A Connectivity-preserving flocking algorithm for multi-agent dynamical systems with bounded potential function

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Abstract: Without assuming that the communication topology can remain its connectivity frequently enough and the potential function can provide an infinite force during the evolution of agents, the flocking problem of multi-agent systems with second-order non-linear dynamics is investigated in this study. By combining the ideas of collective potential functions and velocity consensus, a connectivity-preserving flocking algorithm with bounded potential function is proposed. Using tools from the algebraic graph theory and matrix analysis, it is proved that the designed algorithm can guarantee the group of multiple agents to asymptotically move with the same velocity while preserving the network connectivity if the coupling strength of the velocity consensus term is larger than a threshold value. Furthermore, the flocking algorithm is extended to solve the flocking problem of multi-agent systems with a dynamical virtual leader by adding a navigation feedback term. In this case, each informed agent only has partial velocity information about the leader, yet the present algorithm not only can guarantee the velocity of the whole group to track that of the leader asymptotically, and also can preserve the network connectivity. Finally, some numerical simulations are provided to illustrate the theoretical results.

1 Introduction

Recently, much attention from various disciplines has been attracted to the study of distributed control of multi-agent systems because of its wide applications in the design of sensor networks, control of robot teams, unmanned air vehicles (UAV) formations and so on, with many profound results established [1–8]. One research focus in distributed control of multi-agent systems is to understand how the flocking behaviour emerges as a result of local interactions among multiple mobile individuals [9–15], which is closely related to formation control [3, 4] and consensus problems [7, 8]. Studying along this line not only can help understand some natural behaviours, but also benefit engineers in implementing distributed flocking algorithms in numerous artificial multi-agent systems. Differing from the centralised control problems, the flocking problem in multi-agent dynamical systems is characterised by distributed control using local interactions. Inspired by the pioneering work in [1, 2], various distributed algorithms have been proposed by combining a local artificial potential field and a velocity consensus component to ensure the achievement of flocking in multi-agent systems [3, 4, 9–12]. Under the assumption that the underlying topology can remain its connectivity frequently enough during the motion evolution, the above-mentioned algorithms can successfully guarantee the emergence of flocking in multi-agent dynamical systems. However, this common assumption is not easy or even impossible to be satisfied and verified in practice as the communication topology may change with time subjected to the limited sensing capabilities of agents [12–15]. To avoid this impractical assumption, some efforts have been made to preserve the connectivity of the topology during the evolution of agents. In [15], Ji and Egerstedt introduced a hysteresis in adding new edges and a special potential function to maintain the network connectivity for rendezvous in first-order multi-agent systems. Then, Zavlanos et al. [16] used this hysteresis and an unbounded potential function approach to investigate connectivity-preserving flocking problem of second-order multi-agent systems. Flocking achievement for second-order multi-agent systems with non-linear inner-coupling functions while guaranteeing connectivity was investigated in [17]. In [18, 19], Su et al. proposed a novel connectivity-preserving flocking algorithm for networks of multiple agents with double-integrators dynamics based only on position measurements. It should be noted, however, in [18, 19], some special potential functions were constructed for single-integrator
dynamics and double-integrators dynamics, respectively, where both of them tend to infinity as the distance between two neighbouring agents trends to the sensing radius, which is impractical in real systems as no actuator could provide an infinity control force. It has been observed that most existing works on flocking problem in multi-agent systems focus on agents with integrator-type dynamics. Thus, when flocking is achieved in this case, the final velocity is time invariant since each agent has no intrinsic dynamics. However, multiple agents may be governed by more complicated intrinsic dynamics in real systems [20–26]. Motivated by this fact and based on the above-mentioned works, the connectivity-preserving flocking problem in multi-agent systems with second-order non-linear dynamics is investigated in this paper. To avoid fragmentation, the hysteresis in adding new edges together with a potential function method are employed. Particularly, a new kind of bounded potential function is constructed in this paper. In this case, the equilibrium of velocities is time-varying and evolves according to the non-linear intrinsic dynamics of agents. By using tools from algebraic theory and matrix analysis, it is shown that the present algorithm can guarantee the group of multiple agents to asymptotically move with the same velocity in the condition of preserving the connectivity of the whole network if the algebraic connectivity of the initial network is larger than a threshold value. In addition, without assuming that each informed agent can sense all the states of the leader, flocking in non-linear multi-agent systems with a dynamical virtual leader is investigated. Finally, some numerical simulations are performed to illustrate the theoretical results.

The remainder of the paper is organised as follows. In Section 2, some preliminaries and the model description are given. The main theoretical results are given in Section 3. In Section 4, numerical simulations are performed to validate the effectiveness of the theoretical results. Concluding remarks are finally drawn in Section 5.

2 Preliminaries and model description

2.1 Preliminaries

Let \( \mathcal{G} = (V, E, A) \) be an undirected graph of order \( N \), with a set of vertices \( V = \{v_1, v_2, \ldots, v_N\} \), a set of undirected edges \( E \subseteq V \times V \), and a adjacency matrix \( A = [a_{ij}]_{N \times N} \). An edge \( e_{ij} \) in graph \( \mathcal{G} \) is denoted by \( (v_i, v_j) \). If there exists an edge between vertex \( i \) and vertex \( j \), then \( a_{ij} = a_{ji} > 0 \); otherwise \( a_{ij} = 0 \). A path from vertex \( v_i \) to \( v_j \) is a sequence of edges, \( (v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_{m-1}}, v_{i_m}) \), with distinct vertices \( v_{i_k}, m = 1, 2, \ldots, l \). An undirected graph is called connected if there is a path between each pair of distinct vertices. As usual, only simple graph is considered in this paper. The Laplacian matrix \( L = [l_{ij}]_{N \times N} \) of \( \mathcal{G} \) is defined as \( l_{ij} = \sum_{k \neq j} a_{ik} \) for \( i = j \), \( i, j \neq j, i, j = 1, 2, \ldots, N \). Clearly, \( \sum_{j \neq i} l_{ij} = 0 \) for all \( i = 1, 2, \ldots, N \).

Furthermore, the Laplacian matrix \( L \) of graph \( \mathcal{G} \) has the following properties:

Lemma 1 [27]: The eigenvalues of \( L \) satisfy \( 0 = \lambda_1(L) \leq \cdots \leq \lambda_N(L) \), and the algebraic connectivity \( \lambda_2(L) > 0 \), if only if \( \mathcal{G} \) is connected.

2.2 Model description

Consider a multi-agent system of \( N \) agents, labelled as \( 1, 2, \ldots, N \), moving in an \( n \)-dimensional Euclidean space. The motion of agent \( i \) is governed by

\[
\begin{align*}
\dot{q}_i &= p_i \\
\dot{p}_i &= f(p_i) + u_i
\end{align*}
\]

where \( q_i \in \mathbb{R}^n \) is the position vector of agent \( i \), \( p_i \in \mathbb{R}^n \) is its velocity vector, \( f(p_i) \in \mathbb{R}^n \) is its intrinsic non-linear dynamics, and \( u_i \in \mathbb{R}^n \) is the control input acting on agent \( i, i = 1, 2, \ldots, N \). Suppose that all agents have the same sensing radius \( r, r > 0 \). Let \( \mathcal{G}(t) \) be an undirected dynamic graph with a set of vertices \( V = \{1, 2, \ldots, N\} \) indexed by the set of agents and a time-varying set of links \( E(t) = \{(i,j)|i,j \in V\} \) such that

1. Initial links are generated by \( E(0) = \{\epsilon_1 < \|q_i(0) - q_j(0)\| < r - \epsilon_0, i,j \in V\} \), where \( \epsilon_1 \) and \( \epsilon_0 \) are positive constants satisfying \( \epsilon_1 < r - \epsilon_0 \).
2. If \( (i,j) \not\in E(t^-) \) and \( \|q_i(t) - q_j(t)\| < r - \epsilon_2 \), then \( (i,j) \) is a new link being added to \( E(t) \), where the positive constant \( \epsilon_2 < r \).
3. If \( \|q_i(t) - q_j(t)\| \geq r, \) then \( (i,j) \not\in E(t) \).

Note that the hysteresis in the foregoing process is crucial in preserving the connectivity of a dynamic proximity network [15, 18, 19]. The adjacent matrix \( A(t) = [A_{ij}(t)]_{N \times N} \) of graph \( \mathcal{G}(t) \) is defined as \( A_{ij} = a_{ij} \) for \( (i,j) \in E(t) \) and \( a_{ij} = 0 \), otherwise. Furthermore, denote the Laplacian matrix of graph \( \mathcal{G}(t) \) by \( L(t) \).

The control objective is to make the multi-agent system (1) converge to a stable flocking motion while preserving the connectivity of the dynamic network and avoiding collision among agents. To achieve this goal, the control law \( u_i \) for agent \( i \) is designed as

\[
u_i = \alpha_i + \beta_i \tag{2}\]

where \( \alpha_i \in \mathbb{R}^n \) is the gradient-based term, which is used to avoid collision with nearby agents and also to maintain the connectivity of the network, and \( \beta_i \in \mathbb{R}^n \) is the velocity consensus term, which regulates the velocity of each agent to a common vector. In some cases, the flocking of multi-agent systems is designed for some certain purposes, for example, tracking a desired common time-varying velocity [28–31]. In the situation where there is a dynamic virtual leader, the control law (2) is modified by

\[
u_i = \alpha_i + \beta_i + \gamma_i \tag{3}\]

where \( \gamma_i \in \mathbb{R}^n \) is the navigational feedback term driving agent \( i \) to track the virtual leader. Before moving on, the following lemma is introduced, which will be used in the derivation of the main results.

Lemma 2 [32]: Suppose \( \mathcal{G} \) be an undirected graph of order \( N \), and \( \mathcal{G}_1 \) be the undirected graph by adding some
Assumption 1: There exists a positive constant $\rho$ such that
\[
\|f(x) - f(y)\| \leq \rho\|x - y\|, \quad \forall x, y \in \mathbb{R}^n
\]  
(4)

Assumption 1 is a Lipschitz condition, which is satisfied by, for example, all piecewise linear functions, such as the chaotic Chua circuit [33].

### 3 Main results

In this section, the flocking problems in multi-agent system (1) without and with a virtual leader are investigated.

**Assumption 1**: There exists a positive constant $\rho$ such that
\[
\|f(x) - f(y)\| \leq \rho\|x - y\|, \quad \forall x, y \in \mathbb{R}^n
\]  
(4)

Assumption 1 is a Lipschitz condition, which is satisfied by, for example, all piecewise linear functions, such as the chaotic Chua circuit [33].

#### 3.1 Flocking in multi-agent systems without a virtual leader

In this subsection, the problem of flocking control in multi-agent system (1) without a virtual leader is investigated.

Denote the position and velocity of the centre of mass (COM) of all agents in the system (1) by $\bar{q} = \frac{\sum_{i=1}^{N} q_i}{N}$ and $\bar{\dot{q}} = \frac{\sum_{i=1}^{N} \dot{q}_i}{N}$, respectively. Then, the position and the velocity differences between agent $i$ and the COM are given as $\hat{q}_i = q_i - \bar{q}$ and $\hat{\dot{q}}_i = \dot{q}_i - \bar{\dot{q}}$, respectively. The control law (2) is designed by

\[
\begin{align*}
    u_i &= -\sum_{j \in N_i(t)} \nabla_{q_j} \psi(\|q_j\|) - c \sum_{j \in N_i(t)} a_{ij}(p_i - p_j) \\
    &\quad - \sum_{j \in N_i(t)} \hat{\dot{q}}_j \nabla_{p_j} \psi(\|q_j\|) 
\end{align*}
\]  
(5)

where $q_i = q_i(t), \quad \forall i, j \in V$, coupling strength $c > 0$, and $N_i(t)$ is the neighborhood of agent $i$ at time $t$. The non-negative potential $\psi(\|q_j\|)$ is a bound function of the distance $\|q_j\|$ between agents $i$ and $j$, which is differential with respect to $\|q_j\| \in (0, r)$, such that

1. $\frac{\partial \psi(\|q_j\|)}{\partial q_j} < 0$, for $\|q_j\| \in (0, r_0)$, and $\frac{\partial \psi(\|q_j\|)}{\partial q_j} > 0$ for $\|q_j\| \in (r_0, r)$, where $r_1 < r_0 < r - \epsilon_2$;

2. $\psi(0) = k_1 + Q_{\text{max}}$, $\psi(r) = k_2 + Q_{\text{max}}$, where $k_1, k_2 \geq 0$ and

\[
Q_{\text{max}} = \frac{1}{2} \sum_{i=1}^{N} \hat{\dot{q}}_i^2(0) + \frac{N(N - 1)}{2} \psi_{\text{max}}
\]  
(6)

\[
\psi_{\text{max}} = \max(\psi(||r - \epsilon_2||), \psi(||\epsilon_1||)), \quad \epsilon_1 = \min_{i,j \in V} \|q_i(0) - q_j(0)\|
\]

Condition (1) indicates that the potential function $\psi(\|q_j\|)$ is decreasing when $\|q_j\| \in [\epsilon_1, r_0)$, while increasing when $\|q_j\| \in [r_0, r - \epsilon_2)$. Clearly, the potential function $\psi(\|q_j\|)$ reaches its minimum at $\|q_j\| = r_0$. Condition (2) states that the potential between two agents will be sufficiently large when the distance between the two agents reaches zero or sensing radius $r$, which, respectively, guarantees the avoidance of collision and the preservation of all existing edges. One example of such potential functions is the following (see Fig. 1)

\[
\psi(\|q_j\|) = \frac{\|q_j\| - r_0^2}{\|q_j\| + \frac{r_0^2}{k_1 + Q_{\text{max}}}} + \frac{\|q_j\| (\|q_j\| - r_0)^2}{r - \|q_j\| + \frac{2(\|q_j\| - r_0)^2}{k_2 + Q_{\text{max}}}}
\]  
(7)

The sum of the total artificial potential energy and the total relative kinetic energy between agents and the COM is defined as

\[
Q = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in N_i(t)} \psi(\|\hat{q}_j\|) + \hat{\dot{q}}_j^2
\]  
(8)

with initial energy $Q(0)$. Obviously, $Q$ is a positive semi-definite function. The main results of this subsection are given in the following theorem:

**Theorem 1**: Consider a network of $N$ agents with dynamics (1) steered by protocol (5). Suppose that the initial network $G(0)$ is connected and the coupling strength $c > 2p_1/\lambda_2(L(0))$. Furthermore, the initial energy $Q(0)$ is finite. Then, the followings hold:

1. $G(t)$ is connected for all $t \geq 0$;

2. The velocity of each agent approaches the same value asymptotically;

3. Almost every final configuration locally minimises each agent’s global potential $\sum_{j \in N_i(t)} \nabla_{q_j} \psi(\|q_j\|)$;

4. Collisions between agents are avoided.

**Proof**: We first prove part 1 of Theorem 1. Denote the position and the velocity differences between agent $i$ and the COM as $\hat{q}_i = q_i - \bar{q}$ and $\hat{\dot{q}}_i = \dot{q}_i - \bar{\dot{q}}$, respectively. Simple calculations give that

\[
\hat{\dot{q}}_i = \hat{\dot{p}}_i
\]

\[
\hat{\dot{p}}_i = f(p_i) - \sum_{j \in N_i(t)} \nabla_{p_j} \psi(\|q_j\|) - c \sum_{j \in N_i(t)} a_{ij}(p_i - p_j)
\]  
(9)

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which can be rewritten as
\[
\dot{q}_i = \ddot{p}_i \\
\dot{p}_i = f(p_i) - \frac{1}{N} \sum_{j=1}^{N} f(p_j) - \sum_{j \in N(i)} \nabla q_j \psi(\|\ddot{q}_j\|)
\]
\[-c \sum_{j \in \bar{N}(i)} a_{ij}(\dot{p}_i - \dot{p}_j)
\tag{10}
\]
Assume that \(G(t)\) switches at time \(t_k\), \(k = 1, 2, \ldots\), and keeps fixed over each time interval \([t_k, t_{k+1})\). Without loss of generality, assuming that \(t_0 = 0\). Then, one has that \(Q(t_0)\) is finite. Taking the time derivative of \(Q(t)\) on \([t_0, t_1)\) gives
\[
\dot{Q} = \sum_{i=1}^{N} \ddot{p}_i f(p_i) - \frac{1}{N} \sum_{i=1}^{N} f(p_i) - \sum_{i=1}^{N} \sum_{j \in N(i)} a_{ij}(\dot{p}_i - \dot{p}_j)
\]
\[+ \sum_{i=1}^{N} \ddot{p}_i f(p_i) - \frac{1}{N} \sum_{i=1}^{N} f(p_i)
\tag{11}
\]
According to Assumption 1 and (11), one has
\[
\dot{Q} \leq \sum_{i=1}^{N} \rho \ddot{p}_i \dot{p}_i + \frac{1}{N} \sum_{i=1}^{N} ||\ddot{p}_i||^2 N \sum_{i=1}^{N} \rho ||\ddot{p}_i||
\]
\[-c \sum_{i=1}^{N} \ddot{p}_i \sum_{j \in \bar{N}(i)} a_{ij}(\dot{p}_i - \dot{p}_j)
\]
\[\leq \sum_{i=1}^{N} \rho \ddot{p}_i \dot{p}_i + \lambda_{\max}(1, \frac{1}{N}) \rho ||\ddot{q}||^2
\]
\[-c \sum_{i=1}^{N} \ddot{p}_i \sum_{j \in \bar{N}(i)} a_{ij}(\dot{p}_i - \dot{p}_j)
\]
\[= 2 \sum_{i=1}^{N} \rho \ddot{p}_i \dot{p}_i - c \ddot{p}_i^T (L(t_0) \otimes L_0) \ddot{p}
\tag{12}
\]
From the Courant–Fischer minimum–maximum theorem [34] and (12), one obtains
\[
\dot{Q} \leq -[c \lambda_2(L(t_0)) - 2 \rho] \ddot{p}^T \ddot{p}
\tag{13}
\]
where \(\ddot{p} = (\ddot{p}_1^T, \ldots, \ddot{p}_N^T)^T\). According to the condition \(c > 2 \rho / \lambda_2(L(t_0))\), it then follows from (13) that
\[
\dot{Q}(t) \leq 0, \quad \forall \ t \in [t_0, t_1)
\tag{14}
\]
which implies that \(Q(t) \leq Q(t_0), \forall \ t \in [t_0, t_1)\). From the definition of the potential function, one has \(\psi(r) \geq \max \geq Q(t_0)\). Therefore no edge-distance will tend to \(r\) for \(t \in [t_0, t_1)\). Hence, new edges must be added into the network at the switching time \(t_1\). Without loss of generality, assume that there are \(m_1\) new links being added to the interaction network at time \(t_1\). Clearly, \(0 < m_1 < M\). From (8) and (15), one has
\[
Q(t) \leq Q(t_0) + (m_1 + \cdots + m_k) \psi(||r - \varepsilon||) \leq Q_{\max}
\tag{16}
\]
Since there are at most \(M\) new edges that can be added to \(G(t)\), one has \(k \leq M\) and \(Q(t) \leq Q_{\max}\) for all \(t \geq t_0\). Therefore the number of switching times \(k\) of the system (1) with control (5) is finite, which implies the interaction network \(G(t)\) eventually becomes fixed. Thus, the rest analysis can be restricted on the time interval \([t_0, +\infty)\). Note that the distance of the edge is not larger than \(\max(\psi^{-1}(Q_{\max}))\) and not less than \(\min(\psi^{-1}(Q_{\max}))\). Hence, the set
\[
\Omega = \{\ddot{q} \in D, \ddot{p} \in R^{M} | Q(\ddot{q}, \ddot{p}) \leq Q_{\max}\}
\tag{17}
\]
is positively invariant, where
\[
D = \{\ddot{q} \in R^{M} | ||\ddot{q}|| \leq \min(\psi^{-1}(Q_{\max}))\}, \quad \max(\psi^{-1}(Q_{\max})), \forall i, j \in E(t)
\tag{18}
\]
\[
\ddot{q} = (\ddot{q}_1, \ldots, \ddot{q}_M, \ddot{q}_1, \ldots, \ddot{q}_M, \ddot{q}_1, \ldots, \ddot{q}_M)^T \in R^{M}, \quad \ddot{p} = (\ddot{p}_1, \ldots, \ddot{p}_M, \ddot{p}_1, \ldots, \ddot{p}_M) \in R^{2M}.
\]
Since \(G(t)\) is connected for all \(t \geq t_0\), one has \(||\ddot{q}|| < (N - 1)r, \forall i, j \in E(t)\). Since \(Q(t) \leq Q_{\max}\), one has \(\ddot{p}_1 \leq 2Q_{\max}\), and thus \(||\ddot{p}|| \leq \sqrt{2Q_{\max}}\). Therefore the set \(\Omega\) defined by (17) satisfying \(Q(t) \leq Q_{\max}\) is compact. Thus, the LaSalle’s invariance principle can be applied to infer that if the initial conditions of the system lies in \(\Omega\), all trajectories will converge to the largest invariant set inside the region \(S = \{\ddot{q} \in D, \ddot{p} \in R^{M} | Q(\ddot{q}, \ddot{p}) = 0\}\). From (13), one has that \(Q(0) = 0\), if only if \(\ddot{p}_1 = \cdots = \ddot{p}_N\), which implies that all the agents asymptotically move with the same velocity. We now proceed to prove part (3) of Theorem 1. In the steady state, one has \(\ddot{p}_1 = 0, \forall i \in V\), thus
\[
- \sum_{j \in \bar{N}(i)} \nabla q_j \psi(\|q_j\|) = 0
\tag{19}
\]
which implies that the multi-agent system converges asymptotically to a fixed configuration corresponding to an extremum of agent’s global potential. Note that not all solutions of (1) converge to local minima. However, every point but local minima is an unstable equilibrium [11]. Thus, almost every final configuration locally minimizes each agent’s global potential \(\sum_{i \in \bar{N}(i)} \nabla q \psi(\|q_i\|)\). Finally, we prove part (4) of Theorem 1. In view of (17), \(Q(t) \leq Q_{\max}\) for all \(t \geq t_0\). However, from the definition of potential function, we have \(\lim_{t \to 0} Q(t) = Q_0 > Q_{\max}\). Therefore collisions among agents.
are avoided. This completes the proof of part (4), hence Theorem 1.

Remark 1: The qualitative analysis on the convergence rate of multi-agent system (1) with control law (5) is provided as follows. According to the assumption \( \lambda_2(L(t_0)) > 2\rho/c \) and (15), one obtains that \( \dot{Q}(t) = 0 \) if and only if \( \hat{p}_i = \tilde{p}_i = \cdots = \tilde{p}_N \) and
\[
\dot{Q}(t) \leq -c\lambda_2(L(t)) - 2\rho\|\tilde{p}\|^2
\] (20)

Let \( \varphi(t) = c\lambda_2(L(t)) - 2\rho \). Then, \( \varphi(t) \) characterises the convergence rate of the system at time \( t \). Furthermore, from Lemma 2, one has that the non-zero eigenvalues of \( L(t) \) grow monotonically with the number of added edges. Thus, \( \lambda_2(0) \leq \lambda_2(t) \), for all \( t \geq 0 \), showing that \( \varphi \) characterises the convergence rate of multi-agent system (1) with control law (5). On the other hand, from the fact that the algebraic connectivity \( \lambda_2(L(t)) \) attains its maximum value \( N \) when the network is fully connected [35]. One has that \( \varphi = cN - 2\rho \) characterises the upper bound of the convergence rate of the closed-loop system (1) with control law (5).

Remark 2: From the proof of Theorem 1, one has that the velocity of each agent \( i, i = 1, 2, \ldots, N \), approaches that of the COM asymptotically. In case of \( c = 0 \), from the control law (5), one has
\[
\hat{u} = \hat{p} = -\frac{1}{N} \left( \sum_{j \in N(i)} \nabla q_i \psi(\|q_j\|) + c \sum_{j \in N(i)} a_j(p_i - p_j) \right) = 0
\] (21)

which indicates that the flocking with preserved connectivity can be achieved with a constant velocity if the initial network is connected. Owing to the non-linear dynamics \( f \), the velocity of COM has its own non-linear dynamics \( \hat{p} = \frac{1}{N} \sum_{j \in N(i)} f(p_j) \) with initial condition \( \hat{p}(0) = \frac{1}{N} \sum_{j \in N(i)} f(p_j(0)) \). Thus, the flocking in multi-agent system (1) can be achieved with time-varying velocities.

### 3.2 Leader-flocking in multi-agent systems with partial leader’s information

In this subsection, the problem of flocking control in multi-agent system (1) with a virtual leader is investigated.

Suppose the leader moves according to the following dynamics
\[
\begin{align*}
\dot{q}_i &= p_i, \\
\dot{p}_i &= f(p_i)
\end{align*}
\] (22)

where \( q_i \in \mathbb{R}^n \) and \( p_i \in \mathbb{R}^n \) are its position and velocity vectors, respectively, and \( f(p_i) \in \mathbb{R}^n \) is its non-linear dynamics. In many real cases, few individuals may have some pertinent information, such as knowledge of the location of a food source or a migration route [28, 36, 37].

Motivated by this observation, it is assumed that only a small fraction \( 0 < \delta < 1 \) of the agents can sense the leader and obtain its information. Without loss of generality, assume that the first \( \delta N \) agents can sense the leader and obtain its information. Moreover, in the real situations, it is impractical to assume that an informed agent can observe all the velocity state components of the leader. Thus, assume that the first \( l = \delta N \) agents can only measure partial velocity state components of the leader by
\[
\bar{p}_i = C_i p_i, \quad i = 1, \ldots, l
\] (23)

where \( \bar{p}_i \in \mathbb{R}^m \) is the velocity measurement of \( p_i \) that sensed by agent \( i \) through transmission channel \( C_i \in \mathbb{R}^{m \times n} \), for \( i = 1, \ldots, l \). Clearly, \( C_i = I_{m \times n}, i = l + 1, \ldots, N \), indicates that the first \( l \) agents can sense all the velocity state components of the leader, which is one of the common assumptions made in many previous works, such as [12, 18]. However, this assumption is removed in this paper. The navigational feedback is designed as
\[
\begin{align*}
g_i &= -B_i(C_i p_i - \bar{p}_i), \quad i = 1, \ldots, l \quad (24) \\
g_i &= 0, \quad i = l + 1, \ldots, N
\end{align*}
\]

where \( B_i \in \mathbb{R}^{m \times n} \), \( i = 1, \ldots, l \), are the feedback control gain matrices. In this case, the control law (3) is specified as
\[
\begin{align*}
u_i &= -\sum_{j \in N(i)} \nabla q_i \psi(\|q_j\|) - c \sum_{j \in N(i)} a_j(p_i - p_j) \\
&\quad - \frac{1}{N} \sum_{j \in N(i)} g_j
\end{align*}
\] (25)

where the coupling strength \( c > 0 \), \( \bar{B}_i = B_i \) for \( i = 1, \ldots, l \), and \( \bar{B}_i = 0 \) for \( i = l + 1, \ldots, N \).

**Definition 1:** The velocity state component \( i \) \((1 \leq i \leq n)\) of the leader is said to be observable by an agent \( j \) \((1 \leq j \leq l)\), if there exist a gain matrix \( B_j \) and a positive constant \( \vartheta_j \), such that
\[
-\vartheta_j \left( \frac{B_j C_j + C_j B_j^T}{2} \right) s \
\leq -\vartheta_j \delta_i^2
\] (26)

for all \( s = (s_1, \ldots, s_n)^T \in \mathbb{R}^n \).

**Definition 2:** The velocity state component \( i \) \((1 \leq i \leq n)\) of the leader is said to be observable by a multi-agent system consisting of \( N \) agents if it is observable by an agent \( j_k \) \((1 \leq j_k \leq l)\).

**Definition 3:** The velocity state of the leader is said to be observable by a multi-agent system consisting of \( N \) agents if each of its position state component is observable by the multi-agent system.

**Assumption 2:** Suppose that the velocity of the leader is observable by a multi-agent system of \( N \) agents and, without loss of generality, suppose that the velocity state component \( i, i \in \{1, \ldots, n\} \), is observable by agents \( j_1, \ldots, j_l \), where \( z \leq l \). The velocity state of the leader is observable by agents \( j_1, j_2, \ldots, j_k \), where \( \{j_1, j_2, \ldots, j_k\} = \bigcup_{i=1}^{z} \{j_1, j_2, \ldots, j_{z}\} \).

Let \( \hat{\bar{p}} = (\hat{\bar{p}}_1^T, \ldots, \hat{\bar{p}}_{z}^T)^T \), where \( \hat{\bar{p}}_k = \bar{p}_k - p_k, k = 1, \ldots, N \).

According to Assumption 2, one has that there exist matrices \( B_{k,h}, h \in \{1, \ldots, z\} \), such that
\[
\begin{align*}
-\sum_{h=1}^{z} \vartheta_h \hat{\bar{p}}_h^T \left( \frac{B_{k,h} C_{k,h} + C_{k,h} B_{k,h}^T}{2} \right) \hat{\bar{p}}_h &\leq -\sum_{h=1}^{z} \vartheta_h \vartheta_i \delta_i^2 \hat{\bar{p}}_h^T \quad (27)
\end{align*}
\]

where the velocity state component \( i, i \in \{1, \ldots, n\} \), of the leader is observable by agents \( j_h, h \in \{1, \ldots, z\} \). Here, \( \hat{\bar{p}}_h^T \)
represents the $i$th component of $\hat{p}_i$. Then, one obtains that

$$-\sum_{i=1}^{n} \sum_{j=1}^{N} \left[ B_j C_j + C_j^T B_j^T \right] \hat{p}_i \leq -\sum_{i=1}^{n} \sum_{j=1}^{N} \theta_{ij} \hat{p}_{ij}^2 \tag{28}$$

According to the fact \( \{j_1, j_2, \ldots, j_n\} = \bigcup_{i=1}^{n} \{j_1, j_2, \ldots, j_n\} \) and (28), one has

$$-\sum_{i=1}^{N} \sum_{j=1}^{N} \left[ B_j C_j + C_j^T B_j^T \right] \hat{p}_i \leq -\sum_{i=1}^{n} \sum_{j=1}^{N} \theta_{ij} \hat{p}_{ij}^2 \tag{30}$$

Define the sum of the total artificial potential function and the total relative kinetic energy function as follows

$$\hat{Q} = \frac{1}{2} \sum_{i=1}^{N} \left( \sum_{j \in N_i} \psi(||q_{ij}||) + \hat{p}_i^T \hat{p}_i \right) \tag{31}$$

where $\hat{q}_i = \hat{q}_i - \hat{q}_j$, $\hat{q}_j = q_i - q_j$, and $\hat{q}_i = q_i - q_j$, $i, j = 1, 2, \ldots, N$. The main results of this subsection are given in the following theorem.

**Theorem 2:** Consider a network of $N$ agents with dynamics (1), each steered by protocol (25), which satisfy Assumption 2. Suppose that the initial network $G(0)$ is connected and the initial energy $\hat{Q}_0$ is finite. If

$$\rho I_N - cL(0) - \Theta_i < 0 \tag{32}$$

where

$$\Theta_i = \text{diag}(0, 0, \ldots, 0, \theta_{i1}, 0, \ldots, 0, \theta_{i2}, 0, \ldots, 0, \ldots, 0) \in R^{N \times N}, i = 1, 2, \ldots, N.$$ \(\theta_{ij}, \theta_{ji} \in R \) hold:

1. $G(t)$ is connected for all $t \geq 0$.
2. The velocity of each agent approaches the desired velocity $v$, asymptotically.
3. Almost every final configuration locally minimises each agent’s global potential $\sum_{j \in N_i} \nabla q_i \psi(||q_i||)$.
4. Collisions between agents are avoided.

**Proof:** We first prove part 1) of Theorem 2. Denote the position and the velocity differences between agent $i$ and the leader as $\hat{q}_i = q_i - q_\ell$ and $\hat{p}_i = p_i - p_\ell$, respectively. Then, one has

$$\hat{q}_i = \hat{q}_i$$
$$\hat{p}_i = f(p_i) - f(p_\ell) - \sum_{j \in N_i} \nabla q_j \psi(||q_j||)$$
$$- c \sum_{j \in N_i} a_{ij} (\hat{p}_i - \hat{p}_j) - \hat{B}_j C_j \hat{p}_i \tag{33}$$

Assume that $G(t)$ switches at time $t_k$, $k = 1, 2, \ldots$, and keeps fixed over each time interval $[t_k, t_{k+1})$. Without loss of generality, assume that $t_0 = 0$. Taking the time derivative of $\hat{Q}(t)$ on $[t_0, t_1)$ gives

$$\hat{\dot{Q}} \leq \hat{p}_i^T [(\rho I_N - cL(t_0)) \otimes I_n] \hat{p}_i - \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{ij} \hat{p}_{ij}^2 \tag{34}$$

According to (30) and the assumption that the velocity state component $i$ (1 $\leq i \leq n$) of the leader is observable by agents $j_1, j_2, \ldots, j_n$, it then follows from (34) that

$$\hat{\dot{Q}} \leq \hat{p}_i^T [(\rho I_N - cL(t_0)) \otimes I_n] \hat{p}_i - \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{ij} \hat{p}_{ij}^2 \tag{35}$$

Let $\hat{p}_i = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N)^T \in R^N$, one has

$$\hat{\dot{Q}} \leq \rho I_N - cL(t_0) \otimes I_n \hat{p}_i - \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{ij} \hat{p}_{ij}^2$$

and $\hat{\dot{Q}} \leq \hat{p}_i^T [(\rho I_N - cL(t_0)) \otimes I_n] \hat{p}_i - \Theta_i \hat{p}_i \tag{36}$

where (see (37))

$\hat{Q} \leq 0$, for $t \in [t_0, t_1)$. Then, Theorem 2 can be proved by following the proof of Theorem 1.

**Remark 3:** The qualitative analysis on the convergence rate of multi-agent system (1) with control law (25) is provided as follows. According to (32) and (36), one may obtain that

$$\hat{\dot{Q}} \leq -\sum_{i=1}^{n} \lambda_{\min} [\rho I_N + cL(t_0) + \Theta_i] \hat{p}_i^T \hat{p}_i, \quad t \in [t_0, t_1) \tag{38}$$

where $\Theta_i$ is defined by (37). According to Theorem 2, the network remains connected all the time and some new edge(s) are added into the network at switching times $t_k$.

$$\Theta_i = \text{diag}
\begin{bmatrix}
0, 0, \ldots, 0, \theta_{i1}, 0, \ldots, 0, \theta_{i2}, 0, \ldots, 0, \\
\theta_{i1}, 0, \ldots, 0, \theta_{i2}, 0, \ldots, 0, \\
\theta_{i1}, 0, \ldots, 0, \theta_{i2}, 0, \ldots, 0,
\end{bmatrix}
\in R^{N \times N} \tag{37}$$
\[ k = 1, 2, \ldots \]. Thus, according to Lemma 2, one has
\[
\hat{Q} \leq -\sum_{i=1}^{n} \lambda_{\min}[-\rho I_N + cL(t) + \Theta_i]\tilde{\nabla}_i, \quad t \geq t_0
\] (39)

Based on the above analysis, some simple calculations give that
\[
\hat{Q} \leq -\xi_{\min}\tilde{p}^T\tilde{p}, \quad t \geq t_0
\] (40)

where positive constants \( \xi_{\max} = \min_i\{\lambda_{\min}[-\rho I_N + cL(t) + \Theta_i]\} \), \( i = 1, 2, \ldots, n \). This indicates that \( \xi_{\min} \) characterises the convergence rate of multi-agent system (1) with control law (25). Similar to the analysis in Remark 1, \( \xi \) characterises the upper bound of the convergence rate of multi-agent system (1) with control law (25), where \( \xi = \min_i\{\lambda_{\min}[-\rho I_N + cL + \Theta_i]\} \), \( \tilde{L} \) is the Laplacian matrix of the fully connected graph with order \( N, i = 1, 2, \ldots, n \).

Remark 4: Indeed, even if the initial network is not connected, one may also investigate the connectivity-preserving flocking problem in the connected components of the network by using the present approach. Similarly to the proof of Theorem 2, some corresponding theoretical results can be established, which are omitted here for brevity.

Remark 5: Without assuming that each informed agent can sense all the velocity states of the virtual leader, leader-following flocking with a preserved connectivity in closed-loop system (1) with protocol (25) may still be guaranteed under some suitable conditions. The reason why we can do this lies in the fact that each individual agent can communicate with the neighbours by sharing its velocity states as the system evolves. Thus, one may conclude that Assumption 2 is a prerequisite condition for achieving the leader-following flocking of system (1).

4 Simulation studies

In this section, some numerical simulations are performed to demonstrate the validity and effectiveness of the theoretical analysis.

4.1 Flocking without a virtual leader

Consider a multi-agent system
\[
\dot{q}_i = \dot{p}_i
\]
\[
\dot{p}_i = f(p_i) - \sum_{j \in N(i)} \nabla_q \psi(||q_i||) - c \sum_{j \in N(i)} a_i(p_i - p_j),
\]
\( i = 1, 2, \ldots, 8 \)

where \( q_i, p_i \in \mathbb{R}^2, f(p_i) = (-0.25 \sin(p_{i2}), 0.25 \cos(p_{i1}))^T, \)
\( i = 1, 2, \ldots, 8 \). The potential function \( \psi \) is defined in (7) with \( r_0 = 2 \) and \( r = 5 \). The initial configuration of the multi-agent system is given in Fig. 2 with \( \varepsilon_1 = 1.4247 \) where the solid lines represent the neighbouring relations, and the solid lines with arrows represent the velocity vector. Let \( \varepsilon_0 = \varepsilon_2 = 0.5 \), some simple calculations give that

\[ \psi_{\max} = \psi(||r - \varepsilon_2||) \]. According to (6), one has
\[
Q_{\max} \leq \frac{N(N-1)}{2} \psi(||r - \varepsilon_2||) + \frac{N}{2} \max_{0 \leq i \leq \rho^T(0)p_i(0)}^{\text{max}}
\]

Thus, one obtains that \( Q_{\max} \leq 37.30 \). Choose \( \hat{Q}_0 = Q_0 = 600 \), the potential function (7) can be explicitly described as
\[
\psi(||q_0||) = \frac{(|q_0| - 2)^2(5 - |q_0|)}{5 - |q_0| + \frac{9q_0^2}{425}} + \frac{9q_0^2}{1000}
\]

In the simulation, the weights \( a_{ij} = 1 \) for all \( (i, j) \in E(t) \) and \( c = 6 \). From Theorem 1, one knows that all agents move with the same velocity, as shown in Fig. 3. The relative velocity states between agent \( i, i = 2, 3, \ldots, 8 \), and agent 1 are shown in Fig. 4.
4.2 Flocking with a virtual leader

Consider a multi-agent system

\[
\begin{align*}
\dot{q}_i &= p_i \\
\dot{p}_i &= f(p_i) - \sum_{j \in N(i)} \nabla q_j(\|q_j\|) - c \sum_{j \in N(i)} a_{ij}(p_i - p_j) \\
&\quad - B_i C_i(p_i - p_l), \quad i = 1, \ldots, 6 \\
\dot{p}_i &= f(p_i) - \sum_{j \in N(i)} \nabla q_j(\|q_j\|) - c \sum_{j \in N(i)} a_{ij}(p_i - p_j), \quad i = 7, 8
\end{align*}
\]

(42)

where \( C_i = (1 \ 0) \), for \( k = 1, 2, 3 \); \( C_i = (0 \ 1) \), for \( l = 4, 5, 6 \). Let \( B_i = (1 \ 0)^T \), for \( m = 1, 2, 3 \); \( B_i = (0 \ 1)^T \), for \( z = 4, 5, 6 \). Clearly, the first three agents can observe the first velocity state component of the leader, while the agents \( i \), for \( i = 4, 5, 6 \), can observe the second velocity state component of the leader. Suppose that the virtual leader has the same non-linear dynamics with multi-agent system (42). Furthermore, the potential function \( \psi \) is defined in (7) with \( r_0 = 2 \) and \( r = 5 \). The initial configuration of the multi-agent system is given in Fig. 5 with \( \epsilon_1 = 1.5787 \), where the solid lines represent the neighbouring relations, and the solid lines with arrows represent the velocity vector. The initial position and velocity states of the virtual leader are chosen by \( q_1(0) = (0, 0)^T \) and \( p_1(0) = (-0.15, 0.15)^T \), as shown in the red colour in Fig. 5. Let \( \epsilon_0 = \epsilon_2 = 0.5 \), some simple calculations give that \( \psi_{\text{max}} = \psi(\|r - \epsilon_2\|) \). According to (6), one has

\[
Q_{\text{max}} \leq \frac{N(N - 1)}{2} \psi(\|r - \epsilon_2\|) + \frac{N}{2} \max_{i} (\hat{p}_i(0)\hat{p}_i(0))
\]

Thus, one obtains that \( Q_{\text{max}} \leq 38 \). Choose \( \widehat{Q}_0 = \widehat{Q}_e = 800 \), the potential function (7) can be explicitly described as

\[
\psi(\|q_j\|) = \frac{\|q_j\| - 2}{\|q_j\| + \frac{(5 - \|q_j\|)}{200}} + \|q_j\| \frac{(\|q_j\| - 2)^2}{5 - \|q_j\| + \frac{971}{800}}
\]

Furthermore, the weights \( a_{ij} = 1 \) for all \((i, j) \in E(t)\) and the coupling strength \( c = 6 \). From Theorem 2, one knows

**Fig. 4** Relative velocity states between agent \( i, i = 2, 3, \ldots, 8 \) and agent 1

**Fig. 5** Initial states

**Fig. 6** Final states

**Fig. 7** Relative velocity states between agent \( i, i = 1, 2, \ldots, 8 \) and the virtual leader
that the followers approaches the desired velocity $p_i$, asymptotically, as shown in Fig. 6. The relative velocity states between agent $i, i = 1, 2, \ldots, 8$, and the virtual leader are shown in Fig. 7. It is easy to see that the numerical simulation verifies the theoretical analysis very well.

5 Concluding remarks

In this paper, a novel connectivity-preserving flocking algorithm with bounded potential function has been proposed to guarantee the multi-agent system with second-order non-linear dynamics to achieve flocking with preserved connectivity. By using tools from algebraic graph theory and matrix analysis, it has been shown that the present algorithm can enable the group of multiple agents to move with the same velocity while preserving the connectivity of the whole network if the initial network is connected and the coupling strength of the velocity consensus term is larger than a threshold value. The results are then extended to flocking with a dynamical virtual leader where each informed agent only has partial velocity information about the leader. Finally, some numerical simulations are provided to illustrate the theoretical results. Some future works are as follows: (i) multi-agent systems with non-linear couplings and heterogeneous autonomous agents will be studied. (ii) Based on the pinning-based flocking algorithms present in this paper, flocking in networks of agents with unknown but bounded transmission disturbances will be studied in the future.

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7 References

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