STATE FEEDBACK $H_\infty$ CONTROL FOR NETWORKED CONTROL SYSTEMS WITH QUANTIZATION AND RANDOM COMMUNICATION DELAYS

DANLI WEN$^{1,2}$ AND GUANGHONG YANG$^{2,3}$

$^1$Shenyang Normal University
No. 253, Huanghe North Street, Huanggu Region, Shenyang 110034, P. R. China
wdl0119@163.com

$^2$College of Information Science and Engineering

$^3$Key Laboratory of Integrated Automation of Process Industry
Northeastern University
No.11, Lane 3, WenHua Road, HePing District, Shenyang, 110819, P. R. China
yangguanghong@ise.neu.edu.cn

Received September 2009; revised February 2010

ABSTRACT. In this paper, the state feedback $H_\infty$ control problem for networked control systems with quantization and random communication delays is discussed. The random delays from the sensor to the controller and from the controller to the actuator are considered. The quantizer considered here is dynamic and composed of an adjustable zoom parameter and a static quantizer. A novel quantized random delay model is proposed, and by using this model, the relationship of the quantizations, delays and the system performance is studied. The state feedback $H_\infty$ controller can be constructed via solving a linear matrix inequality. With a condition on the quantization range and the error bound satisfied, a quantized $H_\infty$ control strategy is derived such that the closed-loop system with quantization and random delays exponentially mean-square stable and with a prescribed $H_\infty$ performance bound. An example is presented to illustrate the effectiveness of the proposed method.

Keywords: Network control systems, Quantization, Random communication delays, Linear matrix inequality, Quantized $H_\infty$ control strategy

1. Introduction. Networked control systems (NCSs) have received increasing attention in recent years due to many advantages including lower cost, easier installation, maintenance and higher reliability. They have found successfully industrial applications in automobiles, manufacturing plants, aircraft and unmanned vehicles. However, there are also challenging problems in NCSs, such as network-induced delays and packet dropouts [16-22], which are mainly caused by the limited bandwidth. Signal quantization is also a difficult issue, which has significant impact on the performance of the networked control [1-8]. There are also many results considering the quantization and delays or packet dropouts simultaneity, such as [14,15].

For NCSs with delays or packet dropouts, a class of NCSs with arbitrary but finitelength packet dropouts was described by [18]. [17] further extended this result to general arbitrary packet-dropout process and the Markovian bounded packet-dropout case. [22] is discussed $H_2$ control for NCSs with Markovian data losses and delays. For NCSs with quantization, [4] concerned with global asymptotic stabilization of continuous-time systems subject to the state quantization, the measured output quantization and the control input quantization, respectively. In [1], the quantized state feedback and quantized control input were considered for discrete-time systems and the methods proposed could not
be directly applied to NCSs when considering the effect of the random delays. For NCSs with quantization and with delays or packet dropouts, [7] addressed $H_\infty$ problem and developed a quantized state feedback strategy based on [4] for continuous-time systems over digital communication networks with data dropouts. [14] proposed an NCS system model which considers the effects of both network-induced delays and quantization levels.

It is quite common in practice that the time-delays occur in a random way, rather than a deterministic way, for a number of engineering applications such as real-time distributed decision making and multiplexed data communication networks, see [11]. Based on a similar Bernoulli random delay model, [13] studied the controller design problem for NCSs with sensor-to-controller and controller-to-actuator delays, [12] investigated an $H_\infty$ filtering problem for discrete-time systems with randomly sensor delays, and similar random model had been used in [9,10]. However, to the best of the authors’ knowledge, there is no work which considers the random delays and the quantization at the same time.

Motivated by the above mentioned problem, this paper is concerned with the state feedback $H_\infty$ control problem for networked control systems with quantization and random communication delays. The random delays from the sensor to the controller and from the controller to the actuator are considered, and they are modeled as a function of stochastic variables satisfying Bernoulli distributed white sequence. A novel quantized random delay model is proposed, and by using this model, the relationship of the quantizations, delays and the system performance is studied. The quantizer considered here is dynamic and composed of an adjustable zoom parameter and a static quantizer. The state feedback $H_\infty$ controller can be constructed via solving a linear matrix inequality. A quantized $H_\infty$ control strategy is derived, with a condition on the quantization range and the error bound satisfied, such that the closed-loop system with quantization and random delays is exponentially mean-square stable and with a prescribed $H_\infty$ performance bound. An example is presented to illustrate the effectiveness of the proposed method.

The organization of this paper is as follows. Section 2 presents the problem statement and preliminaries. Section 3 gives a design method of the controller gain and a quantized $H_\infty$ control strategy. In Section 4, an example is presented to illustrate the effectiveness of the proposed method. Section 5 gives some conclusions.

2. Problem Statement and Preliminaries.

2.1. Problem statement. Consider the networked control system in Figure 1, where the plan is a discrete-time LTI system. Assume that the sensor, the controller and the actuator are clock-driven. The random communication delays, which are from the sensor to the controller and from the controller to the actuator, are less than one sampling period, respectively.

![Figure 1. Structure of a networked control system](image-url)
Consider the discrete-time LTI model described by
\[\begin{align*}
x(k + 1) &= Ax(k) + B_2 u(k) + B_1 \omega(k) \\
z(k) &= Cx(k) + Du(k)
\end{align*}\]
where \(x(k) \in \mathbb{R}^n\) is the state, \(z(k) \in \mathbb{R}^q\) is the regulated output, \(u(k) \in \mathbb{R}^m\) is the control input, and \(\omega(k) \in \mathbb{R}^l\) is the disturbance input. \(A, B_1, B_2, C\) and \(D\) are known constant matrices of appropriate dimensions. The pair \((A, B_2)\) is assumed to be stabilizable.

The quantizer considered in this paper is defined with general form as in [4]. Let \(z \in \mathbb{R}^l\) be the variable being quantized. By a quantizer, we mean a piecewise constant function \(q : \mathbb{R}^l \to D\), where \(D\) is a finite subset of \(\mathbb{R}^l\).

Assume that there exist positive real numbers \(M\) and \(\Delta\) such that the following condition holds:
\[\begin{align*}
&\text{If } |z| \leq M \\
&\text{then } |q(z) - z| \leq \Delta
\end{align*}\]
This condition gives a bound on the quantization error when the quantizer does not saturate. \(M\) and \(\Delta\) represent the quantization range and the quantization error, respectively. Assume that \(q(x) = 0\) for \(x\) in some neighborhood of the origin, i.e., the origin lies in the interior of the set \(\{x : q(x) = 0\}\). In the control strategy to be developed below, we will use the one-parameter family of quantizers
\[q_\mu(z) = \mu q \left( \frac{z}{\mu} \right)\]
where \(\mu > 0\) is the quantizer’s parameter. The quantization range is \(M\mu\) and the quantization error is \(\Delta\mu\). \(\mu\) can be seen as the zoom variable: increasing \(\mu\) corresponds to zooming out and essentially obtaining a new quantizer with larger range and larger quantization error such that any signals can be adequately measured, while decreasing \(\mu\) corresponds to zooming in and obtaining a quantizer with smaller range but also smaller quantization error such that the signals can be driven to 0.

On the other hand, the random communication delays considered in this paper are modeled as a function of stochastic variables satisfying Bernoulli distributed white sequence. \(\delta_i\) \((i = 1, 2)\) denote the stochastic variables, which are independent and are Bernoulli distributed white sequence taking the values of 0 and 1 with
\[\begin{align*}
Pr\{\delta_i = 1\} &= E\{\delta_i\} = \rho_i, \quad Pr\{\delta_i = 0\} = E\{(1 - \delta_i)\} = 1 - \rho_i \\
E\{(\rho_i - \delta_i)^2\} &= \rho_i(1 - \rho_i), \quad E\{(\rho_1 - \delta_1)(\rho_2 - \delta_2)\} = 0
\end{align*}\]
where \(\rho_i\) \((i = 1, 2)\) are known constant.

Let the stochastic variable \(\delta_i\) be
\[\delta_i = \begin{cases} 1, & \text{occurs random delay} \\ 0, & \text{otherwise} \end{cases}\]

With the above defined quantizer, for system (1), a quantized state with random delays is
\[x_c(k) = (1 - \delta_1)q_{\mu_k}^z(x(k)) + \delta_1 q_{\mu_{k-1}}^z(x(k - 1))\]
\[= (1 - \delta_1)\mu_k^z q_1 \left( \frac{x(k)}{\mu_k^z} \right) + \delta_1 \mu_{k-1}^z q_1 \left( \frac{x(k - 1)}{\mu_{k-1}^z} \right),\]
and a quantized control input with random delays is
\[u(k) = (1 - \delta_2)u_c(k) + \delta_2 u_c(k - 1)\]
with \( u_c(k) = K x_c(k) \), then

\[
\begin{align*}
u(k) &= (1 - \delta_2) K x_c(k) + \delta_2 K x_c(k - 1) \\
&= (1 - \delta_1)(1 - \delta_2) q_{\mu_{k-1}}^u \left( K \mu_k x q_1 \left( x(k-1) \mu_k^{-1} \right) \right) + (\delta_1(1 - \delta_2) + \delta_2(1 - \delta_1)) \\
&\quad \times q_{\mu_{k-1}}^u \left( K \mu_{k-1} x q_1 \left( x(k-1) \mu_{k-1}^{-1} \right) \right) + \delta_1 \delta_2 q_{\mu_{k-2}} v_2 \left( K \mu_{k-2} x q_1 \left( x(k-2) \mu_{k-2}^{-1} \right) \right) \\
&= (1 - \delta_1)(1 - \delta_2) \mu_k q_2 \left( K \mu_k x q_1 \left( x(k) \mu_k^{-1} \right) \right) + (\delta_1(1 - \delta_2) + \delta_2(1 - \delta_1)) \\
&\quad \times \mu_{k-1} q_2 \left( K \mu_{k-1} x q_1 \left( x(k-1) \mu_{k-1}^{-1} \right) \right) + \delta_1 \delta_2 \mu_{k-2} q_2 \left( K \mu_{k-2} x q_1 \left( x(k-2) \mu_{k-2}^{-1} \right) \right)
\end{align*}
\]

(4)

where the quantizers \( q_{\mu_k}^x(\cdot) \) and \( q_{\mu_k}^u(\cdot) \) are dynamic and defined by (3), the quantizers \( q_1(\cdot) \) and \( q_2(\cdot) \) are static and defined by (2). \( q_1(\cdot) \) is a state quantizer with range \( M_1 \) and error bound \( \Delta_1 \), \( q_2(\cdot) \) is a control input quantizer with range \( M_2 \) and error bound \( \Delta_2 \). We consider a strategy always satisfying

\[
\mu_k^u = \theta \mu_k^x, \quad \mathbf{M}_2 = \frac{||K|| (M_1 + \Delta_1)}{\theta}
\]

(5)

Its derivation will be proved in the following section, where \( \theta \) is a positive constant and should be adjusted in real applications.

Combining controller (4) with system (1), the following closed-loop system with quantization and random delays is obtained:

\[
\begin{align*}
x(k + 1) &= \tilde{A} x(k) + h_2 B_2 K x(k - 1) + h_3 B_2 K x(k - 2) + B_{2\rho} \eta(k) + B_{1\omega}(k) \\
&\quad + H_1 B_2 K x(k) - H_2 B_2 K x(k - 1) - H_3 B_2 K x(k - 2) + B_{2\delta} \eta(k) \\
z(k) &= \tilde{C} x(k) + h_2 D K x(k - 1) + h_3 D K x(k - 2) + C_{\rho} \eta(k) \\
&\quad + H_1 D K x(k) - H_2 D K x(k - 1) - H_3 D K x(k - 2) + C_{\delta} \eta(k)
\end{align*}
\]

(6)

where

\[
\begin{align*}
h_1 &= (1 - \rho_1)(1 - \rho_2), \quad h_2 = \rho_1(1 - \rho_2) + \rho_2(1 - \rho_1), \quad h_3 = \rho_1 \rho_2, \\
H_1 &= (\rho_1 - \delta_1)(\rho_2 - \delta_2) + (1 - \rho_2)(\rho_1 - \delta_1) + (1 - \rho_1)(\rho_2 - \delta_2), \\
H_2 &= 2(\rho_1 - \delta_1)(\rho_2 - \delta_2) + (1 - 2\rho_2)(\rho_1 - \delta_1) + (1 - 2\rho_1)(\rho_2 - \delta_2), \\
H_3 &= -(\rho_1 - \delta_1)(\rho_2 - \delta_2) + \rho_2(\rho_1 - \delta_1) + \rho_1(\rho_2 - \delta_2), \quad \tilde{A} = A + h_1 B_2 K, \\
\tilde{C} &= C + h_1 D K, \quad B_{2\rho} = \left[ h_1 B_2 \quad h_2 B_2 \quad h_3 B_2 \right], \quad B_{2\delta} = \left[ H_1 B_2 \quad -H_2 B_2 \quad -H_3 B_2 \right], \\
C_{\rho} &= \left[ h_1 D \quad h_2 D \quad h_3 D \right], \quad C_{\delta} = \left[ H_1 D \quad -H_2 D \quad -H_3 D \right], \\
\eta(k) &= \left[ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \right] \quad \text{with} \quad e_1 = e_1^a + K e_1^x, \\
&\quad e_2 = e_2^a + K e_2^x, \quad \text{and} \quad e_3 = e_3^a + K e_3^x, \\
\eta_x^a &= \mu_k^x q_1 \left( x(k) x(k) \mu_k^{-1} \right) - x(k), \quad \eta_x^a = \mu_{k-1} q_1 \left( x(k-1) x(k-1) \mu_{k-1}^{-1} \right) - x(k-1)
\end{align*}
\]
\[ e_3^* = \mu_{k-2} (q_1 \left( \frac{x(k-2)}{\mu_{k-2}} \right) - \frac{x(k-2)}{\mu_{k-2}}) , \quad e_1^u = \mu_k^u \left( q_2 \left( K \mu_k^x q_1 \left( \frac{x(k)}{\mu_k^u} \right) \right) - K \mu_k^x q_1 \left( \frac{x(k)}{\mu_k^u} \right) \right) , \]

\[ e_2^u = \mu_{k-1} \left( q_2 \left( K \mu_{k-1}^x q_1 \left( \frac{x(k-1)}{\mu_{k-1}^u} \right) \right) - K \mu_{k-1}^x q_1 \left( \frac{x(k-1)}{\mu_{k-1}^u} \right) \right) , \]

\[ e_3^u = \mu_{k-2} \left( q_2 \left( K \mu_{k-2}^x q_1 \left( \frac{x(k-2)}{\mu_{k-2}^u} \right) \right) - K \mu_{k-2}^x q_1 \left( \frac{x(k-2)}{\mu_{k-2}^u} \right) \right) . \]

Remark 2.1. The model (6) with quantization and random delays are motivated by the random delay model in [13], and the quantization model in [1], etc. The difference is that our model (6) considers the effect of the quantization and the random delays simultaneously.

Definition 2.1. The quantized closed-loop system (6) is said to be exponentially mean-square stable if with \( \omega(k) = 0 \), there exist constants \( \alpha > 0 \) and \( \tau \in (0,1) \), such that

\[ E\{\|x(k)\|^2\} \leq \alpha \tau^k E\{\|x(0)\|^2\} \]

By Definition 2.1, due to the effect of quantization errors and random delays, the problem addressed in this paper is as follows:

Problem 2.1. Design a quantized \( H_\infty \) control strategy such that the closed-loop system (6) satisfies the following two requirements simultaneously.

R1) The closed-loop system (6) is exponentially mean-square stable.

R2) For a given scalar \( \gamma > 0 \) and all nonzero \( \omega(k) \), under the zero-initial condition, the control output \( z(k) \) satisfies

\[ \sum_{k=0}^{\infty} E\{\|z(k)\|^2\} \leq \gamma^2 \sum_{k=0}^{\infty} \|\omega(k)\|^2 \]

Remark 2.2. As mentioned in [12], the performance criterion (7) with deterministic signal \( \omega(k) \) can be easily modified to deal with the case where the disturbance signals may be random ones.

2.2. Preliminaries. The following lemma presented will be used in the sequel.

Lemma 2.1. [13] Let \( V(\zeta_k) \) be a Lyapunov functional. If there exist real scalars \( \lambda \geq 0 \), \( \mu > 0 \), \( \nu > 0 \) and \( 0 < \psi < 1 \) such that \( \mu\|\zeta_k\|^2 \leq V(\zeta_k) \leq \nu \|\zeta_k\|^2 \) and

\[ E\{V(\zeta_{k+1})|\zeta_k\} - V(\zeta_k) \leq \lambda - \psi V(\zeta_k) \]

then the sequence \( \zeta_k \) satisfies

\[ E\{\|\zeta_k\|^2\} \leq \frac{\nu}{\mu} \|\zeta_0\|^2 (1 - \psi)^k + \frac{\lambda}{\mu \psi} \]

3. \( H_\infty \) Controller Design with Quantization and Random Delays. In this section, a design method of the controller gain is given, and then a quantized \( H_\infty \) control strategy is presented to guarantee the closed-loop system (6) exponentially mean-square stable and with the prescribed \( H_\infty \) performance bound.
Theorem 3.1 is presented to design the controller gain. Denote

\[ \Pi_1 = \text{diag}\{P_2 - P_1, P_3 - P_2, -P_3, -\gamma^2 I + B_1^T P_1 B_1\} \]

\[ \Pi_2 = \begin{bmatrix} \bar{A} & h_2 B_2 K & h_3 B_2 K & B_1 \\ \varepsilon_1 B_2 K & -\varepsilon_2 B_2 K & -\varepsilon_3 B_2 K & 0 \\ \bar{C} & h_2 D K & h_3 D K & 0 \\ \varepsilon_1 D K & -\varepsilon_2 D K & -\varepsilon_3 D K & 0 \end{bmatrix} \]

\[ \Pi_3 = \text{diag}\{P_1, P_1, I, I\} \]

where

\[ \varepsilon_1 = \sqrt{(1 - \rho_2)^2 \rho_1 (1 - \rho_1) + (1 - \rho_1)^2 \rho_2 (1 - \rho_2)}, \]

\[ \varepsilon_2 = \sqrt{(1 - 2\rho_2)^2 \rho_1 (1 - \rho_1) + (1 - 2\rho_1)^2 \rho_2 (1 - \rho_2)}, \]

\[ \varepsilon_3 = \sqrt{\rho_2^2 \rho_1 (1 - \rho_1) + \rho_1^2 \rho_2 (1 - \rho_2)}. \]

Theorem 3.1. Consider plant (6) with a given scalar \( \gamma > 0 \), assume that there exist matrices \( Y \) and \( X > 0 \), \( P_2 > 0 \), \( P_3 > 0 \) such that the following LMI holds

\[
\begin{bmatrix}
\bar{P}_2 - X & * & * & * & * & * & * & * \\
0 & \bar{P}_3 - \bar{P}_2 & * & * & * & * & * & * \\
0 & 0 & -\bar{P}_3 & * & * & * & * & * \\
0 & 0 & 0 & -\gamma^2 I & * & * & * & * \\
AX + h_1 B_2 Y & h_2 B_2 Y & h_3 B_2 Y & B_1 & -X & * & * & * \\
\varepsilon_1 B_2 Y & -\varepsilon_2 B_2 Y & -\varepsilon_3 B_2 Y & 0 & 0 & -X & * & * \\
CX + h_1 D Y & h_2 D Y & h_3 D Y & 0 & 0 & 0 & -I & * \\
\varepsilon_1 D Y & -\varepsilon_2 D Y & -\varepsilon_3 D Y & 0 & 0 & 0 & 0 & -I & * \\
0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 & -X
\end{bmatrix} < 0 \tag{10}
\]

denote

\[ K = Y X^{-1} \tag{11} \]

Then, there exist symmetric positive definite matrices \( P_1 > 0, P_2 > 0, P_3 > 0 \) and \( \Omega > 0 \) such that the following LMI holds

\[ \Pi_1 + \Pi_2^T \Pi_3 \Pi_2 + \Omega < 0 \tag{12} \]

Proof: Let \( P_1 = X^{-1} \), first we prove that the following inequality holds

\[ \Pi_1 + \Pi_2^T \Pi_3 \Pi_2 < 0 \tag{13} \]

By the Schur complement, (13) is equivalent to the following LMI

\[ \begin{bmatrix} \Pi_1 & \Pi_2^T \Pi_3 \\ * & -\Pi_3 \end{bmatrix} < 0 \tag{14} \]

Using the Schur complement for (14) again, we obtain a new LMI. Performing a congruence transformation with \( \text{diag}\{X, X, X, I, X, I, I, X\} \) on the new LMI and with the gain matrix (11), denote \( \bar{P}_2 = X P_2 X, \bar{P}_3 = X P_3 X \), we obtain (10), which implies that (10) is equivalent to (13). Then, according to (13), there always exist positive definite matrices \( P_1, P_2, P_3 \) and \( \Omega \) such that (12) holds. Thus, the proof is complete. \( \square \)

By using the gain matrix \( K \), the scalar \( \gamma \) and positive matrices \( P_1, P_2, P_3, \Omega \) obtained by Theorem 3.1, the following theorem gives a control strategy such that the closed-loop system (6) is exponentially mean stable and with the prescribed \( H_\infty \) performance bound.
Denote

$$\Phi = \begin{bmatrix} \bar{A}^T P_1 B_{2p} + (B_2 K)^T P_1 B_{2p} + \bar{C}^T C_\rho + (DK)^T C_{\delta}^1 \\ h_2 (B_2 K)^T P_1 B_{2p} - (B_2 K)^T P_1 B_{2p}^T + h_2 (DK)^T C_\rho - (DK)^T C_{\delta}^2 \\ h_3 (B_2 K)^T P_1 B_{2p} - (B_2 K)^T P_1 B_{2p}^T + h_3 (DK)^T C_\rho - (DK)^T C_{\delta}^3 \end{bmatrix}$$

$$\Psi = 2 B_2^T P_1 B_{2p} + \varepsilon_0 B_0 + C_\rho^T C_\rho + \varepsilon_0 C_0$$

where

$$B_{2p}^1 = [\varepsilon_1 \varepsilon_2 B_2 - \varepsilon_1 \varepsilon_2 B_2 - \varepsilon_1 \varepsilon_2 B_2], \quad B_{2p}^2 = [\varepsilon_1 \varepsilon_2 B_2 - \varepsilon_2 \varepsilon_2 B_2 - \varepsilon_2 \varepsilon_2 B_2],$$

$$B_{2p}^3 = [\varepsilon_1 \varepsilon_3 B_2 - \varepsilon_2 \varepsilon_3 B_2 - \varepsilon_3 \varepsilon_3 B_2], \quad C_{\delta}^1 = [\varepsilon_1 \varepsilon_1 D - \varepsilon_1 \varepsilon_2 D - \varepsilon_1 \varepsilon_3 D],$$

$$C_{\delta}^2 = [\varepsilon_1 \varepsilon_2 D - \varepsilon_2 \varepsilon_2 D - \varepsilon_2 \varepsilon_3 D], \quad C_{\delta}^3 = [\varepsilon_1 \varepsilon_3 D - \varepsilon_3 \varepsilon_3 D - \varepsilon_3 \varepsilon_3 D],$$

$$B_0 = \text{diag}\{B_2^T P_1 B_2, B_2^T P_1 B_2, B_2^T P_1 B_2\}, \quad C_0 = \text{diag}\{D^T D, D^T D, D^T D\},$$

with

$$\varepsilon_1 \varepsilon_1 = (1 - \rho_2)^2 \rho_1 (1 - \rho_1) + (1 - \rho_1)^2 \rho_2 (1 - \rho_2),$$

$$\varepsilon_2 \varepsilon_2 = (1 - 2 \rho_2)^2 \rho_1 (1 - \rho_1) + (1 - 2 \rho_1)^2 \rho_2 (1 - \rho_2),$$

$$\varepsilon_3 \varepsilon_3 = \rho_2^2 \rho_1 (1 - \rho_1) + \rho_2^2 \rho_2 (1 - \rho_2),$$

$$\varepsilon_1 \varepsilon_3 = 2 \rho_1 \rho_2 (1 - \rho_1)(1 - \rho_2),$$

$$\varepsilon_1 \varepsilon_2 = \rho_1 (1 - \rho_1)(1 - \rho_2),$$

$$\varepsilon_2 \varepsilon_2 = \rho_2 (1 - \rho_2)(1 - \rho_2),$$

$$\varepsilon_0 = \begin{bmatrix} \varepsilon_1 \varepsilon_1 & -\varepsilon_1 \varepsilon_2 & -\varepsilon_1 \varepsilon_3 \\ -\varepsilon_1 \varepsilon_2 & \varepsilon_2 \varepsilon_2 & \varepsilon_2 \varepsilon_3 \\ -\varepsilon_1 \varepsilon_3 & \varepsilon_2 \varepsilon_3 & \varepsilon_3 \varepsilon_3 \end{bmatrix}.$$

Theorem 3.2. Consider system (6) controlled by the quantized controller (4), assume that $M_1$ is chosen large enough such that

$$M_1 > \frac{\Gamma \Delta}{\lambda_{\text{min}}(\Omega)} \quad (15)$$

where $\Delta = \theta \Delta_2 + \|K\| \Delta_1$, $\Gamma = \|\Phi\| + \sqrt{\|\Phi\|^2 + \|\Psi\| \lambda_{\text{min}}(\Omega)}$.

Then, the control strategy (4) with

$$\mu_k^* = \frac{2|x(k)|}{M_1 + \frac{\Gamma \Delta}{\lambda_{\text{min}}(\Omega)}}, \quad \mu_k^u = \theta \mu_k^*$$

renders the closed-loop system (6) exponentially mean-square stable and with the $H_\infty$ performance bound $\gamma$.

Proof: By using the properties of (2) for the quantizer $q_1(\cdot)$, it is easy to check that whenever $|x(k)| \leq M_1 \mu_k^*$, we have that $|q_1 \left( \frac{x(k)}{\mu_k^*} \right) - \frac{x(k)}{\mu_k^*}| \leq \Delta_1$ and $|q_1 \left( \frac{x(k)}{\mu_k^*} \right)| \leq \frac{|x(k)|}{\mu_k^*} + \Delta_1 \leq M_1 + \Delta_1$, further

$$\left| \frac{K \mu_k^* q_1 \left( \frac{x(k)}{\mu_k^*} \right)}{\mu_k^*} \right| \leq \frac{\|K\|}{\theta} \left| q_1 \left( \frac{x(k)}{\mu_k^*} \right) \right| \leq \frac{\|K\| (M_1 + \Delta_1)}{\theta} = M_2.$$

By using the properties of (2) for the quantizer $q_2(\cdot)$ again, if only $|x(k)| \leq M_1 \mu_k^*$, we can obtain $|e_1| = \mu_k^* \left( q_2 \left( \frac{K \mu_k^* q_1 \left( \frac{x(k)}{\mu_k^*} \right)}{\mu_k^*} - \frac{K \mu_k^* q_1 \left( \frac{x(k)}{\mu_k^*} \right)}{\mu_k^*} \right) + \left( \frac{K \mu_k^* q_1 \left( \frac{x(k)}{\mu_k^*} \right)}{\mu_k^*} - \frac{K x(k)}{\mu_k^*} \right) \right) \leq \mu_k^* (\Delta_2 + \frac{\|K\| \Delta_1}{\theta}) = \mu_k^* (\theta \Delta_2 + \|K\| \Delta_1) = \mu_k^* \Delta$. In the same way, we have $|e_2| \leq \mu_{k-1}^* \Delta$ and
\(|e| \leq \mu_{k-2}\Delta\), then we can obtain

\[
|e| = \sqrt{|e_1|^2 + |e_2|^2 + |e_3|^2} \leq \sqrt{\left(\mu_k^2\Delta\right)^2 + \left(\mu_{k-1}\Delta\right)^2 + \left(\mu_{k-2}\Delta\right)^2}
\]

Consider the Lyapunov function candidate

\[
V(k) = x^T(k)P_1x(k) + x^T(k-1)P_2x(k-1) + x^T(k-2)P_3x(k-2).
\]

Let \(R_k\) be the minimal \(\delta - \text{algebra} \) generated by \(\{x(i), \ 0 \leq i \leq k\}\).

By using (6) and LMI (12), noting that

\[
E\{\rho_1 - \rho_2 - \rho_3\} = 0, \quad \text{and the values of} \ \varepsilon_1, \ \varepsilon_2 \ \text{and} \ \varepsilon_3 \ \text{above}. \quad \text{Denote} \ \xi^T(k) = [x^T(k) \ \ x^T(k-1) \ \ x^T(k-2) \ \ \omega^T(k)], \quad X^T(k) = [x^T(k) \ \ x^T(k-1) \ \ x^T(k-2)], \quad \text{then we have}
\]

\[
E\{V(k+1)|R_k\} - V(k) = E\{z^T(k)z(k)\} - \gamma^2\omega^T(k)\omega(k)
\]

\[
= (Ax(k) + h_2B_2Kx(k-1) + h_3B_2Kx(k-2) + B_2\eta(k) + B_1\omega(k))P_1(Ax(k)
\]

\[
+ h_2B_2Kx(k-1) + h_3B_2Kx(k-2) + B_2\eta(k) + B_1\omega(k)) + E\{(H_1B_2Kx(k
\]

\[
- H_2B_2Kx(k-1) - H_3B_2Kx(k-2) + B_2\eta(k)\}P_1(H_1B_2Kx(k) - H_2B_2Kx(k-1)
\]

\[
- H_3B_2Kx(k-2) + B_2\eta(k)\}) + x^T(k)P_2x(k) + x^T(k-1)P_3x(k-1)
\]

\[
- x^T(k)P_1x(k) - x^T(k-1)P_2x(k-1) - x^T(k-2)P_3x(k-2) - \gamma^2\omega^T(k)\omega(k)
\]

\[
+ (C_1x(k) + h_2DKx(k-1) + h_3DKx(k-2) + C_\rho\eta(k))T(C_1x(k) + h_2DKx(k-1)
\]

\[
+ h_3DKx(k-2) + C_\rho\eta(k)) + E\{(H_1DKx(k) - H_2DKx(k-1) - H_3DKx(k-2)
\]

\[
+ C_\rho\eta(k))\}T(H_1DKx(k) - H_2DKx(k-1) - H_3DKx(k-2) + C_\rho\eta(k))\}
\]

\[
\leq \xi^T(k)\left\{\Pi_1 + \Pi_2^T\Pi_2\right\}\xi(k) + 2X^T(k)\Phi\eta(k) + \eta^T(k)\Phi\eta(k)
\]

\[
\leq -\lambda_{\min}(\Omega)|X(k)|^2 + 2|X(k)||\Phi||\eta(k)| + |\eta(k)|^2||\Psi||
\]

\[
\leq -\lambda_{\min}(\Omega) \left(|X(k)| - \frac{\Gamma|e|}{\lambda_{\min}(\Omega)}\right) \left(|x(k)| - \frac{(||\Phi|| - \sqrt{||\Phi||^2 + ||\Psi||\lambda_{\min}(\Omega)}|e|}{\lambda_{\min}(\Omega)}\right).
\]

(17)

By choosing the dynamic scaling in (16), we have

\[
|x(k)| = \left(M_1 + \frac{\Gamma\Delta}{\lambda_{\min}(\Omega)}\right) \mu_k^2 / 2
\]

(18)

where \(\mu_k^2\) is positive so long as \(x(k)\) is nonzero.

Combining the above inequality with (15), it is clear that there always exists a sufficiently small scalar \(\varepsilon \in (1, 0)\) such that the following LMI holds

\[
\frac{\Gamma\Delta}{\lambda_{\min}(\Omega)(1-\varepsilon)} \mu_k^2 \leq |x(k)| \leq M_1\mu_k^2
\]

(19)

Thus, by using (19) and (17), we obtain

\[
E\{V(k+1)|R_k\} - V(k)
\]

\[
\leq -E\{z^T(k)z(k)\} + \gamma^2\omega^T(k)\omega(k) - \varepsilon^2\lambda_{\min}(\Omega)|\xi(k)|^2
\]

\[
\leq -E\{z^T(k)z(k)\} + \gamma^2\omega^T(k)\omega(k) - \varepsilon^2\alpha|X(k)|^2
\]

where

\[
0 < \alpha < \min\{\lambda_{\min}(\Omega), \kappa\}, \quad \kappa := \max\{\lambda_{\max}(P_i), \ i \in \{1, 2, 3\}\}\}
\]

From (20), we have

\[
E\{V(k+1)|R_k\} - V(k) < -E\{z^T(k)z(k)\} + \gamma^2\omega^T(k)\omega(k) - \varepsilon^2\alpha|X(k)|^2
\]

(20)
By setting \( \omega = 0 \), obviously, \( E\{V(k+1)|R_k\} - V(k) < -\varepsilon^2 2V(k) \). So, by Definition 2.1 and Lemma 2.1, it can be verified that the closed-loop system (6) is exponentially mean-square stable.

In addition, for any \( k > 0 \), we can obtain

\[
E\{V(k+1)|R_k\} - V(k) + E\{z^T(k)z(k)\} - \gamma^2 \omega^T(k)\omega(k) < 0
\]  

(21)

Summing up (21) from 0 to \( \infty \) with respect to \( k \) and assuming zero initial condition, the following is obtained

\[
\sum_{k=0}^{\infty} E\{\|z(k)\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \|\omega(k)\|^2
\]  

(22)

This ends the proof.

Remark 3.1. With the consideration of quantization and random delays at the same time, Theorem 3.2 offers a quantized control strategy for the closed-loop system (6) such that (6) is exponentially mean-square stable and with a prescribed \( H_\infty \) performance level. To the best of the authors’ knowledge, this work has not been done previously.

Remark 3.2. In [1], the quantized state feedback \( H_\infty \) control was considered for discrete-time systems. [13] is concerned with a new observer-based controller design problem for networked systems with random communication delays. The methods proposed in these two paper could not be directly applied to NCSs when considering the effect of the quantization and the random delays. In this paper, our methods proposed is considered the effect of the quantization and random delays simultaneously.

Remark 3.3. Because \( \lambda_{\min}(\Omega) \) has a significant effect on the value of \( \frac{\Delta_1}{\lambda_{\min}(\Omega)} \), the condition \( \Omega(\varsigma) > 0 \) is introduced to restrict the value of \( \lambda_{\min}(\Omega) \), such that \( \lambda_{\min}(\Omega) \geq \varsigma \).

4. Example. In this section, an example is presented to illustrate the effectiveness of the control strategy presented in this paper.

Consider the following parameters studied in [1],

\[
A = \begin{bmatrix} 2 & 0 \\ 0.5 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

By solving the LMI (10) in Theorem 3.1, the values of the controller gain \( K \) and the \( H_\infty \) performance bound \( \gamma \) with various values of \( \rho_1, \rho_2 \) are shown as in Table 1.

**Table 1.** Values of \( \gamma \) and \( K \) for various values of \( \rho_1, \rho_2 \)

<table>
<thead>
<tr>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( \gamma )</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.02</td>
<td>0.03</td>
<td>-1.8644</td>
</tr>
<tr>
<td>0</td>
<td>0.01</td>
<td>0.02</td>
<td>-1.2867</td>
</tr>
<tr>
<td>5.3800</td>
<td>7.1274</td>
<td>11.5080</td>
<td>-1.0377</td>
</tr>
</tbody>
</table>

Let \( \theta = 2 \) and the quantization error \( \Delta_1 = 0.1, \Delta_2 = 0.1 \). By Theorem 3.1 with \( \Omega(\varsigma) > 0 \), and \( \varsigma = 0.1, \rho_1 = 0.03 \) and \( \rho_2 = 0.02 \), for the case of \( \gamma = 13.5080 \), we obtain the value of \( \frac{\Delta_1}{\lambda_{\min}(\Omega)} \) as 1477.1.

Then, by Theorem 3.2, the range of the quantizer \( q(\cdot) \) can be chosen as \( M_1 = 1477.2 \). According to (5), \( M_2 = 767.4936 \).
In order to illustrate the efficiency of the proposed method, the following simulation is given. With the parameters and the result calculated above, given the initial system state as \( x_0 = [1, -1] \), and let the disturbance input \( \omega(k) \) be

\[
\omega(k) = \begin{cases} 
1, & 23 \leq k \leq 24 \text{ (step)} \\
0, & \text{otherwise} 
\end{cases}
\]

The simulation results of system (6) with the controller gain \( K = [-1.0377, 0.0515] \) are depicted in the following figures.

Figure 2 shows state responses of system (6). Figure 3 shows regulated output responses of system (6). From these figures, we can see that the proposed method is efficient.

5. Conclusion. We have studied the \( H_\infty \) control problem for NCSs with both quantization and random communication delays. A quantized \( H_\infty \) control strategy has been derived such that the quantized closed-loop system is exponentially mean-square stable and with a prescribed \( H_\infty \) performance bound. An example has been presented to illustrate the effectiveness of the proposed method.

Acknowledgment. This work was supported in part by the Funds for Creative Research Groups of China (No. 60821063), National 973 Program of China (Grant No. 2009CB320604), the Funds of National Science of China (Grant No. 60974043) and the 111 Project (B08015), the Fundamental Research Funds for the Central Universities (No. N090604001, N090604002). The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

REFERENCES
