda)
\]
for \( t \in (0,T-L/c) \) and, on the interval \((L/c,T)\), the control functions \( f_1, f_2 \) are defined by (3.31) and on \((T-L/c,L/c)\) by (3.30).

For \( p = \infty \), \( C(p) \) can be reduced to the following problem: Minimize
\[
\max\{\|f_1\|_{L^\infty(0,T-L/c)}, \|f_2\|_{L^\infty(0,T-L/c)}, \|G - f_1\|_{L^\infty(0,T-L/c)}, \|H - f_2\|_{L^\infty(0,T-L/c)}\}
\]
s.t. \( f_1, f_2 \in L^\infty(0,T-L/c), \int_0^{T-L/c} (f_1(t) + f_2(t))/2 \, dt = C_0 \).

Again let \( \Delta = T - L/c \). It is easy to see that the functions \( f_1 = f_2 = C_0/\Delta \) on \((0,\Delta)\) satisfy the integral constraint. In fact, the number \( C_0/\Delta \) is a lower bound for the optimal value of \( C(\infty) \). It can happen that the \( L^\infty \)-norm of the control functions is attained in the middle interval \((\Delta,L/c)\), where their values are prescribed by (3.30). These observations yield the following lemma.

Lemma 3.5. Assume that \( L/c < T < 2L/c \) and \( y_1 \) and \( y'_0 \) are in \( L^\infty(0,L) \). Set \( C_1 = C_0/\Delta \). Assume that max\{\(\|G - C_1\|_{L^\infty(0,\Delta)}, \|H - C_1\|_{L^\infty(0,\Delta)}\) \leq C_1 \) or that
\[
\max\{\|G\|_{L^\infty(\Delta,L/c)}, \|H\|_{L^\infty(\Delta,L/c)}\} \geq \max\{\|G - C_1\|_{L^\infty(0,\Delta)}, \|H - C_1\|_{L^\infty(0,\Delta)}\}.
\]
Then a solution of \( C(\infty) \) is \( f_1 = f_2 = C_1 \) on \((0,\Delta)\). On the interval \((L/c,T)\), the control functions \( f_1, f_2 \) are defined by (3.31) and on \((\Delta,L/c)\) by (3.30).

3.8.1. Symmetric targets. In this section we assume that \( y_0 \) and \( y_1 \) are symmetric with respect to \( L/2 \). Then we have \( \bar{D} = 0 \), which implies that \( G = H = \dot{S} \). Theorem 3.4 yields the equation \( f_1 = f_2 \), which is also true for \( p = \infty \) (see [10]).
If \( y_0 \) and \( y_1 \) are constant functions, we have \( \dot{S}(t) = (c/2)y_0^\text{even}(ct) = K \), which is also a constant function. Equation (3.32) in Theorem 3.4 yields \( \psi_K(\lambda) = C_0/(T - L/c) \); hence for \( t \in (0, T - L/c) \) we have \( f_1(t) = C_0/(T - L/c) \), \( f_2(t + L/c) = K - C_0/(T - L/c) \) and \( f_1(t) = K \) for \( t \in (T - L/c, L/c) \). Note that this solution is independent of \( p < \infty \). Since \( f_1 \) and \( f_2 \) are in \( L^\infty(0, L) \), this implies that it is also the solution of \( C(\infty) \) with minimal \( L^2 \)-norm. If \( K = 0 \) (that is, \( y_1 = 0 \)), this yields the bang-off-bang control presented in [2] for the case \( p = \infty \).

### 3.8.2. Antisymmetric targets.

In this section we assume that \( y_0 \) and \( y_1 \) are antisymmetric with respect to \( L/2 \). Then we have \( \dot{S} = 0 \), which implies that \( \dot{D} = G = -H \) and \( y_0^0 = 0 \), and hence \( R_0 = C_0 = 0 \). Since \( \psi_{-a}(0) = -\psi_a(0) \), Theorem 3.4 yields with \( \lambda = 0 \) the equation \( f_1 = -f_2 \), and for \( p < \infty \) the optimal control satisfies \( f_1(t) = \psi_{D(t)}(0) \) on the interval \( (0, T - L/c) \). Since \( h_a(a/2) = 0 \), this yields \( f_1 = \dot{D}/2 \) on \( (0, T - L/c) \cup (L/c, T) \) and \( f_2 = \dot{D} \) on \( (T - L/c, L/c) \). Note that this solution is again independent of \( p < \infty \). If \( y_1 \) and \( y_0^0 \) are in \( L^\infty(0, L) \), this is also the solution with minimal \( L^2 \)-norm of \( C(\infty) \); this follows from the next lemma.

**Lemma 3.6.** If \( f_1, f_2 \) in \( L^\infty(0, T) \) solve \( C(p) \) for all \( p \in [2, \infty) \), then \( f_1, f_2 \) also solve \( C(\infty) \) and are the solution of \( C(\infty) \) with minimal \( L^2 \)-norm.

For antisymmetric targets, we have \( y_0^0 = 0 \). Thus in this case, for the control functions with \( f_1 = -f_2 \), (3.5) is valid. This implies that for this class of target states, controllability is also possible on the time interval \( [0, L/c) \).

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### REFERENCES


