Finite frequency $H_\infty$ filtering for uncertain discrete-time switched linear systems

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Abstract

This paper is concerned with the problem of robust $H_\infty$ filtering for discrete-time switched linear systems with polytopic uncertainties in the finite frequency domain. Based on the generalized Kalman–Yakubovich–Popov (GKYP) lemma and switched parameter-dependent Lyapunov functions, a switched full-order filter is designed such that the corresponding filtering error system is asymptotically stable and satisfies a prescribed finite frequency $H_\infty$ performance index. Compared with the existing full frequency approaches, the proposed finite frequency one receives better results for the cases in which the frequency ranges of noises are known. A numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: Switched systems; Filtering; Finite frequency; GKYP lemma; Linear matrix inequalities (LMIs)

1. Introduction

As an important class of hybrid systems, switched systems consist of a finite number of subsystems and a logical rule that orchestrates switching between these subsystems. Recently, switched systems have received a great deal of attention, see e.g., survey papers [1–3], recent books [4,5] and references therein. The motivation for studying switched systems comes partly from the fact that switched systems and switched multi-controller systems have numerous applications in the control of mechanical systems, process control, the automotive industry, power systems, aircraft and traffic control, and many other fields. The problems encountered in switched systems can be classified into three categories [1]. The first one, which is also of interest in this paper, is to find conditions that guarantee that switched systems are asymptotically stable under the arbitrary switching signal. The second one is to identify certain useful classes of switching signals for which switched systems are asymptotically stable. And the third one is to construct a switching signal that makes switched systems asymptotically stable. In the study of switched systems, various approaches have been proposed such as the multiple Lyapunov function approach [6,7], the dwell (average dwell) time approach [8–10], the switched Lyapunov function approach [11], etc.

In addition, state estimation has been widely studied and has found many practical applications over the past decades. When a priori statistical information on external disturbance signals is unknown, the celebrated Kalman filtering cannot be employed. To address this issue, $H_\infty$ filtering is introduced, in which external disturbance signals are assumed to be energy bounded, and the main objective is to minimize the $H_\infty$ norm from process noise to estimation error, see e.g., [12–17] and references therein. For switched linear systems, the state estimation problem has been...
investigated with different performance indices such as $H_{\infty}$ filtering [18–20], $l_2-l_{\infty}$ filtering [21,22], $H_2/H_{\infty}$ filtering [23], etc. Note that all the mentioned filtering approaches [18–23] are considered in the full frequency domain. However, sometimes the frequency ranges of noises are known beforehand in practice so that, for these cases, designing a full frequency filter may introduce conservativeness to some extent. So in this paper, we design a finite frequency filter for uncertain discrete-time switched linear systems with the aid of the newly-built generalized KYP lemma [24] and switched parameter-dependent Lyapunov functions. The proposed finite frequency filter has better performances than the existing full frequency ones when the frequency ranges of noises are known. An example is given to illustrate its effectiveness.

The remainder of the paper is organized as follows. Section 2 gives the problem statement and preliminaries. Section 3 presents a finite frequency $H_{\infty}$ filtering approach in detail. Section 4 gives an example to illustrate the effectiveness of the proposed method. Finally, conclusions are given in Section 5.

Notations. We use standard notations throughout this paper. For a matrix $M, M^T, M^\perp$ denote its transpose and orthogonal complement, respectively. $M > 0$ ($M < 0$) means that $M$ is positive definite (negative definite). The symbol $\star$ will be used in some matrix expressions to induce a symmetric structure. The Hermitian part of a square matrix $M$ is denoted by $\text{He}(M) := M + M^\ast$. $\sigma_{\max}(G)$ denotes the maximum singular value of the transfer matrix $G$.

2. Problem statement and preliminaries

Consider a class of uncertain discrete-time switched linear systems described by

$$(\Sigma_p) : \dot{x}(k + 1) = A_{\sigma(k)}(\lambda)x(k) + B_{\sigma(k)}(\lambda)w(k)$$

$$(1)$$

$$y(k) = C_{\sigma(k)}(\lambda)x(k) + D_{\sigma(k)}(\lambda)w(k)$$

$$(2)$$

$$z(k) = L_{\sigma(k)}(\lambda)x(k)$$

$$(3)$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^p$ is the measured output, $z(k) \in \mathbb{R}^q$ is the signal to be estimated, and $w(k) \in \mathbb{R}^r$ is the energy bounded noise with known frequency ranges. The piecewise constant function $\sigma(k) : [0, \infty) \rightarrow \mathcal{I} = \{1, \ldots, N\}$ is a switching signal to specify, at each time instant $k$, the index of the active subsystem, i.e., $\sigma(k) = i$ means that the $i$th subsystem $(A_i(\lambda), B_i(\lambda), C_i(\lambda), D_i(\lambda), L_i(\lambda))$ is activated. $N > 1$ is the number of subsystems. As in [11], it is assumed that the sequence of subsystems in the switching signal $\sigma(t)$ is unknown a priori, but its instantaneous value is available in real-time implementation. The matrices of each subsystem have appropriate dimensions and are assumed to belong to a given convex-bounded polyhedral domain described by $s$ vertices in the $i$th subsystem, i.e., $(A_i(\lambda), B_i(\lambda), C_i(\lambda), D_i(\lambda), L_i(\lambda)) \in \mathcal{A}_i, i \in \mathcal{I}$, where

$$\mathcal{A}_i = \left\{ (A_i(\lambda), B_i(\lambda), C_i(\lambda), D_i(\lambda), L_i(\lambda)) | (A_i(\lambda), B_i(\lambda), C_i(\lambda), D_i(\lambda), L_i(\lambda)) \right\}$$

$$C_i(\lambda), D_i(\lambda), L_i(\lambda)) = \sum_{l=1}^{s} \lambda_l (A_{i,l}, B_{i,l}, C_{i,l}, D_{i,l}, L_{i,l});$$

$$\lambda_l \geq 0, \sum_{l=1}^{s} \lambda_l = 1$$

$$(4)$$

Here, we are interested in designing a switched full-order filter of the form

$$(\Sigma_f) : \dot{\hat{x}}(k + 1) = A_{\sigma(k)}(\lambda)\hat{x}(k) + B_{\sigma(k)}(\lambda)w(k)$$

$$\hat{y}(k) = \hat{C}_{\sigma(k)}(\lambda)\hat{x}(k) + \hat{D}_{\sigma(k)}(\lambda)w(k)$$

$$(5)$$

$$\hat{e}(k) = C_{\sigma(k)}(\lambda)\hat{x}(k) + D_{\sigma(k)}(\lambda)w(k)$$

$$(6)$$

where

$$\hat{x}(k) = \begin{bmatrix} x(k) \\ z(k) \end{bmatrix}, \quad \hat{y}(k) = z(k) - \hat{z}(k)$$

$$\hat{A}_{\sigma(k)}(\lambda) = \begin{bmatrix} A_{\sigma(k)}(\lambda) & 0 \\ B_{\sigma(k)}(\lambda)C_{\sigma(k)}(\lambda) & A_{\sigma(k)}(\lambda) \end{bmatrix}, \quad \hat{B}_{\sigma(k)}(\lambda) = \begin{bmatrix} B_{\sigma(k)}(\lambda) \\ B_{\sigma(k)}(\lambda) \end{bmatrix}$$

$$\hat{C}_{\sigma(k)}(\lambda) = [L_{\sigma(k)}(\lambda) - D_{\sigma(k)}(\lambda)C_{\sigma(k)}(\lambda) - C_{\sigma(k)}(\lambda)]$$

$$\hat{D}_{\sigma(k)}(\lambda) = -D_{\sigma(k)}D_{\sigma(k)}(\lambda)$$

$$(7)$$

$$(8)$$

$$(9)$$

Define $G_{\sigma(k)}(\omega^0)$ the transfer function of the $i$th subsystem from the noise input $w(k)$ to the estimation error $\hat{e}(k)$, then the finite frequency $H_{\infty}$ filtering problem is to find a guaranteed estimation performance index $\gamma > 0$ such that

$$\sup_{\sigma_{\max}(G_{\sigma(k)}(\omega^0))} < \gamma, \quad \theta \in \Theta, \quad i \in \mathcal{I}$$

$$(10)$$

where $\theta \in \mathbb{R}, \Theta$ is defined in Table 1, where LF, MF, and HF stands for low-, middle-, and high-frequency ranges, respectively.

Then, the task of the paper can be formulated as follows: Given an uncertain switched system $\Sigma_p$ and a prescribed noise attenuation level $\gamma > 0$, determine a switched full-order filter $\Sigma_f$ such that the filtering error system $\Sigma_r$ is asymptotically stable and satisfies the finite frequency performance index (10).

Remark 1. The full frequency $H_{\infty}$ filtering problem of switched systems is to find a switched filter $\Sigma_f$ to minimize the worst filtering error $\hat{e}(k)$ over energy bounded noise $w(k)$, i.e.,

Table 1 Different frequency ranges.

| $\Theta$ | $|\theta| \leq \vartheta_1$ | $\vartheta_1 \leq \theta \leq \vartheta_2$ | $|\theta| \geq \vartheta_3$ |
\[
\min \sup_{|w|_2 \neq 0} \frac{||e||^2_2}{||w||_2^2}
\]

This is equivalent to minimizing the \(H_\infty\) norm of the transfer function \(G_{we}(e^\theta)\), i.e., to find a noise attenuation level \(\gamma\) such that

\[
\max_{\theta} \|G_{we}(e^\theta)\|_\infty < \gamma, \quad i \in \mathcal{I}
\]
in another form

\[
\sup_{\theta} \sigma_{\max}(G_{we}(e^\theta)) < \gamma, \quad i \in \mathcal{I}
\]

However, in this paper, we consider the frequency filtering problem, so the performance index becomes (10).

The following lemmas are given, which are essential for later developments.

**Lemma 1** (Projection Lemma). Given a symmetric matrix \(\Psi\) and two matrices \(I, A\), the problem

\[
\Psi + IXA^T + AX^T I^T < 0 \quad (11)
\]
is solvable with respect to the decision matrix \(X\) if and only if

\[
I^P \Psi^{-1} I^T < 0 \quad (12)
\]
and

\[
A^P \Psi A < 0 \quad (13)
\]

**Lemma 2** (Finsler’s Lemma). Let \(\eta \in \mathbb{R}^n\), \(\mathcal{P} = \mathcal{P}^T \in \mathbb{R}^{r \times n}\), and \(\mathcal{H} \in \mathbb{R}^{n \times n}\) such that rank \((\mathcal{H}) = r < n\), then the following statements are equivalent:

(i) \(\eta^T \mathcal{P} \eta < 0\), for all \(\eta \neq 0\), \(\mathcal{H} \eta = 0\);

(ii) \(\exists \mathcal{X} \in \mathbb{R}^{n \times n}\) such that \(\mathcal{P} + \mathcal{X} \mathcal{H} + \mathcal{H}^T \mathcal{X}^T < 0\).

**Remark 2.** In Lemma 1, (11) is a sufficient condition of (12) or (13). In Lemma 2, condition (ii) remains sufficient for condition (i) to hold even arbitrary constraints are imposed to the scaling matrix \(\mathcal{X}\).

### 3. Main results

In this section, we give the main results. First, the following lemma is given to provide a condition to the finite frequency performance index (10).

**Lemma 3.** Consider the filtering error system \(\Sigma_e\), for a given symmetric matrix

\[
\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \in \mathbb{R}^{(n+q) \times (n+q)}
\]

the following statements are equivalent:

(i) The finite frequency inequality

\[
\sigma_{\max}(G_{we}(e^\theta)) < \gamma, \quad \theta \in \Theta, \quad i \in \mathcal{I}
\]

where \(\Theta\) is defined in Table 1.

(ii) There exist Hermitian matrix functions \(P_i(\lambda), Q_i(\lambda)\) satisfying \(Q_i(\lambda) > 0\), and

\[
\begin{bmatrix}
\bar{A}_i(\lambda) & \bar{B}_i(\lambda) \\
I & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\bar{C}_i(\lambda) & \bar{D}_i(\lambda) \\
0 & I
\end{bmatrix}
< 0
\]

where

\[
\Xi_i(\lambda) = 
\begin{bmatrix}
-P_i(\lambda) & Q_i(\lambda) \\
Q_i(\lambda) & P_i(\lambda) - 2 \cos(\theta_i) Q_i(\lambda)
\end{bmatrix}
\]

for the low-frequency range \(\theta \leq \theta_i\), and

\[
\Xi_i(\lambda) = 
\begin{bmatrix}
-P_i(\lambda) & -Q_i(\lambda) \\
-Q_i(\lambda) & P_i(\lambda) + 2 \cos(\theta_i) Q_i(\lambda)
\end{bmatrix}
\]

for the high-frequency range \(\theta \geq \theta_i\).

**Proof.** Condition (14) is equivalent to

\[
G_{we}(e^\theta)^T G_{we}(e^\theta) - \gamma^2 I < 0
\]

that can be rewritten as

\[
\begin{bmatrix}
G_{we}(e^\theta)^T \\
I
\end{bmatrix}^T
\begin{bmatrix}
I & 0 \\
0 & -\gamma^2 I
\end{bmatrix}
\begin{bmatrix}
G_{we}(e^\theta) \\
I
\end{bmatrix} < 0
\]

Applying the generalized KYP lemma in [24] directly yields the results.

Based on Lemma 3, three theorems are given to solve the finite frequency \(H_\infty\) filtering problem.

For concise statements subsequently, several matrices are defined as follows

\[
\begin{bmatrix}
P_{i,j} \\
\star \\
\star \\
\star
\end{bmatrix}, \quad
\begin{bmatrix}
Q_{i,j} \\
\star \\
\star \\
\star
\end{bmatrix}
\]

\[
\begin{bmatrix}
P_{i,j} \\
\star \\
\star \\
\star
\end{bmatrix}, \quad \forall i \in \mathcal{I}, \quad 1 \leq i \leq s
\]

**Theorem 1** (Low frequency). Considering the uncertain switched system \(\Sigma_p\), for given scalar \(\theta_i \geq 0, \gamma > 0\), if there exist matrices \(W_{i1}, W_{i2}, W_{i3}, V_{i1}, V_{i2}, A_{fi}, B_{fi}, C_{fi}, D_{fi}\), scalars \(\rho_{i1}, \rho_{i2}\), and symmetric matrices \(P_{i,i}, Q_{i,i} > 0, \tilde{P}_{i,i} > 0\) satisfying the following inequalities
\[
\begin{bmatrix}
-P_{1,i} & -P_{2,i} & \Omega_{13} & Q_{2,13} + W_{12}^T & 0 & 0 \\
\star & -P_{3,i} & \Omega_{23} & Q_{3,13} + W_{13}^T & 0 & 0 \\
\star & \star & \Omega_{33} & \Omega_{35} & \Omega_{36} & -\bar{C}_f^T \\
\star & \star & \star & \Omega_{44} & \Omega_{45} & -\gamma^3 I \\
\star & \star & \star & \star & \star & -I \\
\end{bmatrix}
\begin{bmatrix}
i \\
1 \\
\vdots \\
s \\
\end{bmatrix}
< 0,
\]

(22)

where
\[
\begin{align*}
\Omega_{13} &= Q_{1,i} + W_{11}^T \\
\Omega_{23} &= Q_{2,i} + W_{13}^T \\
\Omega_{33} &= P_{1,i} - 2 \cos(\theta_i) Q_{1,13} - \mathbf{H} \{ W_{11} A_{1,i} + \bar{B}_f C_{1,13} \} \\
\Omega_{34} &= P_{2,i} - 2 \cos(\theta_i) Q_{2,13} - \bar{A}_f - (W_{12} A_{1,i} + \bar{B}_f C_{1,13})^T \\
\Omega_{35} &= -W_{12} B_{1,i} - \bar{B}_f D_{1,i} \\
\Omega_{36} &= L_{1,i} - (\bar{D}_f C_{1,13})^T \\
\Omega_{44} &= P_{3,i} - 2 \cos(\theta_i) Q_{3,13} - \bar{A}_f - \bar{A}_f^T \\
\Omega_{45} &= -W_{12} B_{1,i} - \bar{B}_f D_{1,i} \\
\Omega_{55} &= -\left( \bar{D}_{1,j} D_{1,13} \right)^T
\end{align*}
\]

(23)

\[
\left[ \begin{array}{cccc}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \rho_1 \bar{A}_f - \rho_2 V_{1,13}^T \\
\Phi_{22} & \Phi_{23} & \Phi_{34} & \rho_1 \bar{A}_f - \rho_2 V_{1,13}^T \\
\star & \star & \star & \rho_1 \bar{A}_f + \rho_2 \bar{A}_f^T \\
\end{array} \right] < 0,
\]

(24)

\[\forall (i, j) \in \mathcal{S} \times \mathcal{S}, \quad 1 \leq i \leq s\]

It is easy to obtain
\[
\Gamma_i = \begin{bmatrix}
-\bar{I} \\
\bar{B}_i(\lambda)^T \\
\end{bmatrix}
\]

(25)

It follows from Projection Lemma (Lemma 1) that if the following inequality
\[
\begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I \\
\end{bmatrix}
\bar{Z}(\lambda) \begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\bar{C}_i(\lambda)^T \bar{C}_i(\lambda)^T < 0
\]

holds for given \( A_i \), then (26) holds. Choose
\[A_i = \begin{bmatrix}0 & I & 0 \end{bmatrix}^T\]

and substitute (16), (27) and (29) into (28), by simple matrix manipulations, we can obtain
\[
\begin{bmatrix}
-P(\lambda) Q(\lambda) + W_1^T \\
\star & \bar{Z}_1 \\
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\bar{C}_i(\lambda)^T \bar{C}_i(\lambda)^T \begin{bmatrix}0 & 0 & 0 \end{bmatrix} < 0
\]

(26)

Define matrix functions
\[
P_i(\lambda) = \begin{bmatrix}
P_{1,i}(\lambda) & P_{2,i}(\lambda) \\
\star & P_{3,i}(\lambda) \\
\end{bmatrix}, \quad Q_i(\lambda) = \begin{bmatrix}Q_{1,i}(\lambda) & Q_{2,i}(\lambda) \end{bmatrix}
\]

(27)

matrix variables
\[
W_i = \begin{bmatrix}W_{1,i} & W_{2,i} \end{bmatrix}
\]

(28)
and

\[ \hat{A}_\beta = W_\beta A_{\beta_i}, \quad \hat{B}_\beta = W_\beta B_{\beta_i}, \quad \hat{C}_\beta = C_{\beta_i}, \quad \hat{D}_\beta = D_{\beta_i} \] (34)

After some substitution of the corresponding matrices, it can be readily established that (31) is equivalent to

\[
\begin{bmatrix}
- P_{11}(\lambda) & - P_{12}(\lambda) & A_{11} & A_{14} & 0 & 0 \\
\star & - P_{23}(\lambda) & A_{23} & A_{24} & 0 & 0 \\
\star & \star & A_{33} & A_{34} & A_{35} & A_{36} \\
\star & \star & \star & A_{44} & A_{45} & - \hat{C}_{\beta}^T \\
\star & \star & \star & \star & - \hat{g}^2 I & A_{56} \\
\star & \star & \star & \star & \star & - I
\end{bmatrix} < 0, \quad \forall i \in \mathcal{I}
\]

(35)

where

\[
\begin{align*}
A_{11} &= Q_{11}(\lambda) + W_{11}^T, \quad A_{14} = Q_{23}(\lambda) + W_{12}^T \\
A_{23} &= Q_{33}(\lambda) + W_{23}^T, \quad A_{24} = Q_{33}(\lambda) + W_{23}^T \\
A_{33} &= P_{11}(\lambda) - 2 \cos(\theta) Q_{11}(\lambda) - \text{He}\{W_{11}A_1(\lambda) + B_\beta C(\lambda)\} \\
A_{34} &= - P_{12}(\lambda) - 2 \cos(\theta) Q_{12}(\lambda) - \hat{A}_\beta - (W_{12}A_1(\lambda) + B_\beta C(\lambda))^T \\
A_{35} &= - W_{12}B_2(\lambda) - \hat{B}_\beta D_2(\lambda) \\
A_{36} &= - D_2^T(\lambda) D_2 \\
A_{44} &= - L_{11}(\lambda) - (D_2 H C(\lambda))^T \\
A_{45} &= - W_{23}B_2(\lambda) - \hat{B}_\beta D_2(\lambda) \\
A_{56} &= - D_2^T(\lambda) D_2
\end{align*}
\]

Assuming matrix functions \( P_i(\lambda) \) and \( Q_i(\lambda) \) to be of the following forms

\[
P_i(\lambda) \triangleq \sum_{i=1}^{s} \lambda_i P_{i,l} \] (36)

\[
Q_i(\lambda) \triangleq \sum_{i=1}^{s} \lambda_i Q_{i,l} \] (37)

and multiplying (22) by the uncertain parameter \( \lambda_i \), and summing it over the index \( l \) from 1 to \( s \), we have (35). Therefore, if (22) holds, then (35) holds, which further implies (15) holds. By Lemma 3, the filtering error system \( \Sigma_e \) satisfies the finite frequency performance index (24).

Then, we consider the asymptotical stability of the filtering error system \( \Sigma_e \). Construct a switched parameter-dependent Lyapunov function as

\[
V(k, \tilde{x}(k)) \triangleq \tilde{x}^T(k) \bar{P}_i(\lambda) \tilde{x}(k), \quad \forall i \in \mathcal{I}
\]

(38)

with \( \bar{P}_i(\lambda)^T = \bar{P}_i(\lambda) > 0 \). From the Lyapunov stability theory, if the following inequality

\[
AV \triangleq V(k + 1, \tilde{x}(k + 1)) - V(k, \tilde{x}(k))
\]

\[
= \tilde{x}^T(k + 1) \bar{P}_i(\lambda) \tilde{x}(k + 1) - \tilde{x}^T(k) \bar{P}_i(\lambda) \tilde{x}(k)
\]

\[
< 0, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I}
\]

(39)

holds, then the filtering error system

\[
\tilde{x}(k + 1) = \bar{A}_i(\lambda) \tilde{x}(k)
\]

(40)

is asymptotically stable. Rewrite (39) and (40) in the form

\[
\begin{bmatrix}
\tilde{x}(k + 1) \\
\tilde{x}(k)
\end{bmatrix} =
\begin{bmatrix}
\bar{P}_i(\lambda) & 0 \\
0 & - \bar{P}_i(\lambda)
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(k + 1) \\
\tilde{x}(k)
\end{bmatrix} < 0
\]

(41)

\[
\begin{bmatrix}
\bar{P}_i(\lambda) & 0 \\
0 & - \bar{P}_i(\lambda)
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(k + 1) \\
\tilde{x}(k)
\end{bmatrix} = 0
\]

(42)

Based on Finsler’s lemma (Lemma 2), if the inequality

\[
\begin{bmatrix}
\bar{P}_i(\lambda) & 0 \\
0 & - \bar{P}_i(\lambda)
\end{bmatrix} + \text{He}\left\{\begin{bmatrix}
\rho_1 W_{11} & 0 \\
0 & \rho_2 W_{12}
\end{bmatrix}\right\} < 0
\]

(43)

holds, then (41) holds. (43) can be rewritten as

\[
\begin{bmatrix}
\bar{P}_i(\lambda) & - \rho_1 W_{11} - \rho_1 W_{12}^T \\
\star & - \rho_2 W_{12}^T
\end{bmatrix} < 0
\]

(44)

Define matrix functions

\[
\bar{P}_i(\lambda) \triangleq \begin{bmatrix}
\bar{P}_i(\lambda) \\
\bar{P}_i(\lambda)
\end{bmatrix}
\]

(45)

and matrices

\[
V_i \triangleq \begin{bmatrix}
V_{11} & W_{12} \\
V_{12} & W_{12}
\end{bmatrix}
\]

(46)

after some substitution of the corresponding matrices and by simple matrix manipulations, then it follows that (44) is equivalent to

\[
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \rho_1 \hat{A}_\beta - \rho_2 V_{12} \\
\star & \Sigma_{22} & \Sigma_{23} & \rho_1 \hat{A}_\beta - \rho_2 W_{12}^T \\
\star & \star & \Sigma_{33} & \Sigma_{34} \\
\star & \star & \star & - \bar{P}_i(\lambda) + \rho_2 \hat{A}_\beta + \rho_2 \hat{A}_\beta^T
\end{bmatrix} < 0,
\]

(47)

\[
\forall (i, j) \in \mathcal{I} \times \mathcal{I}, \quad 1 \leq i \leq s
\]

where

\[
\begin{align*}
\Sigma_{11} &= \bar{P}_1(\lambda) - \rho_1 W_{11} - \rho_1 W_{12}^T \\
\Sigma_{12} &= \bar{P}_2(\lambda) - \rho_1 W_{12} - \rho_1 W_{12}^T \\
\Sigma_{13} &= \rho_1 W_{11}A_1(\lambda) + B_{\beta} C(\lambda) - \rho_2 V_{12}^T \\
\Sigma_{22} &= \bar{P}_3(\lambda) - \rho_1 W_{23} - \rho_2 W_{12}^T \\
\Sigma_{23} &= \rho_1 W_{12}A_1(\lambda) + B_{\beta} C(\lambda) - \rho_2 W_{12}^T \\
\Sigma_{33} &= - \bar{P}_1(\lambda) + \rho_2 \text{He}\left\{W_{11}A_1(\lambda) + B_{\beta} C(\lambda)\right\} \\
\Sigma_{34} &= - \bar{P}_2(\lambda) + \rho_2 A_{\beta_i} + \rho_2 \left(W_{12}A_1(\lambda) + B_{\beta} C(\lambda)\right)^T
\end{align*}
\]

Multiply (23) by the uncertain parameter \( \lambda_i \) and sum it over the index \( l \) from 1 to \( s \), then we have (47) with
Theorem 2 (Middle frequency). Considering the uncertain switched system \( \Sigma_\vartheta \), for given scalars \( \vartheta_1, \vartheta_2 \) and \( \gamma > 0 \), if there exist matrices \( W_{11}, W_{12}, W_{13}, V_{11}, V_{12}, A_{\beta_1}, B_{\beta_1}, C_{\beta_1}, D_{\beta_1} \), scalars \( \rho_1, \rho_2, \) and symmetric matrices \( P_{i,i}, Q_{i,i} > 0, P_{i,i} > 0 \) satisfying (23) and

\[
\begin{bmatrix}
-\rho_{1,i} & -\rho_{2,i} & \Psi_{13} & \Psi_{14} & 0 & 0 \\
* & -\rho_{3,i} & \Psi_{33} & \Psi_{34} & \Psi_{35} & \Psi_{36} \\
* & * & \Psi_{44} & \Psi_{45} & -C_{\beta_1}^T \\
* & * & * & * & -\gamma I & D_{\beta_1}^T D_{\beta_1}^T \\
\end{bmatrix} < 0,
\]

\( \forall i \in \mathcal{J}, \ 1 \leq i \leq s \)  

(48)

where

\[
\begin{aligned}
\Psi_{13} &= e^{\vartheta_1}Q_{11,1} + W_{11}^T \\
\Psi_{14} &= e^{\vartheta_1}Q_{12,1} + W_{12}^T \\
\Psi_{23} &= e^{\vartheta_1}Q_{21,1} + W_{13} \\
\Psi_{24} &= e^{\vartheta_1}Q_{22,1} + W_{13}^T \\
\Psi_{33} &= P_{11,1} - 2 \cos(\vartheta_1)Q_{11,1} - \text{He} \left\{ W_{11}A_{\beta_1} + \bar{B}_{\beta_1}C_{\beta_1} \right\} \\
\Psi_{34} &= P_{12,1} - 2 \cos(\vartheta_1)Q_{12,1} - \hat{A}_{\beta_1} - \left( W_{12}A_{\beta_1} + \bar{B}_{\beta_1}C_{\beta_1} \right)^T \\
\Psi_{35} &= -W_{11}B_{\beta_1} - \bar{B}_{\beta_1}D_{\beta_1} \\
\Psi_{36} &= L_{i,i} - \left( \bar{D}_{\beta_1}C_{\beta_1} \right)^T \\
\Psi_{44} &= P_{13,1} - 2 \cos(\vartheta_1)Q_{13,1} - \hat{A}_{\beta_1} - \hat{A}_{\beta_1}^T \\
\Psi_{45} &= -W_{12}B_{\beta_1} - \bar{B}_{\beta_1}D_{\beta_1} \\
\vartheta_1 &= (\vartheta_1 + \vartheta_2)/2, \quad \vartheta_2 = (\vartheta_1 - \vartheta_2)/2
\end{aligned}
\]

then there exists a switched full-order filter \( \Sigma_i \) such that the filtering error system \( \Sigma_e \) is asymptotically stable and satisfies the specification

\[
\sigma_{\max}(G_{\vartheta_1}(e^{\vartheta})) < \gamma, \quad \forall \vartheta_1 \leq \vartheta \leq \vartheta_2, \ i \in \mathcal{J}
\]

(49)

Moreover, if inequalities (23), (48) are feasible, then a suitable filter can be obtained by (25).

Proof. It is easy to complete the proof by following the same lines as that of Theorem 1. □

Remark 3. Theorems 1–3 provide sufficient conditions for the finite frequency \( H_\infty \) filtering for uncertain discrete-time switched linear systems in different frequency ranges. Numerical examples show that the proposed finite frequency approach has better performances than the existing full frequency ones when the frequency ranges of the noises are known.

Remark 4. In Theorems 1–3, when \( \rho_1, \rho_2 \) are set to be fixed parameters, (22), (23), (48), (50) becomes LMIs that can be solved by the LMI toolbox [26,27]. The reasonable choices of \( \rho_1, \rho_2 \) can be obtained by the heuristic method (such as line search), choosing the ones that make the following performance bound \( \gamma \).
as small as possible.

**Remark 5.** When \( N = 1 \), the switched system \( \Sigma_p \) is reduced to a linear system. Thus, the conditions in Theorems 1–3 can be used to design finite frequency filters for linear systems. However, even for the filter design of linear systems, there are still several differences between our results and those in [25]. First, the proposed approach is applicable to linear systems with polytopic uncertainties, which simultaneously emerge in all system matrices, while the estimation output matrix \( L \) is assumed to be known in [25]. Second, the proposed filter is more general, which has four parameters \((A_f, B_f, C_f, D_f)\), while the filter in [25] has three parameters \((A_f, B_f, C_f)\). The freedom provided by parameter \( D_f \) can lead to less conservativeness. Finally, by choosing the special structure of multipliers \( W_i, V_i \) in (33) and (46), the filter design is considerably simplified.

### 4. Example

Consider the switched system \( \Sigma_p \) consisting of two uncertain subsystems, where there are two vertices in subsystem 1:

\[
A_{11} = \begin{bmatrix} -0.08 & 1.0880 \\ -0.744 & -0.06 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0.4 \\ 0.32 \end{bmatrix}, \\
C_{11} = \begin{bmatrix} 0.4, 0.4 \end{bmatrix}, \quad D_{11} = 0.28, \quad L_{11} = \begin{bmatrix} 0.4, 0.4 \end{bmatrix}
\]

and two vertices in subsystem 2:

\[
A_{12} = \begin{bmatrix} -0.16 & 1.112 \\ -0.744 & -0.06 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix}, \\
C_{12} = \begin{bmatrix} 0.4, 0.4 \end{bmatrix}, \quad D_{12} = 0.28, \quad L_{12} = \begin{bmatrix} 0.4, 0.4 \end{bmatrix}
\]

The purpose here is to design a finite frequency \( H_{\infty} \) filter using the proposed method in the previous section and provide a comparison with the existing full frequency methods.

For a given low-frequency range, e.g., \( |\theta| \leq 0.5 \) (rad/s), letting \( \rho_1 = -3, \rho_2 = 1 \), by Theorem 1, we can obtain the following filter parameters:

\[
A_{f1} = \begin{bmatrix} -0.3898 & 0.0701 \\ -0.5028 & -0.6643 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} -1.4449 \\ -1.1852 \end{bmatrix}, \\
C_{f1} = \begin{bmatrix} -0.0912 & -0.0621 \end{bmatrix}, \quad D_{f1} = 0.3361
\]

or (23), (48) for middle frequency

and the performance bound \( \gamma = 0.0968 \).

For a given middle-frequency range, e.g., \( 0.5 \leq \theta \leq 1 \) (rad/s), letting \( \rho_1 = -3, \rho_2 = 1 \), by Theorem 2, we can obtain the following filter parameters:

\[
A_{f1} = \begin{bmatrix} -0.3381 & -0.0976 \\ -0.8245 & -0.2386 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} -2.5530 \\ -0.6801 \end{bmatrix}, \\
C_{f1} = \begin{bmatrix} 0.0158 & 0.1205 \end{bmatrix}, \quad D_{f1} = 0.2287
\]

or (23), (50) for high frequency

and the performance bound \( \gamma = 0.1073 \).

Finally, for a given high-frequency range, e.g., \( |\theta| \geq 1.5 \) (rad/s), letting \( \rho_1 = -3, \rho_2 = 1 \), by Theorem 3, we can obtain the following filter parameters:

\[
A_{f1} = \begin{bmatrix} -0.0991 & 0.1443 \\ -0.3053 & 0.2519 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} -2.7712 \\ 0.6566 \end{bmatrix}, \\
C_{f1} = \begin{bmatrix} -0.4364 & -0.5425 \end{bmatrix}, \quad D_{f1} = -0.3015
\]

or (23), (50) for high frequency

and the performance bound \( \gamma = 0.6628 \). The performance bounds obtained by Theorems 1–3 and that obtained by the full frequency approach [18] are listed in Table 2. It is easily seen from Table 2 that the proposed finite frequency filter in this paper has better performances than the full frequency one in [18].

For a switching signal \( \sigma(k) \) as depicted in Fig. 1, initial conditions \( x(0) = [-0.2, 0.4]^T \), \( \bar{x}(0) = [0, 0]^T \), white noise with noise power 0.001, and a set of uncertain parameters \( \lambda_1 = 0.2, \lambda_2 = 0.8 \), applying the obtained low-frequency filter, the error response of the resulting filtering error system is shown in Fig. 2. It is observed from the simulation results that the filtering error system is stable under the given switching signal. Therefore, the designed low-frequency filter is feasible and effective. In the same way, the designed

<table>
<thead>
<tr>
<th>Methods</th>
<th>Theorem 1</th>
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<th>Theorem 1</th>
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<tr>
<td>(</td>
<td>\theta</td>
<td>\leq 0.5)</td>
<td>(</td>
<td>\theta</td>
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<tr>
<td>(\rho_1 = -3)</td>
<td>(\rho_1 = -3)</td>
<td>(\rho_1 = -3)</td>
<td>(\epsilon_1 = 1)</td>
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<td>(\rho_2 = 1)</td>
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<td>(\rho_2 = 1)</td>
<td>(\epsilon_1 = 1)</td>
<td>(\epsilon_2 = 1)</td>
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<tr>
<td>(\gamma_{\min})</td>
<td>0.0968</td>
<td>0.1073</td>
<td>0.6628</td>
<td>0.9372</td>
</tr>
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</table>
middle-frequency and high-frequency filters also work well. Due to the limit of the space, the simulation results are omitted.

5. Conclusions

In this paper, we have designed a finite frequency $H_\infty$ filter for the uncertain discrete-time switched linear system via the GKYP lemma and switched Lyapunov functions. Compared with the existing full frequency filters, the proposed finite frequency one has better performances when the frequency ranges of external noises are known. An example has been given to illustrate its effectiveness.

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