# GROTHENDIECK CATEGORIES

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ABSTRACT. The general theory of Grothendieck categories is presented. The Popescu-Gabriel generalized theorem and theorems for projective generating families are proved. Due to these results arbitrary Grothendieck categories as quotient categories of $\mathcal{C}_{\mathcal{A}} = (\operatorname{mod} \mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ with $\operatorname{mod} \mathcal{A}^{\mathrm{op}}$ the category of finitely presented objects of some functor category are presented. Also, various finiteness conditions for localizing subcategories are studied in detail, as well as their applications for the theory of rings and modules.

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## Introduction

To investigate abstract properties for the categories of sheaves, Grothendieck [**1**] has introduced the concept of an Ab5-category with the family of generators. Later this class of categories began to call Grothendieck categories. It was firstly not clear, whether can have the Grothendieck categories any interesting applications. The question was resolved with Gabriel's dissertation [**2**] in which the basic tool for the study of Grothendieck categories was proposed: the localization theory. In particular, this theory works in full force if it to apply to the theory of rings and modules. Here we come to deep and strong generalizations of classical results for the theory of fractions, and the tool of torsion/localizing functors has turned in the given context in a standard language of the description. During long time from the moment of its development by Gabriel the localization theory basically was applied in this direction. Monographies where the general theory of fractions is rather full stated: it first of all Stenström's book [**3**], and also books [**4, 5**].

From time of the publication of Ziegler's work [**6**] applications of Grothendieck categories were enriched due to interaction with the model theory. This theory (see also Prest's monography [**7**]) has brought with itself essentially new principles and statements of questions which concern only algebraic objects. For this reason it has caused a number of investigations, which purpose is to translate model-theoretic idioms to algebraic language.

It has turned out that the basic model-theoretic concepts are realized in the category of generalized modules

$$\mathcal{C}_A = (\operatorname{mod} A^{\mathrm{op}}, \mathrm{Ab})$$

that consists of additive covariant functors defined on the category finitely presented left modules $\operatorname{mod} A^{\mathrm{op}}$ with values in the category of Abelian groups Ab [**8**]. The category $\mathcal{C}_A$ has a number of remarkable properties. First, it is locally coherent [**9**] and every Grothendieck category is a quotient category of $\mathcal{C}_A$ [**10**]. In particular, from here interesting applications for some problems of the theory of rings and modules are obtained: the majority of statements for the classes of $FP$-injective and weakly quasi-Frobenius rings work in full force in a context of torsion functors of the category $\mathcal{C}_A$ or even are senseless outside of it [**11, 12**]. Second, many problems for the representation theory of artinian algebras in a natural way admit reformulations in the category $\mathcal{C}_A$. Such an approach to problems of the representation theory has been proposed in seventieth years by Auslander which is until now used by various mathematicians. Here we recommend Krause's dissertation [**13**] in which the reader further on the theme will find references interesting him and results. Thirdly, in $\mathcal{C}_A$ one of fundamental model-theoretic concepts is realized: the Ziegler spectrum of a ring. This topological space was constructed by Ziegler [**6**] in model-theoretic terms, and recently Herzog [**14**] and Krause [**15**] have proposed algebraic definition of the Ziegler topology for arbitrary locally coherent Grothendieck categories. As a whole it is necessary to tell, that the category $\mathcal{C}_A$ plus the pair $(t_{\mathcal{S}}, (-)_{\mathcal{S}})$ of torsion/localizing functors turns out the good tool every time when in the model theory it appears something new about the category of modules. It is the good heuristic method in the given context.

The listed above reasons became the main stimulus for writing the paper, which the basic purpose is to give the closed exposition (in maximum general view) for the basic results of the theory. Actually central theorems in the work are the Popescu-Gabriel generalized theorem (Theorem 4.1) and theorems for projective generating sets (§4.2). Due to these theorems we present any Grothendieck category $\mathcal{C}$ as a quotient category of $\mathcal{C}_{\mathcal{A}} = (\operatorname{mod} \mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$; here $\mathcal{A}$ denotes a preadditive category formed by a family of generators $\mathcal{U}$ of the category $\mathcal{C}$. The big attention in the work is given the localizing subcategories of finite type as many interesting applications are connected to them. We only concern some applications for the theory of rings and modules, at all not mentioning other applications, that considerably would increase our exposition, such as the model theory of locally finitely presented Grothedieck categories or the representation theory. For them further under the text the reader will find the appropriate references.

The work is organized as follows. The first section is preliminary, collecting the necessary category-theoretic background. In the second section Gabriel's topology and localization in the functor categories are discussed. The following section contains a technical material which is required to us for the proof of Popescu-Gabriel's generalized theorem. The principle section is fourth in which Popescu-Gabriel's generalized theorem is proved, and also projective generating sets are considered. In the fifth section we show how various finiteness conditions for a family of generators $\mathcal{U}$ of a Grothendieck category $\mathcal{C}$ reflect some finiteness conditions for localizing subcategories. In particular, theorem of Breitsprecher is proved here. The basic properties for the categories of generalized modules, including duality of Auslander, Gruson and Jensen, and theorem of Herzog, are resulted in the sixth section. In the remaining sections we present the Grothendieck categories as quotient categories of $\mathcal{C}_{\mathcal{A}}$ and illustrate how localizing subcategories of $\mathcal{C}_{\mathcal{A}}$ are used to study rings and modules.

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**Notation.** When there is no doubt about the ring $A$ or the category $\mathcal{B}$, we usually abbreviate $\mathrm{Hom}_A(M,N)$ or $\mathrm{Hom}_{\mathcal{B}}(M,N)$ as $(M,N)$. We shall freely invoke the fact that every object $X \in \mathcal{C}$ of a Grothendieck category $\mathcal{C}$ has an injective envelope $E(X) \in \mathcal{C}$ (for details see [**16**]). This fact is also discussed in section 4. If $\mathcal{B}$ is a category, then by a subcategory $\mathcal{A}$ of $\mathcal{B}$ we shall always mean a full subcategory of $\mathcal{B}$. For concepts such as subobject, epimorphism, injectivity, etc. we shall use the prefix $\mathcal{A}$-subobject or $\mathcal{B}$-subobject to indicate the context. This prefix can be omitted if the concept in question is absolute with respect to the inclusion $\mathcal{A} \subseteq \mathcal{B}$. To indicate the context of an operation, for example $\operatorname{Ker} \mu$, $E(X)$ or $\varinjlim X_i$, we shall use a subscript, for example, $\operatorname{Ker}_{\mathcal{A}} \mu$, $E_{\mathcal{A}}(X)$, etc. which may also be omitted in case of absoluteness.

## 1. Preliminaries

In this preliminary section we collect the basic facts about Grothendieck categories. Some of them can be considered for more general (than Grothendieck) categories. For details and proofs we refer the reader to [**2, 17**].

1.1. **Ab-conditions.** Recall that an Abelian category $\mathcal{C}$ is *cocomplete* or an *Ab3-category* if it has arbitrary direct sums. A cocomplete Abelian category $\mathcal{C}$ is called an *Ab5-category* if for any directed family $\{A_i\}_{i\in I}$ of subobjects of $A$ and for any subobject $B$ of $A$, the relation

$$(\sum_{i\in I} A_i) \cap B = \sum_{i\in I}(A_i \cap B)$$

holds.

The condition Ab3 is equivalent to the existence in $\mathcal{C}$ of arbitrary inductive limits, and the condition Ab5 is equivalent to that the inductive limit functor indexed by the directed set is exact. Namely, if $I$ is the directed set,

$$\varepsilon_i : 0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$

is an exact sequence in $\mathcal{C}$, then $\varinjlim \varepsilon_i$ is also an exact sequence.

Let $\mathcal{C}$ be a category and $\mathcal{U} = \{U_i\}_{i\in I}$ a family of objects of $\mathcal{C}$. The family $\mathcal{U}$ is called the *family of generators* of the category $\mathcal{C}$ if for any object $A$ of $\mathcal{C}$ and any subobject $B$ of $A$ distinct from $A$ there exists at least an index $i \in I$ and a morphism $u : U_i \to A$ that cannot be factorized through the canonical injection $i : B \to A$ of $B$ into $A$. An object $U$ of $\mathcal{C}$ is called a *generator* of the category $\mathcal{C}$ provided that the family $\{U\}$ is the family of generators for the category $\mathcal{C}$.

If $\mathcal{C}$ is an Ab3-category, then $\mathcal{U} = \{U_i\}_{i\in I}$ is a family of generators for $\mathcal{C}$ if and only if $\oplus_{i\in I} U_i$ is a generator [**18**, 5.33]. According to [**18**, 5.35] an Ab3-category $\mathcal{C}$ possessing a family of generators $\mathcal{U}$ is locally finite. Furthemore, every object of the category $\mathcal{C}$ is isomorphic to a quotient object of $\oplus_{j\in J} U_j$ with $J$ some set of indices, $U_j \in \mathcal{U}$ for any $j \in J$ [**18**, 5.34].

The Ab5-categories possessing a family of generators are called *Grothendieck categories.*

**Examples.** (1) The category of right $A$-modules $\operatorname{Mod} A$, where $A$ is a ring with identity, the category of (pre-)sheaves of $A$-modules defined on an arbitrary topological space are Grothendieck categories.

(2) Let $\mathcal{B}$ be a preadditive small category. We denote by $(\mathcal{B}, \mathrm{Ab})$ the category whose objects are the additive functors $F : \mathcal{B} \to \mathrm{Ab}$ from $\mathcal{B}$ to the category of Abelian groups Ab and whose morphisms are the natural transformations between functors. That it is Grothendieck follows from [**3**, V.2.2]. Moreover, the family of representable functors $\{h^B = (B, -)\}_{B\in\mathcal{B}}$ is a family of projective generators for $(\mathcal{B}, \mathrm{Ab})$ [**3**, IV.7.5].

Let $\mathcal{C}$ be an Abelian category and $\mathcal{U} = \{U_i\}_{i\in I}$ some set of objects of $\mathcal{C}$. Consider $\mathcal{U}$ as a small preadditive category and let $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ be the category of the additive contravariant functors from $\mathcal{U}$ to Ab. By $T : \mathcal{C} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ we denote the functor defined as follows.

$$TX = {}_{\mathcal{C}}(-, X),\ Tf = (-, f)$$

with $X \in \mathcal{C}$ and $f$ a morphism of $\mathcal{C}$.

**Proposition 1.1.** *The functor $T : \mathcal{C} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ defined above is faithful if and only if $\mathcal{U}$ is the family of generators for $\mathcal{C}$.*

*Proof.* Assume $T$ is faithful. Let $i : X' \to X$ be a monomorphism of $\mathcal{C}$, which is not an isomorphism and let $j : X \to X/X'$ be the cokernel of $i$. If $Ti$ is an isomorphism, it follows that $Tj = 0$ since $T$ is left exact. Therefore $j = 0$, a contradiction.

Conversely, suppose $\mathcal{U}$ is the family of generators for $\mathcal{C}$ and $f : X \to Y$ is a morphism in $\mathcal{C}$ such that $Tf = 0$. If $f = pj$ is the canonical decomposition of $f$ with $p$ a monomorphism and $j$ an epimorphism, then $Tj = 0$ and if $i : \operatorname{Ker} f \to X$ is the kernel of $j$, we get that $Ti$ is an isomorphism. Since $\mathcal{U}$ is the family of generators, $i$ is an isomorphism and therefore $j = 0$, hence $f = 0$. □

1.2. **Localization in Grothendieck categories.** A subcategory $\mathcal{S}$ of the Grothendieck category $\mathcal{C}$ is said to be *closed under extensions* if for any short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

in $\mathcal{C}$ with $X', X'' \in \mathcal{S}$ the object $X$ belongs to $\mathcal{S}$. The subcategory $\mathcal{S}$ is a *Serre subcategory* provided that it is closed under extensions, subobjects, and quotient objects. The corresponding *quotient category* $\mathcal{C}/\mathcal{S}$ is constructed as follows. The objects of $\mathcal{C}/\mathcal{S}$ are those of $\mathcal{C}$ and

$${}_{\mathcal{C}/\mathcal{S}}(X, Y) = \varinjlim {}_{\mathcal{C}}(X', Y/Y')$$

with $X' \subseteq X$, $Y' \subseteq Y$ and $X/X', Y' \in \mathcal{S}$. The set of such pairs is a partially ordered directed set with respect to the relation $(X', Y/Y') \leq (X'', Y/Y'')$, which holds if and only if $X'' \subseteq X'$ and $Y' \subseteq Y''$. The direct limit is indexed by this partial order. Again $\mathcal{C}/\mathcal{S}$ is Abelian and there is the canonicaly defined *quotient functor* $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ such that $q(X) = X$; it is exact with $\operatorname{Ker} q = \mathcal{S}$ (see [**2, 17**]). Here, by definition, the *kernel* $\operatorname{Ker} f$ of a functor $f : \mathcal{C} \to \mathcal{D}$ is the subcategory of all objects $X$ such that $f(X) = 0$.

A Serre subcategory $\mathcal{S}$ of $\mathcal{C}$ is said to be *localizing* provided that the corresponding quotient functor admits a right adjoint $s : \mathcal{C}/\mathcal{S} \to \mathcal{C}$. Note that $\mathcal{S}$ is localizing if and only if $\mathcal{S}$ is closed under taking coproducts [**17**, 15.11]. In this case, $\mathcal{S}$ and $\mathcal{C}/\mathcal{S}$ are again Grothendieck categories [**2**, III.4.9]. Moreover, the inclusion functor $\mathcal{S} \to \mathcal{C}$ admits a right adjoint $t = t_{\mathcal{S}} : \mathcal{C} \to \mathcal{S}$ that assigns to $X \in \mathcal{C}$ the largest subobject $t(X)$ of $X$ belonging to $\mathcal{S}$ [**15**, 2.1]. The functor $t_{\mathcal{S}}$ is called the *torsion functor generated by* $\mathcal{S}$. An object $X$ is said to be $\mathcal{S}$*-periodic* or simply *periodic* provided that $t_{\mathcal{S}}(X) = X$. Furthermore, for any object $X \in \mathcal{C}$ there is the natural morphism $\lambda_X : X \to sq(X)$ such that $\operatorname{Ker} \lambda_X, \operatorname{Coker} \lambda_X \in \mathcal{S}$ and $\operatorname{Ker} \lambda_X = t_{\mathcal{S}}(X)$ (see [**17**]).

We say that an object $X \in \mathcal{C}$ is $\mathcal{S}$*-closed* (respectively $\mathcal{S}$*-torsionfree*) provided that $\lambda_X$ is an isomorphism (respectively a monomorphism). Thus the *section functor* $s$ induces an equivalence between $\mathcal{C}/\mathcal{S}$ and the subcategory of $\mathcal{S}$-closed objects in $\mathcal{C}$ [**17**, 15.19B]. Furthermore, by [**15**, 2.2] the subcategory of $\mathcal{S}$-closed objects coincides with the *perpendicular category* $\mathcal{S}^{\perp}$ consisting of $X \in \mathcal{C}$ such that ${}_{\mathcal{C}}(S, X) = 0$ and $\operatorname{Ext}^1_{\mathcal{C}}(S, X) = 0$ for any $S \in \mathcal{S}$. Henceforth, the object $sq(X)$ is denoted by $X_{\mathcal{S}}$ and the morphism $sq(\alpha)$ is denoted by $\alpha_{\mathcal{S}}$ for every $X \in \mathcal{C}$ and $\alpha \in \operatorname{Mor} \mathcal{C}$. The morphism

$\lambda_X$ we shall call the $\mathcal{S}$-*envelope* of the object $X$. Thus for any object $X$ of $\mathcal{C}$ there is an exact sequence

$$0 \longrightarrow A' \longrightarrow X \xrightarrow{\lambda_X} X_{\mathcal{S}} \longrightarrow A'' \longrightarrow 0, \tag{1.1}$$

with $A'$, $A'' \in \mathcal{S}$ and $\lambda_X$ the $\mathcal{S}$-envelope for $X$. Note that any two $\mathcal{S}$-envelopes $\lambda^i_X : X \to X_{\mathcal{S}}$, $i = 1, 2$, of $X \in \mathcal{C}$ are isomorphic and $X_{\mathcal{S}} \approx (X_{\mathcal{S}})_{\mathcal{S}}$. Also, note that $X_{\mathcal{S}} = 0$ if and only if the object $X$ belongs to $\mathcal{S}$.

**Proposition 1.2.** [**17**, 15.19C] *Let $X$ be $\mathcal{S}$-torsionfree. A monomorphism $\mu : X \to Y$ is an $\mathcal{S}$-envelope if and only if $Y$ is $\mathcal{S}$-closed and $X/Y \in \mathcal{S}$. In this case, the following properties hold:*

(1) *$\mu$ is an essential monomorphism;*

(2) *if $E$ is an essential extention of $Y$, then both $E$ and $E/Y$ are $\mathcal{S}$-torsionfree.*

*Conversely, if the conditionds* (1) *and* (2) *hold and $Y/X \in \mathcal{S}$, then $\mu$ is an $\mathcal{S}$-envelope. Moreover, if $E(X)$ is an injective envelope of $X$ and $X$ is $\mathcal{S}$-torsionfree, then its $\mathcal{S}$-envelope is the largest subobject $D$ of $E(X)$ containing $X$ such that $D/X \in \mathcal{S}$. Thus an $\mathcal{S}$-torsionfree object $X$ is $\mathcal{S}$-closed if and only if $E(X)/X$ is $\mathcal{S}$-torsionfree.*

Consider the *localizing functor* $(-)_{\mathcal{S}} : \mathcal{C} \to \mathcal{S}^{\perp}$, $(-)_{\mathcal{S}} = sq$, where $\mathcal{S}^{\perp}$ is the subcategory consisting of $\mathcal{S}$-closed objects. Obviously, the inclusion functor $i : \mathcal{S}^{\perp} \to \mathcal{C}$ is fully faithful, and the localizing functor $(-)_{\mathcal{S}}$ is exact, because $q$ is exact and the functor $s$, as we have observed above, induces an equivalence of $\mathcal{S}^{\perp}$ and $\mathcal{C}/\mathcal{S}$. Suppose $X$, $Y$ are some objects of $\mathcal{C}$ and $\alpha \in {}_{\mathcal{C}}(X, Y)$. Then $(-)_{\mathcal{S}}(\alpha) = \alpha_{\mathcal{S}} = (\lambda_Y \alpha)_{\mathcal{S}}$, where $\lambda_Y$ is the $\mathcal{S}$-envelope for $Y$. Clearly, $\alpha_{\mathcal{S}} = 0$ if and only if $\operatorname{Im} \alpha \subseteq t_{\mathcal{S}}(Y)$. From here it easily follows that given $X \in \mathcal{C}$, $Y \in \mathcal{S}^{\perp}$ there is an isomorphism ${}_{\mathcal{C}}(X, Y) \approx {}_{\mathcal{S}^{\perp}}(X_{\mathcal{S}}, Y)$, and, hence, $i$ is right adjoint to the localizing functor $(-)_{\mathcal{S}}$. On the other hand, if $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck categories, $q' : \mathcal{C} \to \mathcal{D}$ is some exact functor, and a functor $s' : \mathcal{D} \to \mathcal{C}$ is fully faithful and right adjoint to $q'$, then $\operatorname{Ker} q'$ is a localizing subcategory and there exists an equivalence $\mathcal{C}/\operatorname{Ker} q' \overset{H}{\approx} \mathcal{D}$ such that $Hq' = q$ with $q$ the canonical functor [**17**, 15.18].

Later the quotient category $\mathcal{C}/\mathcal{S}$ always means the subcategory of $\mathcal{S}$-closed objects $\mathcal{S}^{\perp}$ with the pair of functors $(i, (-)_{\mathcal{S}})$, where $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ is the inclusion functor, $(-)_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is the localizing functor.

**Lemma 1.3.** (1) *$E \in \mathcal{C}/\mathcal{S}$ is a $\mathcal{C}/\mathcal{S}$-injective object if and only if it is $\mathcal{C}$-injective.*

(2) *Every $\mathcal{S}$-torsionfree $\mathcal{C}$-injective object $E$ is $\mathcal{S}$-closed.*

*Proof.* (1). The inclusion functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ preserves injectivity since it is right adjoint to the exact functor $(-)_{\mathcal{S}}$. If $E \in \mathcal{C}/\mathcal{S}$ is $\mathcal{C}$-injective, then any $\mathcal{C}/\mathcal{S}$-monomorphism $\mu : E \to X$ is also a $\mathcal{C}$-monomorphism, and, hence, splits.

The second assertion follows from Proposition 1.2. □

**Lemma 1.4.** *Let $\alpha : X \to Y$ be some morphism of $\mathcal{C}/\mathcal{S}$. Then:*

(1) *the $\mathcal{C}$-kernel of $\alpha$ is an $\mathcal{S}$-closed object;*

(2) *$\alpha$ is a $\mathcal{C}/\mathcal{S}$-epimorphism if and only if $Y/\operatorname{Im}_{\mathcal{C}} \alpha \in \mathcal{S}$.*

*Proof.* (1). It suffices to observe that the inclusion functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ is left exact since it is right adjoint to the localizing functor $(-)_{\mathcal{S}}$.

(2). By localizing the exact sequence

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Y/\operatorname{Im}_{\mathcal{C}} \alpha \longrightarrow 0$$

with $\beta = \operatorname{Coker} \alpha$, we get that $\beta_{\mathcal{S}} = 0$ that implies $(Y/\operatorname{Im}_{\mathcal{C}} \alpha)_{\mathcal{S}} = 0$. □

In particular, a $\mathcal{C}/\mathcal{S}$-morphism is a monomorphism if and only if it is a $\mathcal{C}$-monomorphism. We shall refer to this as the *absoluteness* of monomorphism. So for $A, B \in \mathcal{C}/\mathcal{S}$ the relation $A \leq B$ holds in $\mathcal{C}/\mathcal{S}$ if and only if it holds in $\mathcal{C}$. Also, it is easy to show that for a $\mathcal{C}$-morphism $\alpha : X \to Y$ the $\mathcal{C}/\mathcal{S}$-morphism $\alpha_{\mathcal{S}}$ is a $\mathcal{C}/\mathcal{S}$-monomorphism if and only if $\operatorname{Ker} \alpha \in \mathcal{S}$, and $\alpha_{\mathcal{S}}$ is a $\mathcal{C}/\mathcal{S}$-epimorphism if and only if $Y/\operatorname{Im} \alpha \in \mathcal{S}$. Finally, $\alpha_{\mathcal{S}}$ is a $\mathcal{C}/\mathcal{S}$-isomorphism if and only if $\operatorname{Ker} \alpha \in \mathcal{S}$ and $Y/\operatorname{Im} \alpha \in \mathcal{S}$.

1.3. **Lattices of localizing subcategories.** Let $\mathcal{C}$ be a Grothendieck category with a family of generators $\mathcal{U} = \{U_i\}_{i \in I}$. Denote by $\mathsf{L}(\mathcal{C})$ the lattice consisting of the localizing subcategories of $\mathcal{C}$ ordered by inclusion.

We refer to $X \in \mathcal{C}$ as a *$\mathcal{U}$-finitely generated object* provided that there is an epimorphism $\oplus_{i=1}^n U_i \to X$ with $U_i \in \mathcal{U}$. The subcategory consisting of $\mathcal{U}$-finitely generated objects is denoted by $\operatorname{fg}_{\mathcal{U}} \mathcal{C}$. The fact that $\mathsf{L}(\mathcal{C})$ is a set follows from that any localizing subcategory $\mathcal{S}$ is generated by intersection $\operatorname{fg}_{\mathcal{U}} \mathcal{S} = \mathcal{S} \cap \operatorname{fg}_{\mathcal{U}} \mathcal{C}$. This means that every object $X \in \mathcal{S}$ can be written as a direct union $\sum X_i$ of objects from $\operatorname{fg}_{\mathcal{U}} \mathcal{S}$. Since the category $\operatorname{fg}_{\mathcal{U}} \mathcal{C}$ is skeletally small, $\mathsf{L}(\mathcal{C})$ is indeed a set.

**Proposition 1.5.** [**10**, 2.7] *Suppose $\mathcal{P}$ and $\mathcal{S}$ are localizing subcategories of $\mathcal{C}$; then $\mathcal{P} \subseteq \mathcal{S}$ if and only if $\mathcal{C}/\mathcal{S}$ is the quotient category of $\mathcal{C}/\mathcal{P}$ with respect to the localizing subcategory $\mathcal{S}/\mathcal{P} = \{X \in \mathcal{C}/\mathcal{P} \mid X_{\mathcal{S}} = 0\}$.*

**Proposition 1.6.** *Let $\mathcal{P}$ be a localizing subcategory of $\mathcal{C}$ and $\mathcal{A}$ be a localizing subcategory of $\mathcal{C}/\mathcal{P}$. Then there exists a localizing subcategory $\mathcal{S}$ of $\mathcal{C}$ containing $\mathcal{P}$ such that $\mathcal{S}/\mathcal{P} = \mathcal{A}$.*

*Proof.* Suppose the pair

$$i_{\mathcal{P}} : \mathcal{C}/\mathcal{P} \longrightarrow \mathcal{C},\ (-)_{\mathcal{P}} : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{P}$$

determines $\mathcal{C}/\mathcal{P}$ as a quotient category of $\mathcal{C}$. Also, suppose

$$i_{\mathcal{A}} : (\mathcal{C}/\mathcal{P})/\mathcal{A} \longrightarrow \mathcal{C}/\mathcal{P},\ (-)_{\mathcal{A}} : \mathcal{C}/\mathcal{P} \longrightarrow (\mathcal{C}/\mathcal{P})/\mathcal{A}$$

determines $(\mathcal{C}/\mathcal{P})/\mathcal{A}$ as a quotient category of $\mathcal{C}/\mathcal{P}$.

Denote by

$$Q = (-)_{\mathcal{A}} \circ (-)_{\mathcal{P}} : \mathcal{C} \longrightarrow (\mathcal{C}/\mathcal{P})/\mathcal{A}$$
$$I = i_{\mathcal{P}} \circ i_{\mathcal{A}} : (\mathcal{C}/\mathcal{P})/\mathcal{A} \longrightarrow \mathcal{C}.$$

Being the composition of exact fuctors, $Q$ is an exact functor. Similarly, being the composition of fully faithful fuctors, $I$ is a fully faithful functor. Furthermore, given $X \in \mathcal{C}$ and $Y \in (\mathcal{C}/\mathcal{P})/\mathcal{A}$, we have

$${}_{\mathcal{C}}(X, Y) \approx {}_{\mathcal{C}/\mathcal{P}}(X_{\mathcal{P}}, Y) \approx {}_{(\mathcal{C}/\mathcal{P})/\mathcal{A}}(Q(X), Y).$$

Hence $Q$ is a left adjoint functor to $I$. Thus the pair $(I, Q)$ determines $(\mathcal{C}/\mathcal{P})/\mathcal{A}$ as a quotient category of $\mathcal{C}$ with respect to the localizing subcategory $\mathcal{S} = \operatorname{Ker} Q$. By construction of $\mathcal{S}$ it is easy to show that $\mathcal{P} \subseteq \mathcal{S}$ and $\mathcal{S}/\mathcal{P} = \mathcal{A}$. □

Given a localizing subcategory $\mathcal{P}$ of $\mathcal{C}$, consider the following sublattice of $\mathsf{L}(\mathcal{C})$:

$$\mathsf{L}_{\mathcal{P}}(\mathcal{C}) = \{\mathcal{S} \in \mathsf{L}(\mathcal{C}) \mid \mathcal{S} \supseteq \mathcal{P}\}.$$

**Corollary 1.7.** *If $\mathcal{P}$ is a localizing subcategory of $\mathcal{C}$, then the map*

$$L : \mathsf{L}_{\mathcal{P}}(\mathcal{C}) \longrightarrow \mathsf{L}(\mathcal{C}/\mathcal{P}), \ \mathcal{S} \longmapsto \mathcal{S}/\mathcal{P}$$

*is a lattice isomorphism.*

Also, note that for any $\mathcal{S} \in \mathsf{L}_{\mathcal{P}}(\mathcal{C})$

$$\mathcal{S}/\mathcal{P} = \mathcal{S}_{\mathcal{P}} = \{S_{\mathcal{P}} \mid S \in \mathcal{S}\}.$$

Indeed, clearly, $\mathcal{S}/\mathcal{P} \subset \mathcal{S}_{\mathcal{P}}$. In turn, for $S \in \mathcal{S}$ consider the exact sequence (1.1)

$$0 \longrightarrow A' \longrightarrow S \longrightarrow S_{\mathcal{P}} \longrightarrow A'' \longrightarrow 0$$

with $A'$, $A'' \in \mathcal{P}$. Since $\mathcal{P} \subseteq \mathcal{S}$, it follows that $A'$, $A'' \in \mathcal{S}$. Hence $S_{\mathcal{P}} \in \mathcal{S}$ and since $S_{\mathcal{P}}$ is $\mathcal{P}$-closed, one gets $S_{\mathcal{P}} \in \mathcal{S}/\mathcal{P}$.

1.4. **Locally finitely presented Grothendieck categories.** Throughout this paragraph we fix a Grothendieck category $\mathcal{C}$. We define here the most impotant subcategories of $\mathcal{C}$, essentially used further. Namely, we describe the subcategories consisting of finitely generated, finitely presented and coherent objects respectively. These categories are ordered by inclusion as follows:

$$\mathcal{C} \supseteq \operatorname{fg}\mathcal{C} \supseteq \operatorname{fp}\mathcal{C} \supseteq \operatorname{coh}\mathcal{C}.$$

Recall an object $A \in \mathcal{C}$ is *finitely generated* if whenever there are subobjects $A_i \subseteq A$ for $i \in I$ satisfying $A = \sum_{i\in I} A_i$, then there is a finite subset $J \subset I$ such that $A = \sum_{i\in J} A_i$. The category of finitely generated subobjects of $\mathcal{C}$ is denoted by $\operatorname{fg}\mathcal{C}$. The category is *locally finitely generated* provided that every object $X \in \mathcal{C}$ is a directed sum $X = \sum_{i\in I} X_i$ of finitely generated subobjects $X_i$, or equivalently, $\mathcal{C}$ possesses a family of finitely generated generators.

**Theorem 1.8.** [**3**, V.3.2] *An object $C \in \mathcal{C}$ is finitely generated if and only if the canonical homomorphism $\Phi : \varinjlim {}_{\mathcal{C}}(C, D_i) \to {}_{\mathcal{C}}(C, \sum D_i)$ is an isomorphism for every object $D \in \mathcal{C}$ and every directed family $\{D_i\}_I$ of subobjects of $D$.*

A finitely generated object $B \in \mathcal{C}$ is *finitely presented* provided that every epimorphism $\eta : A \to B$ with $A$ finitely generated has a finitely generated kernel $\operatorname{Ker}\eta$. The subcategory of finitely presented objects of $\mathcal{C}$ is denoted by $\operatorname{fp}\mathcal{C}$. The corresponding categories of finitely presented left and right $A$-modules over the ring $A$ are denoted by $\operatorname{mod} A^{\mathrm{op}} = \operatorname{fp}(\operatorname{Mod} A^{\mathrm{op}})$ and $\operatorname{mod} A = \operatorname{fp}(\operatorname{Mod} A)$, respectively. Note that the subcategory $\operatorname{fp}\mathcal{C}$ of $\mathcal{C}$ is closed under extensions. Moreover, if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence in $\mathcal{C}$ with $B$ finitely presented, then $C$ is finitely presented if and only if $A$ is finitely generated.

The most obvious example of a finitely presented object of $\mathcal{C}$ is a finitely generated projective object $P$. We say that $\mathcal{C}$ has *enough* finitely generated projectives provided that every finitely generated object $A \in \mathcal{C}$ admits an epimorphism $\eta : P \to A$ with $P$ a finitely generated projective object. If $\mathcal{C}$ has enough finitely generated projectives, then by the remarks above, every finitely presented object $B \in \mathcal{C}$ is isomorphic to the cokernel of a morphism between finitely generated projective objects. This is expressed by an exact sequence

$$P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

called a *projective presentation* of $B \in \operatorname{fp}\mathcal{C}$.

**Examples.** The category $\operatorname{Mod} A$ of left $A$-modules has enough finitely generated projectives.

Another example of the category having enough finitely generated projectives is the category of functors $(\mathcal{B}, \mathrm{Ab})$ from a small preadditive category $\mathcal{B}$ to the category of Abelian groups Ab. In this category every finitely generated projective object is a coproduct factor of a finite coproduct of representable objects $\oplus_{i=1}^{n}(B_i, -)$ (see [**14**, §1.2]). In addition, if $\mathcal{B}$ is an additive category, that is $\mathcal{B}$ is preadditive, has finite products/coproducts and idempotents split in $\mathcal{B}$, then every finitely generated projective object in $(\mathcal{B}, \mathrm{Ab})$ is representable [**14**, 2.1].

The category $\mathcal{C}$ is *locally finitely presented* provided that every object $B \in \mathcal{C}$ is a direct limit $B = \varinjlim B_i$ of finitely presented objects $B_i$, or equivalently, $\mathcal{C}$ possesses a family of finitely presented generators. As an example, every locally finitely generated Grothendieck category having enough finitely generated projectives $\{P_i\}_{i \in I}$ is locally finitely presented. In this case, $\{P_i\}_{i \in I}$ are generators for $\mathcal{C}$. For instance, the set of representable functors $\{h^B\}_{B \in \mathcal{B}}$ of the functor category $(\mathcal{B}, \mathrm{Ab})$ with $\mathcal{B}$ a small preadditive category form a family of finitely generated projective generators for $(\mathcal{B}, \mathrm{Ab})$. Therefore $(\mathcal{B}, \mathrm{Ab})$ is a locally finitely presented Grothendieck category (see [**14**, 1.3]).

**Theorem 1.9.** [**3**, V.3.4] *Let $\mathcal{C}$ be a locally finitely generated Grothendieck category. An object $B \in \mathcal{C}$ is finitely presented if and only if the functor ${}_{\mathcal{C}}(B, -) : \mathcal{C} \to \mathrm{Ab}$ commutes with the direct limits.*

A finitely presented object $C \in \mathcal{C}$ is *coherent* provided that every finitely generated subobject $B \subseteq C$ is finitely presented. Evidently, any finitely generated subobject of a coherent object is also coherent. The subcategory of coherent objects of $\mathcal{C}$ is denoted by $\operatorname{coh}\mathcal{C}$. The category $\mathcal{C}$ is *locally coherent* provided that every object of $\mathcal{C}$ is a direct limit of coherent objects. Equivalently, $\operatorname{fp}\mathcal{C}$ is Abelian [**19**, §2] or if $\mathcal{C}$ possesses a family of coherent generators. For example, the category of left $A$-modules is locally coherent if and only if the ring $A$ is left coherent.

In order to characterize the fact that $(\mathcal{B}, \mathrm{Ab})$ with $\mathcal{B}$ an additive category is locally coherent, that is $\operatorname{fp}\mathcal{C} = \operatorname{coh}\mathcal{C}$ [**19**, §2], recall that a morphism $\psi : Y \to Z$ is a *pseudo-cokernel* for $\varphi : X \to Y$ in $\mathcal{B}$ if the sequence $h^Z \overset{(\psi,-)}{\to} h^Y \overset{(\varphi,-)}{\to} h^X$ is exact, i.e., every morphism $\delta : Y \to Z'$ with $\delta\varphi = 0$ factors trough $\psi$.

**Lemma 1.10.** [**13**, C.3] *The following statements are equivalent for $\mathcal{C}$:*

(1) $\operatorname{fp}\mathcal{C}$ *is Abelian.*

(2) *Every morphism in $\mathcal{B}$ has a pseudo-cokernel.*

The classical example of a locally coherent Grothendieck category is the category of right (left) generalized $A$-modules $\mathcal{C}_A = (\operatorname{mod} A^{\mathrm{op}}, \mathrm{Ab})$ (${}_A\mathcal{C} = (\operatorname{mod} A, \mathrm{Ab})$) that consist of covariant additive functors from the category $\operatorname{mod} A^{\mathrm{op}}$ ($\operatorname{mod} A$) of finitely presented left (right) $A$-modules to Ab. By the preceding lemma the category $\operatorname{fp}\mathcal{C}_A$, henceforth the category of coherent functors, is Abelian. As we have already said, the finitely generated projective objects of $\mathcal{C}_A$ are the representable functors $(M,-) = \operatorname{Hom}_A(M,-)$ for some $M \in \operatorname{mod} A^{\mathrm{op}}$ and they are generators for $\mathcal{C}_A$.

There is a natural right exact and fully faithful functor

$$? \otimes_A - : \operatorname{Mod} A \longrightarrow \mathcal{C}_A \tag{1.2}$$

that takes each module $M_A$ to the tensor functor $M \otimes_A -$. Recall that a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

of right $A$-modules is *pure* if for any $M \in \operatorname{mod} A$ the sequence of Abelian groups

$$0 \longrightarrow \operatorname{Hom}_A(M,X) \longrightarrow \operatorname{Hom}_A(M,Y) \longrightarrow \operatorname{Hom}_A(M,Z) \longrightarrow 0$$

is exact. Equivalently, the $\mathcal{C}_A$-sequence

$$0 \longrightarrow X \otimes_A - \longrightarrow Y \otimes_A - \longrightarrow Z \otimes_A - \longrightarrow 0$$

is exact. The module $Q \in \operatorname{Mod} A$ is *pure-injective* if the functor $\operatorname{Hom}_A(-,Q)$ takes the pure-monomorphisms to epimorphisms.

The functor (1.2) identifies the pure-injective $A$-modules with the injective objects of $\mathcal{C}_A$ [**20**, 1.2] (see also [**14**, 4.1]). Furthermore, the functor $M \otimes_A - \in \operatorname{coh}\mathcal{C}_A$ if and only if $M \in \operatorname{mod} A$ [**9, 14**].

The category $\operatorname{coh}\mathcal{C}_A$ has enough injectives and they are precisely the objects of the form $M \otimes_A -$ with $M \in \operatorname{mod} A$ [**14**, 5.2]. Thus every coherent object $C \in \operatorname{coh}\mathcal{C}_A$ has both a projective presentation in $\mathcal{C}_A$

$$(K,-) \longrightarrow (L,-) \longrightarrow C \longrightarrow 0$$

and an injective presentation in $\operatorname{coh}\mathcal{C}_A$

$$0 \longrightarrow C \longrightarrow M \otimes_A - \longrightarrow N \otimes_A -.$$

Here $K$, $L \in \operatorname{mod} A^{\mathrm{op}}$ and $M$, $N \in \operatorname{mod} A$.

It should be remarked that most important for the applications in the representation theory of finite dimensional algebras is the concept of purity because the pure-injective modules play a prominent role among non-finitely generated modules. For this reason many concepts and problems of the theory are naturally formulated and solved in the category $\mathcal{C}_A$. For this subject we recommend the reader Krause's dissertation [**13**].

Another important application of the category $\mathcal{C}_A$ has come from the model theory of modules as its basic conceptions are realized in $\mathcal{C}_A$ (see [**8, 14**]). One of such concepts ("The Ziegler spectrum") we shall discuss in section 5.

## 2. Module categories

The following terminology is inspired from the classical theory of rings and modules. Similar to $\operatorname{Mod} A$ the Grothendieck categories $\mathcal{C}$ possessing a family of finitely generated projective generators $\mathcal{A} = \{P_i\}_{i\in I}$ we denote by $\operatorname{Mod}\mathcal{A}$, and we refer to $\mathcal{A}$ as a *ring of projective generators* $\{P_i\}_{i\in I}$ or just a *ring*. The category $\operatorname{Mod}\mathcal{A}$ we call the *category of right* $\mathcal{A}$*-modules*. Finally, every subobject $\mathfrak{a}$ of the object $P_i \in \mathcal{A}$ we call an *ideal* of the ring $\mathcal{A}$ corresponding to $P_i$.

2.1. **The Gabriel topologies.** Let $\mathfrak{F} = \cup_{i\in I}\mathfrak{F}^i$ be some family of ideals, where $\mathfrak{F}^i$ is some family of ideals corresponding to the object $P_i$. We refer to $\mathfrak{F}$ as a *Gabriel topology* for $\mathcal{A}$ provided that the following axioms hold:

$T1$. $P_i \in \mathfrak{F}^i$ for every $i \in I$;

$T2$. if $\mathfrak{a} \in \mathfrak{F}^i$ and $\mu \in \operatorname{Hom}_{\mathcal{A}}(P_j, P_i)$, $P_j \in \mathcal{A}$, then $\{\mathfrak{a} : \mu\} = \mu^{-1}(\mathfrak{a}) \in \mathfrak{F}^j$;

$T3$. if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathcal{A}$ corresponding to $P_i$ such that $\mathfrak{a} \in \mathfrak{F}^i$ and $\{\mathfrak{b} : \mu\} \in \mathfrak{F}^j$ for any $\mu \in \operatorname{Hom}_{\mathcal{A}}(P_j, P_i)$ with $\operatorname{Im}\mu \subset \mathfrak{a}$, $P_j \in \mathcal{A}$, then $\mathfrak{b} \in \mathfrak{F}^i$.

If $\mathcal{A} = \{A\}$ is a ring and $\mathfrak{a}$ is a right ideal of $A$, then for every endomorphism $\mu : A \to A$ of the module $A_A$

$$\mu^{-1}(\mathfrak{a}) = \{\mathfrak{a} : \mu(1)\} = \{a \in A \mid \mu(1)a \in \mathfrak{a}\}.$$

On the other hand, if $x \in A$, then $\{\mathfrak{a} : x\} = \mu^{-1}(\mathfrak{a})$, where $\mu \in \operatorname{End} A$ is such that $\mu(1) = x$.

**Remark.** Later we use the following properties for Gabriel topologies $\mathfrak{F} = \cup_{i\in I}\mathfrak{F}^i$ of a ring $\mathcal{A} = \{P_i\}_{i\in I}$.

(1). If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}^i$ and $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathfrak{b} \in \mathfrak{F}^i$. Indeed, if $\mu \in (P_j, P_i)$ is such that $\operatorname{Im}\mu \subseteq \mathfrak{a}$, then $\{\mathfrak{b} : \mu\} = P_j \in \mathfrak{F}^j$.

(2). If $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{F}^i$, then $\mathfrak{a} \cap \mathfrak{b} \in \mathfrak{F}^i$. Indeed, since $\{\mathfrak{a} \cap \mathfrak{b} : \mu\} = \{\mathfrak{a} : \mu\} \cap \{\mathfrak{b} : \mu\}$ for any $\mu \in (P_j, P_i)$, we get that $\{\mathfrak{a} \cap \mathfrak{b} : \mu\} = \{\mathfrak{a} : \mu\} \in \mathfrak{F}^j$ for $\mu \in (P_j, P_i)$ with $\operatorname{Im}\mu \subset \mathfrak{b}$.

Thus $\mathfrak{F}^i$, $i \in I$, is a downwards directed system of ideals.

**Theorem 2.1** (Gabriel). *The map*

$$\mathcal{S} \longmapsto \mathfrak{F}(\mathcal{S}) = \{\mathfrak{a} \subseteq P_i \mid i \in I, P_i/\mathfrak{a} \in \mathcal{S}\}$$

*establishes a bijection between the Gabriel topologies for $\mathcal{A}$ and the localizing subcategories of* $\operatorname{Mod}\mathcal{A}$.

*Proof.* Suppose $\mathfrak{F} = \cup_{i\in I}\mathfrak{F}^i$ is a Gabriel topology for $\mathcal{A}$. By $\mathcal{S}$ we denote the following subcategory of $\operatorname{Mod}\mathcal{A}$: $A \in \mathcal{S}$ if and only if for each $\delta : P_i \to A$ the kernel $\operatorname{Ker}\delta \in \mathfrak{F}^i$. We claim that the subcategory $\mathcal{S}$ is localizing. Indeed, let $A \in \mathcal{S}$ and $i : A' \to A$ be a monomorphism, $\delta : P_i \to A'$. Then $\operatorname{Ker}\delta = \operatorname{Ker}(i\delta) \in \mathfrak{F}^i$. Suppose now that $p : A \to A''$ is an epimorphism, $\delta : P_i \to A''$. Since $P_i$ is projective, there exists $\gamma : P_i \to A$ such that $p\gamma = \delta$. Then $\operatorname{Ker}\gamma \subset \operatorname{Ker}\delta$, which yields $\operatorname{Ker}\delta \in \mathfrak{F}^i$.

Now, we show that $\mathcal{S}$ is closed under extensions. To see this, consider an exact sequence

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$$

with $A', A'' \in \mathcal{S}$. Let $\delta : P_i \to A$; then $\mathfrak{a} = \operatorname{Ker}(p\delta) \in \mathfrak{F}^i$. Let $\gamma : P_j \to P_i$ be such that $\operatorname{Im}\gamma \subset \mathfrak{a}$. Since $p\delta\gamma = 0$, there exists $\alpha : P_j \to A'$ such that $\delta\gamma = i\alpha$. Therefore $\{\operatorname{Ker}\delta : \gamma\} = \gamma^{-1}(\operatorname{Ker}\delta) = \operatorname{Ker}(\delta\gamma) = \operatorname{Ker}\alpha \in \mathfrak{F}^j$. From $T3$ it follows that $\operatorname{Ker}\delta \in \mathfrak{F}^i$. So $\mathcal{S}$ is a Serre subcategory. If $\delta : P_i \to \oplus A_k$, $A_k \in \mathcal{S}$, there exist $k_1, \ldots, k_s$ such that $\operatorname{Im}\delta \subset \oplus_{j=1}^s A_{k_j}$ (since $P_i$ is finitely generated). Since $\mathcal{S}$ is a Serre subcategory, any finite product of objects from $\mathcal{S}$ belongs to $\mathcal{S}$, and, hence, $\operatorname{Ker}\delta \in \mathfrak{F}^i$. Therefore $\mathcal{S}$ is a localizing subcategory.

Conversely, assume that $\mathcal{S}$ is a localizing subcategory of $\operatorname{Mod}\mathcal{A}$. Let $\mathfrak{F}^i = \{\mathfrak{a} \subset P_i \mid P_i/\mathfrak{a} \in \mathcal{S}\}$. Obviously, $P_i \in \mathfrak{F}^i$. If $\mathfrak{a} \in \mathfrak{F}^i$ and $\delta : P_j \to P_i$, then $\{\mathfrak{a} : \delta\} = \operatorname{Ker}(p\delta)$, where $p : P_i \to P_i/\mathfrak{a}$ is the canonical epimorphism. Since $P_i/\mathfrak{a} \in \mathcal{S}$, we obtain that $\{\mathfrak{a} : \delta\} \in \mathfrak{F}^j$. It remains to check $T3$. Let $\mathfrak{a} \in \mathfrak{F}^i$ and $\mathfrak{b} \subset P_i$ be such that for any $\mu : P_j \to P_i$ with $\operatorname{Im}\mu \subset \mathfrak{a}$ the object $\mu^{-1}(\mathfrak{b}) \in \mathfrak{F}^j$. Consider an exact sequence

$$0 \longrightarrow \mathfrak{a} + \mathfrak{b}/\mathfrak{b} \longrightarrow P_i/\mathfrak{b} \longrightarrow P_i/\mathfrak{a} + \mathfrak{b} \longrightarrow 0.$$

Since $\mathfrak{a} \subset \mathfrak{a} + \mathfrak{b}$, we have $\mathfrak{a} + \mathfrak{b} \in \mathfrak{F}^i$. Let $p : P_i \to P_i/\mathfrak{b}$ be the canonical epimorphism, $\gamma_\mu = p\mu$ for $\mu \in (P_j, P_i)$. Since $\mu^{-1}(\mathfrak{b}) = \operatorname{Ker}\gamma_\mu$, one gets $P_j/\mu^{-1}(\mathfrak{b}) = \operatorname{Im}\gamma_\mu = p(\mu(P_j) + \mathfrak{b})$. In particular, if $\operatorname{Im}\mu \subset \mathfrak{a}$, then $p(\mu(P_j) + \mathfrak{b}) = P_j/\mu^{-1}(\mathfrak{b}) \in \mathcal{S}$, and then we obtain that

$$\mathfrak{a} + \mathfrak{b}/\mathfrak{b} = \sum_{\substack{\mu \in (P_j, P_i) : \operatorname{Im}\mu \subset \mathfrak{a}, \\ P_j \in \mathcal{A}}} p(\mu(P_j) + \mathfrak{b})$$

belongs to $\mathcal{S}$. Since $\mathcal{S}$ is closed under extensions, we conclude that $P_i/\mathfrak{b} \in \mathcal{S}$. □

**Proposition 2.2** (Freyd [**16**]). *Every module category* $\operatorname{Mod}\mathcal{A}$ *with* $\mathcal{A} = \{P_i\}_{i \in I}$ *the ring of finitely generated projective generators is equivalent via the functor* $M \overset{T}{\mapsto} (-, M)$ *to the functor category* $(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$.

In view of Proposition 2.2, in order to define a right $\mathcal{A}$-module $M$ (respectively an $\mathcal{A}$-homomorphism), it suffices to define $M$ as a functor from $\mathcal{A}^{\mathrm{op}}$ to Ab (respectively as a natural transformation of functors). And conversely, every functor $F : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ (respectively a natural transformation of functors from $(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$) can be considered as a right $\mathcal{A}$-module (respectively an $\mathcal{A}$-homomorphism). Further, we do not distinguish the $\mathcal{A}$-modules and the functors from $(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ and freely use this fact without additional reserves.

2.2. **Localisation in module categories.** Fix a localizing subcategory $\mathcal{S}$ from $\operatorname{Mod}\mathcal{A}$, $\mathcal{A} = \{P_i\}_{i \in I}$. Let $\mathfrak{F} = \cup_{i \in I}\mathfrak{F}^i$ be the corresponding Gabriel topology for $\mathcal{A}$ and $t = t_{\mathcal{S}}$ the corresponding $\mathcal{S}$-torsion functor. As we have already observed on p. 11, $\mathfrak{F}^i$ is a downwards directed system. Let $X$ be some right $\mathcal{A}$-module. For every pair $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{F}^i$ such that $\mathfrak{b} \subset \mathfrak{a}$ there is a homomorphism

$$\operatorname{Hom}_{\mathcal{A}}(\mathfrak{a}, X/t(X)) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(\mathfrak{b}, X/t(X)),$$

induced by inclusion of $\mathfrak{b}$ into $\mathfrak{a}$. Clearly, the Abelian groups $\operatorname{Hom}_{\mathcal{A}}(\mathfrak{a}, X/t(X))$ with these homomorphisms form an inductive system over $\mathfrak{F}^i$.

Consider a functor $H : \operatorname{Mod}\mathcal{A} \to \operatorname{Mod}\mathcal{A}$ defined as follows. For every $M \in \operatorname{Mod}\mathcal{A}$ and every $P_i \in \mathcal{A}$ we put

$$H(M)(P_i) = \varinjlim_{\mathfrak{a} \in \mathfrak{F}^i} \operatorname{Hom}_{\mathcal{A}}(\mathfrak{a}, M/t(M)). \tag{2.1}$$

The equality (2.1) we also call the *Gabriel formula*.

Let us show that the Abelian groups (2.1) determine $H(M)$ as a functor from $\mathcal{A}^{\mathrm{op}}$ to Ab. To see this, we consider a morphism $\mu : P_j \to P_i$ and an element $m \in H(M)(P_i)$. Let $u : \mathfrak{a} \to M/t(M)$ be a morphism representing the element $m$ of the direct limit. We then define $H(M)(\mu)(m) \in H(M)(P_j)$ to be represented by the composed map

$$\mu^{-1}(\mathfrak{a}) \xrightarrow{\bar{\mu}} \mathfrak{a} \xrightarrow{u} M/t(M). \tag{2.2}$$

It is easy to see that $H(M)(\mu)$ is well defined, i.e., is independent of the choice of the representing morphism $u$. Thus $H(M)$ becomes a right $\mathcal{A}$-module.

Now let $f : M \to N$ be a morphism of $\operatorname{Mod}\mathcal{A}$. Obviously, $f(t(M))$ is contained in $t(N)$. Thus $f$ induces the unique morphism $f' : M/t(M) \to N/t(N)$. In turn, $f'$ induces the unique morphism $H(f) : H(M) \to H(N)$, which is a homomorphism of right $\mathcal{A}$-modules. This concludes the construction of the functor $H$.

The Gabriel topology $\mathfrak{F}$ ordered by inclusion is a directed set (see a remark on p. 11). From the fact that Ab satisfies the Ab5-condition and from construction of $H$ we deduce that $H$ is a left exact functor. Moreover, if $M \in \mathcal{S}$, then $H(M) = 0$ since $t(M) = M$.

Let $\zeta_i$ be the canonical morphism from $\operatorname{Hom}_{\mathcal{A}}(P_i, M/t(M))$ into the direct limit $H(M)(P_i)$, $p : M \to M/t(M)$ be the canonical epimorphism. Consider the map of $\mathcal{A}$-modules $\Phi_M : M \to H(M)$ defined as follows. For $P_i \in \mathcal{A}$, $\alpha \in \operatorname{Hom}_{\mathcal{A}}(P_i, M)$ we put $\Phi_M(P_i)(\alpha) = \zeta_i(p\alpha)$. Let $\mu : P_j \to P_i$ be a morphism. From construction of the map $H(M)(\mu)$ it easily follows that the diagram

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{A}}(P_i, M) & \xrightarrow{\Phi_M(P_i)(\mu)} & H(M)(P_i) \\
{\scriptstyle (\mu,M)}\downarrow & & \downarrow{\scriptstyle H(M)(\mu)} \\
\operatorname{Hom}_{\mathcal{A}}(P_j, M) & \xrightarrow[\Phi_M(P_j)(\mu)]{} & H(M)(P_j)
\end{array}$$

is commutative, and, hence, $\Phi_M$ is an $\mathcal{A}$-homomorphism. It is directly verified that $\Phi_M$ is functorial in $M$. Thus we obtain a functorial morphism $\Phi : 1_{\operatorname{Mod}\mathcal{A}} \to H$.

Concerning the functoral morphism $\Phi$ we prove the following.

**Proposition 2.3.** $\operatorname{Ker}\Phi_M$ *and* $\operatorname{Coker}\Phi_M$ *belong to* $\mathcal{S}$ *for every* $M \in \operatorname{Mod}\mathcal{A}$.

*Proof.* As above, we can construct a morphism $\Psi_M : M/t(M) \to H(M)$, $\Psi_M(P_i)(\mu) = \zeta_i(\mu)$ with $\mu \in \operatorname{Hom}_{\mathcal{A}}(P_i, M/t(M))$. One analogously varifies that $\Psi_M$ is an $\mathcal{A}$-homomorphism. Then from the definitions of the morphisms $\Phi_M$ and $\Psi_M$ it easily follows that the diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & t(M) & \longrightarrow & M & \longrightarrow & M/t(M) \\
 & & & & \| & & \downarrow{\scriptstyle \Psi_M} \\
0 & \longrightarrow & \operatorname{Ker}\Phi_M & \longrightarrow & M & \xrightarrow[\Phi_M]{} & H(M)
\end{array}$$

is commutative whence one gets $t(M) \subset \operatorname{Ker}\Phi_M$. Let us show that $\Psi_M$ is a monomorphism, or equivalently, $\Phi_M$ is a monomorphism for any $\mathcal{S}$-torsionfree object $M$. Indeed, assume that $M$ is $\mathcal{S}$-torsionfree and let $\mu : P_i \to M$ be such that $\Phi_M(P_i)(\mu) = 0$. Then there exists an element $\mathfrak{a} \in \mathfrak{F}^i$ such that the restriction $\mu|_{\mathfrak{a}} = 0$. This implies that $\mathfrak{a}$ is contained in $\operatorname{Ker}\mu$ whence $\operatorname{Ker}\mu \in \mathfrak{F}^i$ (see a remark on p. 11), and, hence, $\operatorname{Im}\mu \in \mathcal{S}$. Since $M$ is $\mathcal{S}$-torsionfree, it follows that $\operatorname{Im}\mu = 0$, i.e., $\mu = 0$. Thus $\operatorname{Ker}\Phi_M = t(M)$. Therefore, if $H(M) = 0$, then $M \in \mathcal{S}$. Indeed, we obtain then that $\operatorname{Ker}\Phi_M = M$, hence $M \in \mathcal{S}$. Therefore $H(M) = 0$ if and only if $M \in \mathcal{S}$.

It remains to check that $\operatorname{Coker}\Phi_M \in \mathcal{S}$. Let $\mu : P_i \to H(M)$. We suffice to show that the ideal $\mu^{-1}(\operatorname{Im}\Phi_M)$ is an element of $\mathfrak{F}^i$. Indeed, if $p : P_i \to \operatorname{Coker}\Phi_M$, then there exists $\mu : P_i \to H(M)$ such that $p = \operatorname{Coker}\Phi_M \circ \mu$ since $P_i$ is projective. Further, since the sequence

$$0 \longrightarrow \mu^{-1}(\operatorname{Im}\varPhi_M) \longrightarrow P_i \xrightarrow{p} \operatorname{Coker}\varPhi_M$$

is exact and $\mu^{-1}(\operatorname{Im}\varPhi_M)$, by assumtion, belongs to $\mathfrak{F}^i$, we obtain then that $\operatorname{Coker}\varPhi_M$ belongs to $\mathcal{S}$. Without loss of a generality we can assume that $\varPhi_M$ is a monomorphism and identify $M$ with $\operatorname{Im}\varPhi_M$. We put $\mathfrak{b} = \mu^{-1}(M)$.

Let $u : \mathfrak{a} \to M$ be an $\mathcal{A}$-homomorphism representing $\mu$ in the direct limit $H(M)(P_i)$ and $\xi : P_j \to P_i$ be such that $\operatorname{Im}\xi \subseteq \mathfrak{a}$. Let us consider the following commutative diagram:

$$\begin{array}{ccccc}
\xi^{-1}(\mathfrak{b}) & \longrightarrow & \mathfrak{b} & \longrightarrow & M \\
{\scriptstyle\iota}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\varPhi_M} \\
P_j & \underset{\xi}{\longrightarrow} & P_i & \underset{\mu}{\longrightarrow} & H(M).
\end{array}$$

Recall that the element $\mu\xi$ is represented by the composed morphism $u\bar{\xi} : \{\mathfrak{a} : \xi\} \to M$ (see sequence (2.2)). Since $\operatorname{Im}\xi \subseteq \mathfrak{a}$, we have $\{\mathfrak{a} : \xi\} = P_j$, and therefore $\varPhi_M(P_j)(u\bar{\xi}) = \mu\xi$. Since both squares of the diagram are pullback, it follows that the outer square is pullback, and, hence, there exists a morphism $\varkappa : P_j \to \xi^{-1}(\mathfrak{b})$ such that $\iota\varkappa = 1_{P_j}$ whence $\xi^{-1}(\mathfrak{b}) = P_j \in \mathfrak{F}^j$. By $T3$ we deduce that $\mathfrak{b} \in \mathfrak{F}^i$. □

**Theorem 2.4.** *For an arbitrary right $\mathcal{A}$-module $M$ the module $H(M)$ is $\mathcal{S}$-closed. Moreover, the homomorphism $\varPhi_M$ is an $\mathcal{S}$-envelope for $M$.*

*Proof.* To begin with, we shall show that $H(M)$ is $\mathcal{S}$-torsionfree. Let $S$ be a subobject of $H(M)$ belonging to $\mathcal{S}$, $\mu : P_i \to S$ and let $\mu$ be represented by $u : \mathfrak{a} \to M/t(M)$ in the direct limit $H(M)(P_i)$. Suppose also $\xi : P_j \to P_i$ is such that $\operatorname{Im}\xi \subseteq \operatorname{Ker}\mu$; then the equality $\mu\xi = 0$ implies that the image of the composed map $u\bar{\xi} : \{\mathfrak{a} : \xi\} \to M/t(M)$ in $H(M)(P_i)$ equals zero. So, by properties of direct limits, there is an ideal $\mathfrak{b} \in \mathfrak{F}^j$ such that the restriction $u\bar{\xi}$ representing $\mu\xi$ to $\mathfrak{b}$ equals zero, and, hence, $\mathfrak{b} \subseteq \operatorname{Ker}(u\bar{\xi})$. Then $\operatorname{Ker}(u\bar{\xi}) \in \mathfrak{F}^j$, i.e., $\operatorname{Im}(u\bar{\xi}) \in \mathcal{S}$. But $M/t(M)$ is $\mathcal{S}$-torsionfree whence $\operatorname{Im}(u\bar{\xi}) = 0$, hence $u\bar{\xi} = 0$. Since this holds for every $\xi : P_j \to P_i$ such that $\operatorname{Im}\xi \subseteq \operatorname{Ker}\mu$, we infer that $\operatorname{Ker}\mu \subseteq \operatorname{Ker} u$. Since $\operatorname{Ker}\mu \in \mathfrak{F}^i$, it follows that $\operatorname{Ker} u$ is also an element of $\mathfrak{F}^i$. In that case, being the image of the zero homomorphism $u \circ \operatorname{Ker} u$ from $\operatorname{Hom}_{\mathcal{A}}(\operatorname{Ker} u, M/t(M))$, $\mu$ equals zero. Since this holds for any $\mu \in \operatorname{Hom}_{\mathcal{A}}(P_i, S)$, we deduce that $S = 0$.

Now, we prove that a module $M$ is $\mathcal{S}$-closed if and only if $\varPhi_M$ is an isomorphism. Indeed, if $M$ is $\mathcal{S}$-closed, then, in view of the preceding proposition, $\operatorname{Ker}\varPhi_M = t(M) = 0$, i.e., $\varPhi_M$ is a monomorphism. Since $\operatorname{Ext}^1(\operatorname{Coker}\varPhi_M, M) = 0$, there exists a morphism $\alpha : H(M) \to M$ such that $\alpha\varPhi_M = 1_M$, i.e., $\operatorname{Coker}\varPhi_M$ is a direct summand of $H(M)$. Since $H(M)$ is $\mathcal{S}$-torsionfree, we conclude that $\operatorname{Coker}\varPhi_M = 0$ that implies $\varPhi_M$ is an isomorphism.

On the other hand, if for $M$ the morphism $\varPhi_M$ is an isomorphism, then $t(M) = \operatorname{Ker}\varPhi_M = 0$ whence $M$ is $\mathcal{S}$-torsionfree. If we showed that every short exact sequence

$$0 \longrightarrow M \xrightarrow{i} N \longrightarrow S \longrightarrow 0$$

with $S \in \mathcal{S}$ splits, it would follow then that $\operatorname{Ext}^1(S, M) = 0$ for any $S \in \mathcal{S}$. So $M$ would be $\mathcal{S}$-closed. To see this, consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & N & \longrightarrow & S & \longrightarrow & 0 \\ & & \Phi_M \downarrow & & \downarrow \Phi_N & & & & \\ 0 & \longrightarrow & H(M) & \xrightarrow[H(i)]{} & H(N) & \longrightarrow & H(S) & & \end{array}$$

where the bottom row is exact and $H(S) = 0$. We deduce that $H(i)$ is an isomorphism. Hence $\Phi_M^{-1} H(i)^{-1} \Phi_N i = 1_M$ that implies $i$ is a splitting monomorphism.

Thus to see that $H(M)$ is $\mathcal{S}$-closed, it suffices to show that $\Phi_{H(M)} : H(M) \to H(H(M))$ is an isomorphism. To begin, we prove that $H(\Phi_M)$ is an isomorphism. By construction of $H(M)$ it follows that $H(M) = H(M/t(M))$. Let $p : M \to M/t(M)$ be the canonical epimorphism. If we apply the functor $H$ to the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{p} & M/t(M) \\ \Phi_M \downarrow & & \downarrow \Phi_{M/t(M)} \\ H(M) & = & H(M/t(M)), \end{array}$$

we obtain $H(\Phi_M) = H(\Phi_{M/t(M)})$. Since $H(\operatorname{Coker} \Phi_{M/t(M)}) = 0$, it follows that the morphism $H(\Phi_{M/t(M)})$ is an isomorphism, and, hence, $H(\Phi_M)$ is an isomorphism.

Further, since $\Phi$ is a functorial morphism, the following relations hold:

$$\Phi_{H(M)} \Phi_M = H(\Phi_M) \Phi_M \tag{2.3}$$

and

$$\begin{aligned} \Phi_{H^2(M)} \Phi_{H(M)} &= H(\Phi_{H(M)}) \Phi_{H(M)}, \\ \Phi_{H^2(M)} H(\Phi_M) &= H^2(\Phi_M) \Phi_{H(M)}. \end{aligned} \tag{2.4}$$

Applying the functor $H$ to (2.3), one gets $H(\Phi_{H(M)}) H(\Phi_M) = H^2(\Phi_M) H(\Phi_M)$. Since $H(\Phi_M)$ is an isomorphism, $H(\Phi_{H(M)}) = H^2(\Phi_M)$. From the equalities (2.4) it follows that

$$\Phi_{H^2(M)} \Phi_{H(M)} = \Phi_{H^2(M)} H(\Phi_M). \tag{2.5}$$

Since $H^2(M)$ is $\mathcal{S}$-torsionfree, by the first part of the proof of the preceding proposition $\Phi_{H^2(M)}$ is a monomorphism. Therefore, from (2.5) it follows that $\Phi_{H(M)} = H(\Phi_M)$. So, $\Phi_{H(M)}$ is an isomorphism, and, hence, $H(M)$ is $\mathcal{S}$-closed.

In particular, if we consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} \Phi_M = t(M) \longrightarrow M \xrightarrow{\Phi_M} H(M) \longrightarrow \operatorname{Coker} \Phi_M \longrightarrow 0$$

and apply the exact localizing functor $(-)_{\mathcal{S}}$, one obtains

$$M_{\mathcal{S}} \approx (H(M))_{\mathcal{S}} \approx H(M)$$

whence it immediately follows that $\Phi_M$ is an $\mathcal{S}$-envelope for $M$. $\square$

Suppose now $i : \operatorname{Mod} \mathcal{A}/\mathcal{S} \to \operatorname{Mod} \mathcal{A}$ is the inclusion functor. The functor $H$ is left adjoint to $i$. Unit of adjointness between $i$ and $H$ is given by the functor morphism $\Phi : 1_{\operatorname{Mod} \mathcal{A}} \to H = i \circ H$ constructed above; and counit of adjointness is the functor morphism $\Psi : H \circ i \to 1_{\mathrm{Mod}\,\mathcal{A}/\mathcal{S}}$ defined by the rule $\Psi_M = (\Phi_M)^{-1}$ for all $M \in \mathrm{Mod}\,\mathcal{A}/\mathcal{S}$. The fact that $\Psi_{H(M)} H(\Phi_M) = 1_{H(M)}$ follows from the equality $\Phi_{H(M)} = H(\Phi_M)$ proved in the preceding theorem. The equality $i(\Psi_M)\Phi_{i(M)} = 1_{i(M)}$ is trivial. So $H$ is indeed left adjoint to $i$. Since the localizing functor $(-)_{\mathcal{S}}$ is also left adjoint to $i$, it follows that the functors $(-)_{\mathcal{S}}$ and $H$ are equivalent.

## 3. Negligible objects and covering morphisms

Let $\mathcal{C}$ be an Abelian category, $\mathcal{M}$ some class of objects from $\mathcal{C}$, and $\mathcal{S} = \{C \in \mathcal{C} \mid {}_{\mathcal{C}}(C, M) = 0, \mathrm{Ext}^1_{\mathcal{C}}(C, M) = 0 \text{ for every } M \in \mathcal{M}\}$.

**Proposition 3.1.** (1) *$\mathcal{S}$ is closed under extensions. An object $S \in \mathcal{S}$ if and only if for every $M \in \mathcal{M}$, $X \in \mathcal{C}$ and every epimorphism $f : X \to S$ the canonical homomorphism $(X, M) \to (\mathrm{Ker}\, f, M)$ is an isomorphism. Furthermore, if $\mathcal{C}$ is cocomplete, then $\mathcal{S}$ is closed under coproducts.*

(2) *For $\mathcal{S}$ the following statements are equivalent:*
- (a) *$\mathcal{S}$ is a Serre subcategory.*
- (b) *$\mathcal{S}$ is closed under subobjects.*
- (c) *$S \in \mathcal{S}$ if and only if for every $M \in \mathcal{M}$ and every morphism $f : X \to S$, $X \in \mathcal{C}$, the canonical homomorphism $(X, M) \to (\mathrm{Ker}\, f, M)$ is an isomorphism.*

*Moreover, if $\mathcal{C}$ is a Grothendieck category , then $\mathcal{S}$ is a localizing subcategory.*

*Proof.* (1). Consider a short exact sequence in $\mathcal{C}$

$$0 \longrightarrow S' \xrightarrow{i} S \xrightarrow{j} S'' \longrightarrow 0. \tag{3.1}$$

It induces the exact sequence

$$0 \to (S'', M) \to (S, M) \to (S', M) \to \mathrm{Ext}^1(S'', M) \to \mathrm{Ext}^1(S, M) \to \mathrm{Ext}^1(S', M). \tag{3.2}$$

If $M \in \mathcal{M}$ and $S', S'' \in \mathcal{S}$, it is easy to see that $S \in \mathcal{S}$. So $\mathcal{S}$ is closed under extensions.

Let $S \in \mathcal{S}$, $M \in \mathcal{M}$, $X \in \mathcal{C}$, $f : X \to S$ is an arbitrary epimorphism. If we consider the exact sequence

$$0 \to (S, M) = 0 \to (X, M) \to (\mathrm{Ker}\, f, M) \to \mathrm{Ext}^1(S, M) = 0, \tag{3.3}$$

we obtain that $(X, M) \approx (\mathrm{Ker}\, f, M)$.

Conversely, consider the identity morphism $1_S : S \to S$. Then

$$0 = (\mathrm{Ker}\, 1_S, M) \approx (S, M).$$

It remains to check that $\mathrm{Ext}^1(S, M) = 0$. For this, we show that any short exact sequence

$$0 \longrightarrow M \xrightarrow{h} P \xrightarrow{f} S \longrightarrow 0$$

splits. By assumption, for $1_M : M \to M$ there exists $g : P \to M$ such that $gh = 1_M$, i.e., $h$ splits.

In turn, if $\mathcal{C}$ is cocomplete, then $\mathcal{S}$ is closed under copruducts since the functor $\mathrm{Ext}^1_{\mathcal{C}}(-, M)$ commutes with coproducts.

(2). (a) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (c). Suppose $S \in \mathcal{S}$, $f : X \to S$ is an arbitrary morphism. Then, by assumption, $\operatorname{Im} f$ belongs to $\mathcal{S}$. Replacing $S$ by $\operatorname{Im} f$ in (3.3), we get that for any $M \in \mathcal{M}$ the canonical morphism $(X, M) \to (\operatorname{Ker} f, M)$ is an isomorphism. The proof of the converse assertion is similar to (1).

(c) $\Rightarrow$ (a). By the first statement the subcategory $\mathcal{S}$ is closed under extensions. Suppose now that in the exact sequence (3.1) the object $S \in \mathcal{S}$ and $f : X \to S'$, $X \in \mathcal{C}$; then $\operatorname{Ker} f = \operatorname{Ker}(if)$. Hence $S' \in \mathcal{S}$. Futher, if we consider the exact sequence (3.2) with $S', S \in \mathcal{S}$, one also gets that $S'' \in \mathcal{S}$.

If $\mathcal{C}$ is a Grothendieck category, then $\mathcal{S}$ is closed coproducts, and, hence, it is a localizing subcategory by [**17**, 15.11]. $\square$

Let $\mathcal{C}$ be a Grothendieck category, $\mathcal{M}$ be some class of objects of the category $\mathcal{C}$; then an object $S \in \mathcal{C}$ is $\mathcal{M}$-*negligible* provided that for any $M \in \mathcal{M}$, $X \in \mathcal{C}$, and $f : X \to S$ the canonical homomorphism $(X, M) \to (\operatorname{Ker} f, M)$ is an isomorphism.

**Example.** Every localizing subcategory $\mathcal{S}$ of $\mathcal{C}$ is the subcategory consisting of $\mathcal{C}/\mathcal{S}$-negligible objects, where $\mathcal{C}/\mathcal{S}$ is the quotient category of $\mathcal{C}$ with respect to $\mathcal{S}$.

**Lemma 3.2.** *The subcategory $\mathcal{S}$ of $\mathcal{C}$ consisting of $\mathcal{M}$-negligible objects is the largest localizing subcategory for which all $M \in \mathcal{M}$ are $\mathcal{S}$-closed.*

*Proof.* Indeed, if $\mathcal{P}$ is a localizing subcategory such that every object $M \in \mathcal{M}$ is $\mathcal{P}$-closed, then for any $f : X \to S$, $S \in \mathcal{P}$, the object $\operatorname{Im} f \in \mathcal{P}$. Hence, if we consider the exact sequence (3.3) with $S = \operatorname{Im} f$, we get that the homomorphism $(X, M) \to (\operatorname{Ker} f, M)$ is an isomorphism. Therefore, in view of Proposition 3.1, $S \in \mathcal{S}$ that implies $\mathcal{P} \subset \mathcal{S}$. $\square$

Let $\mathcal{S}$ be a localizing subcategory of the category $\mathcal{C}$. We say that an object $M \in \mathcal{C}$ *cogenerates* $\mathcal{S}$ provided that $\mathcal{S} = \{C \in \mathcal{C} \mid {}_{\mathcal{C}}(C, M) = 0\}$.

**Lemma 3.3.** *The localizing subcategory $\mathcal{S}$ consisting of $\mathcal{M}$-negligible objects is cogenerated by the objects $E(M) \oplus E(E(M)/M)$ where $M \in \mathcal{M}$, $E(M)$ is the injective envelope for $M$.*

*Proof.* Denote by $\mathcal{P}$ the localizing subcategory cogenerated by $E(M) \oplus E(E(M)/M)$. We must show that $\mathcal{S} = \mathcal{P}$.

Let $M \in \mathcal{M}$. Consider the short exact sequence

$$0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0.$$

For each $S \in \mathcal{C}$ it induces the exact sequence

$$0 \to (S, M) \to (S, E(M)) \to (S, E(M)/M) \to \operatorname{Ext}^1(S, M) \to 0.$$

If $S \in \mathcal{S}$, then $\operatorname{Ext}^1(S, M) = 0$ and also $(S, E(M)) = 0$ since $M$ is an $\mathcal{S}$-torsionfree object. Therefore $(S, E(M)/M) = 0$, and, hence, $(S, E(E(M)/M)) = 0$ that implies $\mathcal{S} \subset \mathcal{P}$.

On the other hand, suppose $S \in \mathcal{P}$; then $(S, E(M)/M) = 0$ since $E(E(M)/M)$ is a $\mathcal{P}$-torsionfree object. Hence we obtain that $(S, M) = 0$ and $\operatorname{Ext}^1(S, M) = 0$. Thus $M$ is $\mathcal{P}$-closed. But, by assumption, $\mathcal{S}$ is the largest localizing subcategory such that each $M$ is $\mathcal{S}$-closed. Hence $\mathcal{P} \subset \mathcal{S}$. $\square$

In prooving the lemma we have used the fact that an object $M \in \mathcal{C}$ is $\mathcal{S}$-torsionfree if and only if its injective envelope $E(M)$ is $\mathcal{S}$-torsionfree.

We say that an object $C$ of the category $\mathcal{C}$ is *cyclic* provided that there is an epimorphism $f : U_i \to C$ for some $U_i \in \mathcal{U}$.

**Lemma 3.4.** *Suppose $C, D$ are objects of $\mathcal{C}$; then ${}_{\mathcal{C}}(C, E(D)) = 0$ if and only if ${}_{\mathcal{C}}(C', D) = 0$ for every cyclic subobject $C'$ of $C$.*

*Proof.* Easy. □

**Proposition 3.5.** *Let* Mod $\mathcal{A}$ *be the category of right $\mathcal{A}$-modules, where $\mathcal{A}$ is the ring of finitely generated projective generators $\{P_i\}_{i\in I}$, and let $\mathcal{M}$ be some class of $\mathcal{A}$-modules. Then an object $S$ is $\mathcal{M}$-negligible if and only if for an arbitrary morphism $f : P_i \to S$, $P_i \in \mathcal{A}$, the canonical homomorphism $(P_i, M) \to (\operatorname{Ker} f, M)$ with $M \in \mathcal{M}$ is an isomorphism.*

*Proof.* The necessary condition is obvious. Let $S'$ be an arbitrary cyclic subobject of the object $S$; then $(S', M) = 0$. Indeed, there exists an epimorphism $f : P_i \to S'$ for some $P_i \in \mathcal{A}$. If we consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} f \xrightarrow{\ i\ } P_i \xrightarrow{\ f\ } S' \longrightarrow 0,$$

we get $(S', M) = 0$, since, by assumption, the homomorphism $(i, M)$ is an isomorphism. Taking Lemma 3.4 into account, we get $(S, E(M)) = 0$.

Now we consider the exact sequence

$$0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0.$$

It induces the exact sequence

$$(S', E(M)) \to (S', E(M)/M) \to \operatorname{Ext}^1(S', M) \to 0.$$

Since $(S, E(M)) = 0$, it follows that $(S', E(M)) = 0$, hence $(S', E(M)/M) \approx \operatorname{Ext}^1(S', M)$. Our proof would be finished if we showed that any short exact sequence

$$0 \longrightarrow M \xrightarrow{\ g\ } N \xrightarrow{\ h\ } S' \longrightarrow 0$$

splits. To see this, consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & \operatorname{Ker} f & = & \operatorname{Ker} f \\
 & & & & {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\
0 & \longrightarrow & M & \xrightarrow{\ \nu\ } & N\Pi_{S'}P_i & \xrightarrow{\ \pi\ } & P_i \\
 & & \| & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
0 & \longrightarrow & M & \xrightarrow[g]{} & N & \xrightarrow[h]{} & S',
\end{array}$$

where the lower right square is pullback. Since $f$ and $h$ are epimorphisms, $f'$ and $\pi$ are also epimorphisms. Since $P_i$ is a projective $\mathcal{A}$-module, $\nu$ is a splitting monomorphism. Let $\pi'$ be the canonical projection onto $M$, i.e., $\pi'\nu = 1_M$. By assumption, there exists a morphism $\beta : P_i \to M$ such that $\pi' i' = \beta i$. Since $\beta i = \beta\pi i'$, we have $(\pi' - \beta\pi)i' = 0$. Then there exists $q : N \to M$ such that $\pi' - \beta\pi = qf'$, and, hence, $1_M = \pi'\nu = qf'\nu = qg$. Thus $g$ is a splitting monomorphism that finishes the proof. □

Let $\mathcal{M}$ be some class of objects of a Grothendieck category $\mathcal{C}$. We refer to a morphism $u : X \to Y$ as $\mathcal{M}$-*covering* provided that $\mathrm{Coker}\, u$ is an $\mathcal{M}$-negligible object.

Let $\mathrm{Mod}\,\mathcal{A}$ be the category of right $\mathcal{A}$-modules, where $\mathcal{A} = \{P_i\}_{i\in I}$ is a ring. We consider morphisms $u : X \to Y$ and $f : P_i \to Y$, and also a commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & & & \mathrm{Ker}(pf) & & & & \\
 & & & & & & \downarrow{\scriptstyle h} & & & & \\
0 & \longrightarrow & \mathrm{Ker}\, u & \xrightarrow{i'} & X\Pi_Y P_i & \xrightarrow{u'} & P_i & & & & \\
 & & \| & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & & & \\
0 & \longrightarrow & \mathrm{Ker}\, u & \xrightarrow[i]{} & X & \xrightarrow[u]{} & Y & \xrightarrow[p]{} & \mathrm{Coker}\, u & \longrightarrow & 0,
\end{array}$$

in which the rows are exact, the couple $(\mathrm{Ker}(pf), h)$ is the kernel of $pf$.

**Lemma 3.6.** *The morphism $u : X \to Y$ is $\mathcal{M}$-covering if and only if for every $P_i \in \mathcal{A}$, $M \in \mathcal{M}$ and every morphism $f : P_i \to Y$ the sequence*

$$0 \to \mathrm{Hom}_{\mathcal{A}}(P_i, M) \stackrel{\varphi}{\to} \mathrm{Hom}_{\mathcal{A}}(X\Pi_Y P_i, M) \stackrel{\psi}{\to} \mathrm{Hom}_{\mathcal{A}}(\mathrm{Ker}\, u, M)$$

*induced by $u'$ and $i'$ is exact.*

*Proof.* Since $pfu' = puf' = 0$, there exists the unique morphism $q : X\Pi_Y P_i \to \mathrm{Ker}(pf)$ such that $u' = hq$. We claim that $q$ is an epimorphism. Indeed, consider the folowing diagram:

$$\begin{array}{ccccccc}
X\Pi_Y P_i & \xrightarrow{q} & \mathrm{Ker}(pf) & \xrightarrow{h} & P_i & & \\
{\scriptstyle f'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
X & \xrightarrow[n]{} & \mathrm{Ker}\, p & \xrightarrow[k]{} & Y & \xrightarrow[p]{} & \mathrm{Coker}\, u,
\end{array}$$

where $(\mathrm{Ker}\, p, k)$ is the kernel of $p$, $kn = u$, and the middle arrow is the unique morphism that makes the diagram commute. As the outer and right squares are pullback, it follows that also the left square is pullback. Then the epimorphism $n : M \to \mathrm{Im}\, u = \mathrm{Ker}\, p$ implies $q$ is an epimorphism.

Suppose that $u$ is $\mathcal{M}$-covering and $\alpha : P_i \to M$ a morphism with $\alpha u' = 0$. Then $\alpha hq = 0$, and so $\alpha h = 0$. Since $\mathrm{Coker}\, u$ is $\mathcal{M}$-negligible, it follows that $\alpha = 0$ . Thus the homomorphism $\varphi$ induced by $u'$ is a monomorphism. Suppose now that a morphism $\alpha : X\Pi_Y P_i \to M$ satisfies $\alpha i' = 0$. Since $(\mathrm{Ker}(pf), q)$ is the cokernel of $i'$, we deduce that there exists a morphism $t : \mathrm{Ker}(pf) \to M$ such that $tq = \alpha$. Since $\mathrm{Coker}\, u$ is $\mathcal{M}$-negligible, there exists $l : P_i \to M$ such that $lh = t$. Then $lu' = lhq = tq = \alpha$.

Conversely, suppose the sequence of lemma is exact and $g : P_i \to \mathrm{Coker}\, u$ is an arbitrary morphism. Since $P_i$ is projective, there exists a morphism $f : P_i \to Y$ such that $pf = g$, so that $(\mathrm{Ker}(pf), h)$ is the kernel of $g$. If a morphism $\alpha : P_i \to M$ satisfies $\alpha h = 0$, then $\alpha u' = \alpha hq = 0$. Whence $\alpha = 0$ since $\varphi$ is a monomorphism. Now, let $\beta \in \mathrm{Hom}_{\mathcal{A}}(\mathrm{Ker}\, g, M)$. Since $\beta q i' = 0$ and $\mathrm{Ker}\, \psi = \mathrm{Im}\, \varphi$, it follows that there exists $\alpha : P_i \to M$ such that $\beta q = \alpha u'$, i.e., $\beta q = \alpha hq$. Therefore $\beta = \alpha h$. Thus the canonical homomorphism $\mathrm{Hom}_{\mathcal{A}}(P_i, M) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Ker}\, g, M)$ is an isomorphism. Proposition 3.5 implies $\mathrm{Coker}\, u$ is $\mathcal{M}$-negligible. $\square$

## 4. The Popescu-Gabriel generalized theorem

Consider a Grothendieck category $\mathcal{C}$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be some family of generators for $\mathcal{C}$. In section 1.1 we have defined the functor $T : \mathcal{C} \to \operatorname{Mod}\mathcal{A} = (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ that takes each object $X \in \mathcal{C}$ to $TX = (-, X)$ (here $\mathcal{A}$ denotes the ring of projective generators $h_{U_i} = (-, U_i)$, $i \in I$, of the functor category $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$). The next theorem has also been obtained by Prest [**21**].

**Theorem 4.1** (Popescu, Gabriel)**.** *Let $\mathcal{C}$ be a Grothendieck category with a family of generators $\mathcal{U} = \{U_i\}_{i\in I}$ and $T$ be the functor defined above. Then:*

*(1) $T$ is full and faithful;*

*(2) $T$ induces an equivalence between $\mathcal{C}$ and the quotient category $\operatorname{Mod}\mathcal{A}/\mathcal{S}$, where $\mathcal{S}$ denotes the largest localizing subcategory in $\operatorname{Mod}\mathcal{A}$ for which all modules $TX = (-, X)$ are $\mathcal{S}$-closed.*

*Proof.* (1). By Proposition 1.1 the fuctor $T$ is faithful. To see that it is full, we must show that if $X$ and $Y$ are objects in $\mathcal{C}$ and $\Phi : (-, X) \to (-, Y)$ is a functor, then $\Phi$ if of the form $\Phi(f) = \varphi f$ for some $\varphi : X \to Y$. Denote by $\Lambda_i = \operatorname{Hom}_{\mathcal{A}}(U_i, X)$ the set of all morphisms $U_i \to X$ and put $\Lambda = \cup_{i\in I}\Lambda_i$. Given $\alpha \in \Lambda_i$ by $i_\alpha : U_i \to U_\Lambda = \oplus_{\beta\in\Lambda}U_\beta$, where $U_\beta = U_i$ iff $\beta \in \Lambda_i$, we denote the corresponding injection. There exists the unique morphism $\lambda : U_\Lambda \to X$ such that $\lambda i_\alpha = \alpha$ for each $\alpha \in \Lambda$, and $\lambda$ is an epimorphism since $\mathcal{U}$ is a family of generators. Similarly, there exists the unique morphism $\mu : U_\Lambda \to Y$ such that $\mu i_\alpha = \Phi(U_i)(\alpha)$ for every $\alpha \in \Lambda$. Let $\varkappa : K \to U_\Lambda$ be the kernel of $\lambda$. We can show that $\mu\varkappa = 0$, then $\mu$ factors as $\mu = \varphi\lambda$ for some $\varphi : X \to Y$ and for each $i \in I$, $\alpha : U_i \to X$ we get $\Phi(U_i)(\alpha) = \mu i_\alpha = \varphi\lambda i_\alpha = \varphi\alpha$, and our assertion would be proved.

So we need to check that $\mu\varkappa = 0$. For each finite subset $J$ of $\Lambda$ and each $\alpha \in J$ there are the canonical morphisms $\pi'_\alpha : U_J = \oplus_{\beta\in J}U_\beta \to U_\alpha$, $i'_\alpha : U_\alpha \to U_J$ and $i_J : U_J \to U_\Lambda$; here $U_\beta = U_i$ (respectively $U_\alpha = U_i$) if $\beta \in \Lambda_i$ (respectively if $\alpha \in \Lambda_i$). Let $\varkappa_J : K_J \to U_J$ be the kernel of the composed morphism $\lambda i_J : U_J \to X$. Since $K$ is the direct limit of the kernels $K_J$ for all finite subsets $J$ of $\Lambda$, it suffices to show that $\mu i_J\varkappa_J = 0$. Now for each $U_i \in \mathcal{U}$, $\beta : U_i \to K_J$ we have, using the fact that $\Phi$ is a functor, that

$$\begin{aligned}\mu i_J\varkappa_J\beta &= \mu i_J(\sum_{\alpha\in J} i'_\alpha\pi'_\alpha)\varkappa_J\beta = \sum_{\alpha\in J}\mu i_\alpha\pi'_\alpha\varkappa_J\beta = \sum_{\alpha\in J}\Phi(U_\alpha)(\alpha)\pi'_\alpha\varkappa_J\beta \\ &= \sum_{\alpha\in J}\Phi(U_i)(\alpha\pi'_\alpha\varkappa_J\beta) = \Phi(U_i)(\sum_{\alpha\in J}\lambda i_J i'_\alpha\pi'_\alpha\varkappa_J\beta) = \Phi(U_i)(\lambda i_J\varkappa_J\beta) = 0\end{aligned}$$

since $\lambda i_J\varkappa_J = 0$. Since this holds for arbitrary $\beta : U_i \to K_J$, it follows that $\mu i_J\varkappa_J = 0$.

(2). Let $\mathcal{S}$ be the largest localizing subcategory in $\operatorname{Mod}\mathcal{A}$ for which all modules of the form $TX = (-, X)$ are $\mathcal{S}$-closed. This subcategory exists by Lemmas 3.2 and 3.3 and it is cogenerated by the class of injective modules of the form $E(TX) \oplus E(E(TX)/TX)$. Let $\mathcal{T} = \{TX\}_{X\in\mathcal{C}}$; then the corresponding $\mathcal{T}$-negligible objects and $\mathcal{T}$-covering morphisms will be referred to as negligible and covering respectively, omitting the prefix $\mathcal{T}$. Since every module $TX$ is $\mathcal{S}$-closed, there is a functor $T' : \mathcal{C} \to \operatorname{Mod}\mathcal{A}/\mathcal{S}$ such that $T = iT'$ with $i : \operatorname{Mod}\mathcal{A}/\mathcal{S} \to \operatorname{Mod}\mathcal{A}$ the inclusion functor. We must show that $T'$ is an equivalence. Since $iT' = T$ is full and faithful by (1), also $T'$ is full and faithful, and thus it suffices to show that every $\mathcal{S}$-closed module is isomorphic to a module of the form $TX$.

We choose for an $\mathcal{S}$-closed $\mathcal{A}$-module $M$ an exact sequence in $\operatorname{Mod}\mathcal{A}/\mathcal{S}$

$$\oplus_{k\in K} h_{V_k} \xrightarrow{\alpha} \oplus_{j\in J} h_{V_j} \longrightarrow M \longrightarrow 0$$

with $V_k, V_j \in \mathcal{U}$, $K, J$ some sets of indices.

If we knew that $T'$ preserves colimits, we would have then the following isomorphisms:

$$\begin{aligned}(\oplus_{k\in K} h_{V_k}, \oplus_{j\in J} h_{V_j}) &\approx (\oplus_{k\in K} T'V_k, \oplus_{j\in J} T'V_j) \\ &\approx (T'(\oplus_{k\in K} V_k), T'(\oplus_{j\in J} V_j)) \approx {}_{\mathcal{C}}(\oplus_{k\in K} V_k, \oplus_{j\in J} V_j).\end{aligned}$$

In that case, there exists $\beta$ such that $T'\beta = \alpha$.

Now let us define $\widetilde{M}$ by the exact sequence

$$\oplus_{k\in K} V_k \xrightarrow{\beta} \oplus_{j\in J} V_j \longrightarrow \widetilde{M} \longrightarrow 0. \tag{4.1}$$

We now apply the functor $T'$ to (4.1). If we showed that $T'$ is exact, we would obtain the following commutative diagram with exact rows in $\operatorname{Mod}\mathcal{A}/\mathcal{S}$

$$\begin{array}{ccccccc}\oplus_{k\in K} h_{V_k} & \xrightarrow{\alpha} & \oplus_{j\in J} h_{V_j} & \longrightarrow & M & \longrightarrow & 0 \\ \| & & \| & & & & \\ \oplus_{k\in K} h_{V_k} & \xrightarrow[T'\beta=\alpha]{} & \oplus_{j\in J} h_{V_j} & \longrightarrow & T'(\widetilde{M}) & \longrightarrow & 0.\end{array}$$

In that case, $M = T'(\widetilde{M})$. To conclude the proof it thus remains to show:

**Lemma 4.2.** *The functor $T' : \mathcal{C} \to \operatorname{Mod}\mathcal{A}/\mathcal{S}$ is exact and preserves direct sums.*

*Proof.* First, we prove the exactness of $T'$. The functor $T' = (-)_{\mathcal{S}}T$, where $(-)_{\mathcal{S}}$ is the corresponding localizing functor, is obviously left exact, and so it suffices to prove that $T'$ preserves epimorphisms. This means that if $u : X \to Y$ is an epimorphism in $\mathcal{C}$, then the morphism $Tu$ of $\operatorname{Mod}\mathcal{A}$ is covering, i.e., in view of Lemma 3.6, that for any object $Z$ of $\mathcal{C}$ and $U_i \in \mathcal{U}$ we have the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(h_{U_i}, TZ) \to \operatorname{Hom}_{\mathcal{A}}(h_{U_i}\Pi_{TY}TX, TZ) \to \operatorname{Hom}_{\mathcal{A}}(\operatorname{Ker} Tu, TZ) \tag{4.2}$$

induced by any $\mathcal{A}$-homomorphism $f : h_{U_i} \to TY$. Since $T$ is full and faithful and $h_{U_i} = TU_i$, we deduce that there exists the morphism $g : U_i \to Y$ such that $f = Tg$. Therefore we have the commutative diagram

$$\begin{array}{ccccccccc}0 & \longrightarrow & \operatorname{Ker} u & \xrightarrow{i'} & U_i\Pi_Y X & \xrightarrow{u'} & U_i & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow g & & \\ & & & & X & \xrightarrow[u]{} & Y & \longrightarrow & 0\end{array}$$

with exact rows. The short exact sequence

$$0 \longrightarrow \operatorname{Ker} u \longrightarrow U_i\Pi_Y X \longrightarrow U_i \longrightarrow 0$$

induces the exact sequence

$$0 \longrightarrow {}_{\mathcal{C}}(U_i, Z) \longrightarrow {}_{\mathcal{C}}(U_i \Pi_Y X, Z) \longrightarrow {}_{\mathcal{C}}(\mathrm{Ker}\, u, Z).$$

As $T$ is fully faithful, we have the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(h_{U_i}, TZ) \to \mathrm{Hom}_{\mathcal{A}}(T(U_i \Pi_Y X), TZ) \to \mathrm{Hom}_{\mathcal{A}}(T(\mathrm{Ker}\, u), TZ).$$

Since $T$ is left exact, $T(\mathrm{Ker}\, u)$ is isomorphic to $\mathrm{Ker}(Tu)$, $T(U_i \Pi_Y X)$ is obviously isomorphic to $h_{U_i} \Pi_{TY} TX$ and these isomorphisms are functorial. Thus the sequence (4.2) is exact for all $Z \in \mathcal{C}$, $U_i \in \mathcal{U}$ and now it suffices to apply Lemma 3.6.

It remains to prove that $T'$ preserves direct sums. Actually we prove a little more, namely that it preserves direct unions. Suppose $\{X_\gamma\}_{\gamma \in \Gamma}$ is a directed family of subobjects of $X \in \mathcal{C}$ such that $X = \sum_{\gamma \in \Gamma} X_\gamma$ with $\Gamma$ some set of indices. We need to show that the canonical monomorphism

$$u : \sum_{\gamma \in \Gamma} TX_\gamma \longrightarrow T(\sum_{\gamma \in \Gamma} X_\gamma)$$

is covering. Let $q$ be an arbitrary morphism $TU_i \to \mathrm{Coker}\, u$. Since $TU_i = h_{U_i}$ is a projective object of $\mathrm{Mod}\, \mathcal{A}$, there exists a morphism $g : TU_i \to TX$ such that $q = vg$ where $v = \mathrm{Coker}\, u$. Since $T$ is fully faithful, there exists a morphism $f : U_i \to X$ such that $g = Tf$. Let $X_\gamma \Pi_X U_i$ be the fibered product associated to the diagram

$$\begin{array}{ccc} & & U_i \\ & & \big\downarrow f \\ X_\gamma & \xrightarrow[u_\gamma]{} & X \end{array}$$

with $u_\gamma$ the canonical monomorphism. Since $T$ is left exact, it follows that $T(X_\gamma \Pi_X U_i)$ is the fibered product associated to the diagram

$$\begin{array}{ccc} & & TU_i \\ & & \big\downarrow Tf \\ TX_\gamma & \xrightarrow[Tu_\gamma]{} & TX. \end{array}$$

We obtain thus that $\sum\limits_{\gamma \in \Gamma} T(X_\gamma \Pi_X U_i)$ is the fibered product associated to the diagram

$$\begin{array}{ccc} & & TU_i \\ & & \big\downarrow Tf \\ \sum\limits_{\gamma \in \Gamma} TX_\gamma & \xrightarrow[Tu]{} & TX \end{array}$$

that implies $\sum\limits_{\gamma \in \Gamma} T(X_\gamma \Pi_X U_i) \approx (\sum\limits_{\gamma \in \Gamma} TX_\gamma) \Pi_{TX} TU_i$. Since

$$\sum\nolimits_{\mathcal{C}} (X_\gamma \Pi_X U_i) \approx (\sum\nolimits_{\mathcal{C}} X_\gamma) \Pi_X U_i = X \Pi_X U_i \approx U_i$$

and $T$ is fully faithful, one obtains

$$\mathrm{Hom}_{\mathcal{A}}(\sum_{\gamma\in\Gamma} T(X_\gamma \Pi_X U_i), TZ) \approx \varprojlim \mathrm{Hom}_{\mathcal{A}}(T(X_\gamma \Pi_X U_i), TZ)$$
$$\approx \varprojlim {}_{\mathcal{C}}((X_\gamma \Pi_X U_i), Z) \approx {}_{\mathcal{C}}((\sum_{\gamma\in\Gamma} X_\gamma \Pi_X U_i), Z) \approx {}_{\mathcal{C}}(U_i, Z) \approx \mathrm{Hom}_{\mathcal{A}}(h_{U_i}, TZ)$$

for all $Z \in \mathcal{C}$. Hence we get, in view of Lemma 3.6, that $\mathrm{Coker}\, u$ is negligible.

Thus the functor $T'$ is an equivalence. This concludes the proof of the Popescu-Gabriel generalized theorem. □

**Corollary 4.3.** *Let $\mathcal{C}$ be a Grothendieck category with a family of generators $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{M} = \{M_j\}_{j\in J}$ an arbitrary family of objects of $\mathcal{C}$. Let $\overline{\mathcal{U}} = \mathcal{U} \cup \mathcal{M}$ and $\mathcal{A} = \{h_U = (-, U)\}_{U\in\overline{\mathcal{U}}}$. Then the functor $T : \mathcal{C} \to \mathrm{Mod}\,\mathcal{A}$, $TX = (-, X)$, determines an equivalence of $\mathcal{C}$ and the quotient category $\mathrm{Mod}\,\mathcal{A}/\mathcal{S}$, where $\mathcal{S}$ denotes the largest localizing subcategory in $\mathrm{Mod}\,\mathcal{A}$ for which all modules $TX$ are $\mathcal{S}$-closed.*

*Proof.* It suffices to observe that $\overline{\mathcal{U}}$ is again a family of generators for $\mathcal{C}$. □

**Corollary 4.4** (Popescu, Gabriel [**22**])**.** *If $\mathcal{C}$ is a Grothendieck category, $U$ is a generator of $\mathcal{C}$, then the functor ${}_{\mathcal{C}}(U, -)$ establishes an equivalence between the categories $\mathcal{C}$ and $\mathrm{Mod}\, A/\mathcal{S}$ with $A = \mathrm{End}\, U$ the endomorphism ring of $U$, $\mathcal{S}$ some localizing subcategory of $\mathrm{Mod}\, A$.*

*Proof.* By Theorem 4.1 the category $\mathcal{C}$ is equivalent to the quotient category of $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ with $\mathcal{U} = \{U\}$. Since the representable functor $h_U$ is a finitely generated projective generator for $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$, by the Mitchell theorem, it follows that $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is equivalent to the category of right $A$-modules $\mathrm{Mod}\, A$. □

**Remark.** In proving Lemma 4.2 we essentially have used Lemma 3.6, which, in turn, uses the fact that every $\mathcal{A}$-module has an injective envelope. There are different ways of (independent of the Popescu-Gabriel theorem [**22**]) proving this fact (see, e.g., [**16**] or [**5**, §III.3.10]).

4.1. **The Gabriel formula.** Let $\mathcal{P}$ be a localizing subcategory of $\mathcal{C}$. We shall identify via the functor $T : \mathcal{C} \to \mathrm{Mod}\,\mathcal{A}$, $\mathcal{A} = \{h_{U_i}\}_{i\in I}$, the category $\mathcal{C}$ and the quotient category $\mathrm{Mod}\,\mathcal{A}/\mathcal{S}$. By Proposition 1.6 there exists the localizing subcategory $\mathcal{T}$ of $\mathrm{Mod}\,\mathcal{A}$ such that $\mathcal{T} \supseteq \mathcal{S}$ and $\mathcal{T}/\mathcal{S} = \mathcal{P}$. Moreover, the quotient category $\mathcal{C}/\mathcal{P}$ is equivalent to the quotient category $\mathrm{Mod}\,\mathcal{A}/\mathcal{T}$.

Let $\mathfrak{F}$ be the Gabriel topology of $\mathcal{A}$ that corresponds to $\mathcal{T}$. Similar to the module categories the family $\mathfrak{G} = \cup_{i\in I}\mathfrak{G}^i$

$$\mathfrak{G}^i = \{X \mid X \subseteq U_i, U_i/X \in \mathcal{P}\}$$

we call the *Gabriel topology* for $\mathcal{U}$. One easily verifies that $\mathfrak{G} \subseteq \mathfrak{F}$ and

$$\mathfrak{G} = \mathfrak{F}_{\mathcal{S}} = \{\mathfrak{a}_{\mathcal{S}} \mid \mathfrak{a} \in \mathfrak{F}\}.$$

**Proposition 4.5.** *For a Gabriel topology $\mathfrak{G}$ of $\mathcal{U}$ the following conditions hold:*

*T1′. $U_i \in \mathfrak{G}^i$ for each $i \in I$;*

*T2′. if $\mathfrak{a} \in \mathfrak{G}^i$ and $\mu \in {}_{\mathcal{C}}(U_j, U_i)$, $U_j \in \mathcal{U}$, then $\{\mathfrak{a} : \mu\} = \mu^{-1}(\mathfrak{a}) \in \mathfrak{G}^j$;*

*T3′. if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are subobjects of* $U_i$ *such that* $\mathfrak{a} \in \mathfrak{G}^i$ *and* $\{\mathfrak{b} : \mu\} \in \mathfrak{G}^j$ *for every* $\mu \in {}_{\mathcal{C}}(U_j, U_i)$ *with* $\operatorname{Im}\mu \subset \mathfrak{a}$, $U_j \in \mathcal{U}$*, then* $\mathfrak{b} \in \mathfrak{G}^i$.

*Proof.* $T1'$. Obvious.

$T2'$. Since the left square of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu^{-1}(\mathfrak{a}) & \longrightarrow & U_j & \longrightarrow & U_j/\mu^{-1}(\mathfrak{a}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & U_i & \longrightarrow & U_i/\mathfrak{a} & \longrightarrow & 0 \end{array}$$

is pullback, $\delta$ is a monomorphism. Since $\mathcal{P}$ is closed under subobjects, it follows that $U_j/\mu^{-1}(\mathfrak{a}) \in \mathcal{P}$.

$T3'$. Let $\mathfrak{a} \in \mathfrak{G}^i$ and $\mu : h_{U_j} \to h_{U_i}$ be an $\mathcal{A}$-homomorphism such that $\operatorname{Im}\mu \subseteq \mathfrak{a}$; then $\mathfrak{a} \in \mathfrak{F}$. Since $(-)_{\mathcal{T}} = (-)_{\mathcal{P}}(-)_{\mathcal{S}}$, we get that

$$(h_{U_j}/\mu^{-1}(\mathfrak{b}))_{\mathcal{T}} = (U_j/\mu_{\mathcal{S}}^{-1}(\mathfrak{b}))_{\mathcal{P}} = 0,$$

because $U_j/\mu_{\mathcal{S}}^{-1}(\mathfrak{b}) \in \mathcal{P}$. Therefore $\mu^{-1}(\mathfrak{b}) \in \mathfrak{F}$ and $T3$ implies that $\mathfrak{b} \in \mathfrak{F}$. But $\mathfrak{b}$ is $\mathcal{S}$-closed, and, hence, $\mathfrak{b} \in \mathfrak{G}$. □

**Remark.** In contrast to the Gabriel topology defined for a ring $\mathcal{A} = \{P_i\}_{i\in I}$ it is not enough the conditions $T1' - T3'$ that the family $\mathfrak{G} = \cup_{i\in I}\mathfrak{G}^i$, satisfying these conditions, determines some localizing subcategory in $\mathcal{C}$. The precise conditions (namely, the axioms $T1' - T2'$ and a little bit strengthened axiom $T3'$) when $\mathfrak{G}$ determines localization in $\mathcal{C}$ the reader can find in [**23**].

Further, construction of the $\mathcal{P}$-envelope $X_{\mathcal{P}}$ of an object $X$ from $\mathcal{C}$ is similar to $\operatorname{Mod}\mathcal{A}$ (see also [**24**]). Namely, since the localizing functor $(-)_{\mathcal{T}}$ factors as $(-)_{\mathcal{T}} = (-)_{\mathcal{P}}(-)_{\mathcal{S}}$, for the localization $X_{\mathcal{P}}$ of $X$ the Gabriel formula

$$X_{\mathcal{P}}(U_i) = X_{\mathcal{T}}(U_i) = \varinjlim_{\mathfrak{a}\in\mathfrak{F}^i}(\mathfrak{a}, X/t_{\mathcal{T}}(X)) = \\ \varinjlim_{\mathfrak{a}\in\mathfrak{F}^i}(\mathfrak{a}, X/t_{\mathcal{P}}(X)) \approx \varinjlim_{\mathfrak{a}_{\mathcal{S}}\in\mathfrak{G}^i}(\mathfrak{a}_{\mathcal{S}}, X/t_{\mathcal{P}}(X))$$

holds. This isomorphism is functorial in both arguments. Here, we have used the fact that $t_{\mathcal{T}}(X) = t_{\mathcal{P}}(X)$ for every $X \in \mathcal{C}$.

Using Proposition 3.5 and properties of localizing subcategories, we can easily prove the following statement.

**Proposition 4.6.** *Let* $\mathcal{M}$ *be some class of objects of* $\mathcal{C}$*. Then an object* $S$ *is* $\mathcal{M}$*-negligible if and only if for an arbitrary morphism* $f : U_i \to S$, $U_i \in \mathcal{U}$*, the canonical homomorphism* $(U_i, M) \to (\operatorname{Ker} f, M)$ *with* $M \in \mathcal{M}$ *is an isomorphism.*

4.2. **Projective generating families.** Suppose $\mathcal{U} = \{U_i\}_{i\in I}$ is some family of objects from $\mathcal{C}$. We denote by $\mathcal{G}_{\mathcal{U}}$ the subcategory of the category $\mathcal{C}$ that consists of the objects generated by the family $\mathcal{U}$. Namely, $C \in \mathcal{G}_{\mathcal{U}}$ if and only if for $C$ there is a presentation

$$\oplus_K V_k \longrightarrow \oplus_J V_j \longrightarrow C \longrightarrow 0$$

with $V_k$, $V_j \in \mathcal{U}$. We refer to $\mathcal{U}$ as a *generating family* for $\mathcal{G}_{\mathcal{U}}$. When $\mathcal{U} = \{U\}$, we write $\mathcal{G}_U = \mathcal{G}_{\mathcal{U}}$. Clearly, $\mathcal{G}_{\mathcal{U}} = \mathcal{C}$ if and only if $\mathcal{U}$ is the family of generators for $\mathcal{C}$.

We consider the categories $\mathcal{G}_{\mathcal{U}}$ generated by projective families $\mathcal{U}$. So, let $P$ be a projective object of $\mathcal{C}$. For every $M \in \mathcal{G}_P$ there is a projective presentation

$$\oplus_I P_i \longrightarrow \oplus_J P_j \longrightarrow M \longrightarrow 0 \tag{4.3}$$

with $I$, $J$ some sets of indices, $P_i = P_j = P$ for any $i, j$.

**Theorem 4.7.** *Let $\mathcal{C}$ be a Grothendieck category, $P$ some projective object of $\mathcal{C}$. Then the subcategory $\mathcal{S} = \{C \in \mathcal{C} \mid {}_{\mathcal{C}}(P, C) = 0\}$ is localizing and the localized object $P_{\mathcal{S}}$ is a $\mathcal{C}/\mathcal{S}$-projective generator. Moreover, the localizing functor $(-)_{\mathcal{S}}$ induces an equivalence of the categories $\mathcal{C}/\mathcal{S}$ and $\mathcal{G}_P$.*

*Proof.* First, we show that $\mathcal{S}$ is localizing. Let

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

be a short exact sequence of $\mathcal{C}$. If we apply the exact functor ${}_{\mathcal{C}}(P, -)$, we obtain the short exact sequence

$$0 \longrightarrow {}_{\mathcal{C}}(P, A') \longrightarrow {}_{\mathcal{C}}(P, A) \longrightarrow {}_{\mathcal{C}}(P, A'') \longrightarrow 0,$$

whence it follows that ${}_{\mathcal{C}}(P, A) = 0$ if and only if ${}_{\mathcal{C}}(P, A') = 0$ and ${}_{\mathcal{C}}(P, A'') = 0$, i.e., $\mathcal{S}$ is a Serre subcategory. Further, if we consider the map

$${}_{\mathcal{C}}(P, \oplus A_i) \xrightarrow{\varphi} {}_{\mathcal{C}}(P, \prod A_i) \approx \prod {}_{\mathcal{C}}(P, A_i) = 0,$$

where $A_i \in \mathcal{S}$ and $\varphi$ is the monomorphism induced by the canonical monomorphism $\oplus A_i \to \prod A_i$, we get that $\mathcal{S}$ is closed under coproducts. So, $\mathcal{S}$ is a localizing subcategory.

In the rest of the proof we shall show that $P_{\mathcal{S}}$ is a projective generator for $\mathcal{C}/\mathcal{S}$. First, we consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{C}/\mathcal{S}$. It induces the exact sequence in $\mathcal{C}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow S \longrightarrow 0$$

with $S \in \mathcal{S}$. If we apply the exact functor ${}_{\mathcal{C}}(P, -)$, we get the following commutative diagram of Abelian groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_{\mathcal{C}}(P, A) & \longrightarrow & {}_{\mathcal{C}}(P, B) & \longrightarrow & {}_{\mathcal{C}}(P, C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}_{\mathcal{C}/\mathcal{S}}(P_{\mathcal{S}}, A) & \longrightarrow & {}_{\mathcal{C}/\mathcal{S}}(P_{\mathcal{S}}, B) & \longrightarrow & {}_{\mathcal{C}/\mathcal{S}}(P_{\mathcal{S}}, C) & \longrightarrow & 0 \end{array}$$

where the vertical arrows are isomorphisms. We see that $P_{\mathcal{S}}$ is a $\mathcal{C}/\mathcal{S}$-projective object.

It remains to check that $P_{\mathcal{S}}$ is a generator for $\mathcal{C}/\mathcal{S}$. Let $A$ be an arbitrary object of $\mathcal{C}/\mathcal{S}$, $I = {}_{\mathcal{C}}(P, A)$; then there exists the morphism $u : \oplus_{i \in I} P_i \to A$ with $P_i = P$, $u_i = i$ for all $i \in I$. We have the exact sequence in $\mathcal{C}$

$$\oplus P_i \xrightarrow{u} A \longrightarrow \operatorname{Coker} u \longrightarrow 0.$$

Let $w : P \to A$ be an arbitrary morphism. By construction of $u$ we have that $\operatorname{Im} w \subseteq \operatorname{Im} u$. Therefore there exists a morphism $p : P \to \oplus P_i$ such that $w = up$ (since $P$ is projective) that implies ${}_{\mathcal{C}}(P, \operatorname{Coker} u) = 0$, i.e., $\operatorname{Coker} u \in \mathcal{S}$.

Thus,

$$u_{\mathcal{S}} : (\oplus P_i)_{\mathcal{S}} \approx \oplus (P_i)_{\mathcal{S}} \longrightarrow A_{\mathcal{S}}$$

is a $\mathcal{C}/\mathcal{S}$-epimorphism. Thus any object $A \in \mathcal{C}/\mathcal{S}$ is a quotient object of $\oplus(P_i)_{\mathcal{S}}$, and so $P_{\mathcal{S}}$ is a generator for $\mathcal{C}/\mathcal{S}$.

Now, show that the restriction of $(-)_{\mathcal{S}}$ to $\mathcal{G}_P$ determines an equivalence of $\mathcal{C}/\mathcal{S}$ and $\mathcal{G}_P$. If $M \in \mathcal{C}$ there is the exact sequence of the form (1.1)

$$0 \longrightarrow A' \longrightarrow M \xrightarrow{\lambda} M_{\mathcal{S}} \longrightarrow A'' \longrightarrow 0$$

with $A'$, $A'' \in \mathcal{S}$. Hence $\lambda$ induces an isomorphism:

$${}_{\mathcal{C}}(P, M) \approx {}_{\mathcal{C}}(P, M_{\mathcal{S}}) \approx {}_{\mathcal{C}/\mathcal{S}}(P_{\mathcal{S}}, M_{\mathcal{S}}). \tag{4.4}$$

Then

$${}_{\mathcal{C}}(\oplus P_i, M) \approx \prod {}_{\mathcal{C}}(P_i, M) \approx {}_{\mathcal{C}}(\oplus P_i, M_{\mathcal{S}}) \approx {}_{\mathcal{C}/\mathcal{S}}(\oplus (P_i)_{\mathcal{S}}, M_{\mathcal{S}})$$

with $P_i = P$ for any $i \in I$. Now, if we consider the projective presentation (4.3) for $M \in \mathcal{G}_P$, we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_{\mathcal{C}}(M, N) & \longrightarrow & {}_{\mathcal{C}}(\oplus P_j, N) & \longrightarrow & {}_{\mathcal{C}}(\oplus P_i, N) \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & {}_{\mathcal{C}}(M, N_{\mathcal{S}}) & \longrightarrow & {}_{\mathcal{C}}(\oplus P_j, N_{\mathcal{S}}) & \longrightarrow & {}_{\mathcal{C}}(\oplus P_i, N_{\mathcal{S}}) \end{array}$$

with exact rows and the vertical arrows isomorphisms. Thus,

$${}_{\mathcal{C}}(M, N) \approx {}_{\mathcal{C}}(M, N_{\mathcal{S}}) \approx {}_{\mathcal{C}/\mathcal{S}}(M_{\mathcal{S}}, N_{\mathcal{S}})$$

that implies $(-)_{\mathcal{S}}|_{\mathcal{G}_P}$ is a fully faithful functor. Finally, let $N \in \mathcal{C}/\mathcal{S}$. Consider a $\mathcal{C}/\mathcal{S}$-projective presentation of $N$

$$\oplus (P_i)_{\mathcal{S}} \xrightarrow{\alpha} \oplus (P_j)_{\mathcal{S}} \longrightarrow N \longrightarrow 0$$

where $P_i = P_j = P$. Then there exists $\beta : \oplus P_i \to \oplus P_j$ such that $\alpha = \beta_{\mathcal{S}}$, hence $N \approx (\operatorname{Coker} \beta)_{\mathcal{S}}$, as was to be proved. □

**Corollary 4.8.** *Under the conditions of Theorem 4.7 the quotient category $\mathcal{C}/\mathcal{S}$ is equivalent to a quotient category* $\operatorname{Mod} A/\mathcal{P}$, *where* $A = \operatorname{End} P$ *is the endomorphism ring of $P$, $\mathcal{P}$ is some localizing subcategory of* $\operatorname{Mod} \mathcal{A}$.

*Proof.* It suffices to observe that the isomorphisms (4.4) induce a ring isomorphism

$$A = {}_{\mathcal{C}}(P, P) \approx {}_{\mathcal{C}}(P, P_{\mathcal{S}}) \approx {}_{\mathcal{C}/\mathcal{S}}(P_{\mathcal{S}}, P_{\mathcal{S}}).$$

Since $P_{\mathcal{S}}$ is a generator for $\mathcal{C}/\mathcal{S}$, our assertion follows from Corollary 4.4. □

Now, we consider an arbitrary family of projective objects $\mathcal{U} = \{P_i\}_{i \in I}$. Let $\mathcal{G}_{\mathcal{U}}$ be the category of $\mathcal{C}$ generated by $\mathcal{U}$.

**Corollary 4.9.** *The subcategory $\mathcal{S} = \{C \in \mathcal{C} \mid {}_{\mathcal{C}}(P, C) = 0$ for all $P \in \mathcal{U}\}$ is localizing and $\mathcal{C}/\mathcal{S}$ is equivalent to the quotient category* $\operatorname{Mod} \mathcal{A}/\mathcal{P}$ *with $\mathcal{A} = \{h_P\}_{P \in \mathcal{U}}$, $\mathcal{P}$ some localizing subcategory of* $\operatorname{Mod} \mathcal{A}$. *Moreover, the functor $(-)_{\mathcal{S}}$ induces an equivalence of the categories $\mathcal{C}/\mathcal{S}$ and $\mathcal{G}_{\mathcal{U}}$.*

*Proof.* Denote by $Q = \oplus_{P\in\mathcal{U}}P$; then $\mathcal{S} = \{C \in \mathcal{C} \mid {}_{\mathcal{C}}(Q, C) = 0\}$. The preceding theorem implies that $\mathcal{S}$ is localizing and $Q_{\mathcal{S}} = \oplus_{P\in\mathcal{U}}P_{\mathcal{S}}$ is a $\mathcal{C}/\mathcal{S}$-projective generator. Therefore $\mathcal{U}_{\mathcal{S}} = \{P_{\mathcal{S}} \mid P \in \mathcal{U}\}$ is a family of projective generators for $\mathcal{C}/\mathcal{S}$.

In view of the isomorphisms (4.4), there is an equivalence of $\mathcal{U}$ and $\mathcal{U}_{\mathcal{S}}$. Now our assertion follows from Theorem 4.1. □

**Remark.** Under the conditions of Corollary 4.9 the localizing functor $(-)_{\mathcal{S}}$ factors through $\mathcal{G}_{\mathcal{U}}$

$$\mathcal{C}[2]e, t(-)_{\mathcal{S}}se, bF\mathcal{C}/\mathcal{S}[2]\mathcal{G}_{\mathcal{U}}, ne, bG$$

where $G$ is the restriction of $(-)_{\mathcal{S}}$ to $\mathcal{G}_{\mathcal{U}}$ and $F$ is constructed as follows. If $C \in \mathcal{C}$ and $I = \cup_{P\in\mathcal{U}}I_P$ with $I_P = {}_{\mathcal{C}}(P, C)$, there is the morphism $\varphi : \oplus_{\mu\in I}P_\mu \to C$ such that $\varphi_\mu = \mu$. In a similar way we can construct the morphism $\psi : \oplus_{\nu\in J}P_\nu \to \operatorname{Ker}\varphi$ with $J = \cup_{P\in\mathcal{U}}J_P$, $J_P = {}_{\mathcal{C}}(P, \operatorname{Ker}\varphi)$. So, we have the following commutative diagram:

$$\oplus_{\nu\in J}P_\nu[2]e, t\zeta se, b\psi\oplus_{\mu\in I}P_\mu e, t\varphi C[2]\operatorname{Ker}\varphi, ne, bi$$

where $i$ is the kernel of $\varphi$. By definition, we put $F(C) = \operatorname{Coker}\zeta$.

Let $\mathcal{A} = \{P_i\}_{i\in I}$ be a family of finitely generated projective objects of $\mathcal{C}$. Taking the preceding arguments into account, we have an isomorphism $\mathcal{G}_{\mathcal{A}} \approx (\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$.

**Theorem 4.10.** *Suppose $\mathcal{C}$ is a Grothendieck category and $\mathcal{A} = \{P_i\}_{i\in I}$ is some family of finitely generated projective objects for $\mathcal{C}$. Then the subcategory $\mathcal{S} = \{C \in \mathcal{C} \mid {}_{\mathcal{C}}(P_i, C) = 0$ for all $P_i \in \mathcal{A}\}$ is localizing and $\mathcal{A}_{\mathcal{S}} = \{(P_i)_{\mathcal{S}}\}_{i\in I}$ is a family of finitely generated projective generators for the quotient category $\mathcal{C}/\mathcal{S}$. Moreover, the localizing functor $(-)_{\mathcal{S}}$ induces an equivalence of $\mathcal{C}/\mathcal{S}$ and* $\operatorname{Mod}\mathcal{A}$.

*Proof.* By Corollary 4.9 $\mathcal{A}_{\mathcal{S}}$ is a family of projective generators for $\mathcal{C}/\mathcal{S}$. It thus remains to show that every $(P_i)_{\mathcal{S}} \in \mathcal{A}_{\mathcal{S}}$ is $\mathcal{C}/\mathcal{S}$-finitely generated.

To see this, we consider an object $X$ of $\mathcal{C}/\mathcal{S}$. Let $\{X_j\}_{j\in J}$ be a directed family of $\mathcal{C}/\mathcal{S}$-subobjects of $X$ such that $X = \sum_{\mathcal{C}/\mathcal{S}} X_j$. Since $X_j$ are also $\mathcal{C}$-subobjects, it follows that the $\mathcal{C}$-direct union $\sum_{\mathcal{C}} X_j$ is a subobject of $X$ and the quotient object $A = X/\sum_{\mathcal{C}} X_j$ belongs to $\mathcal{S}$. Indeed, if we apply the exact localizing functor $(-)_{\mathcal{S}}$, commuted with the direct limits, to the short exact sequence

$$0 \longrightarrow \sum\nolimits_{\mathcal{C}} X_j \longrightarrow X \longrightarrow A \longrightarrow 0$$

we shall obtain the short exact sequence

$$0 \longrightarrow (\sum\nolimits_{\mathcal{C}} X_j)_{\mathcal{S}} = \sum\nolimits_{\mathcal{C}/\mathcal{S}} X_j \longrightarrow X \longrightarrow A_{\mathcal{S}} \longrightarrow 0.$$

Whence $A_{\mathcal{S}} = 0$, i.e., $A \in \mathcal{S}$. For any $P_i \in \mathcal{A}$ one has then

$${}_{\mathcal{C}}(P_i, \sum\nolimits_{\mathcal{C}} X_j) \approx {}_{\mathcal{C}}(P_i, X) \approx {}_{\mathcal{C}/\mathcal{S}}((P_i)_{\mathcal{S}}, \sum\nolimits_{\mathcal{C}/\mathcal{S}} X_j).$$

Thus,

$${}_{\mathcal{C}/\mathcal{S}}((P_i)_{\mathcal{S}}, \sum\nolimits_{\mathcal{C}/\mathcal{S}} X_j) \approx {}_{\mathcal{C}}(P_i, \sum\nolimits_{\mathcal{C}} X_j) \approx \varinjlim {}_{\mathcal{C}}(P_i, X_j) \approx \varinjlim {}_{\mathcal{C}/\mathcal{S}}((P_i)_{\mathcal{S}}, X_j).$$

By Theorem 1.8 $(P_i)_{\mathcal{S}} \in \operatorname{fg}\mathcal{C}/\mathcal{S}$.

The proof of the fact that the localizing functor $(-)_{\mathcal{S}}$ induces an equivalence between $\mathcal{C}/\mathcal{S}$ and $\operatorname{Mod}\mathcal{A}$ is similar to that of Theorem 4.7. □

**Corollary 4.11.** [**10**, 2.1] *Let $\mathcal{C}$ be a Grothendieck category and $P$ be some finitely generated projective object of $\mathcal{C}$. Then the subcategory $\mathcal{S} = \{C \in \mathcal{C} \mid {}_{\mathcal{C}}(P, C) = 0\}$ is localizing and the functor ${}_{\mathcal{C}/\mathcal{S}}(P_{\mathcal{S}}, -)$ determines an equivalence of the categories $\mathcal{C}/\mathcal{S}$ and $\operatorname{Mod} A$ with $A = \operatorname{End} P$ the endomorphism ring of the object $P$.*

*Proof.* According to the preceding theorem the localized object $P_{\mathcal{S}}$ is a finitely generated projective generator. By the Mitchell theorem and Corollary 4.8, the quotient category $\mathcal{C}/\mathcal{S}$ is equivalent to the category of modules over the ring $A = \operatorname{End}_{\mathcal{C}} P$. □

**Example.** Consider the category of generalized right $A$-modules $\mathcal{C}_A = (\operatorname{mod} A^{\mathrm{op}}, \mathrm{Ab})$. Let $M \in \operatorname{mod} A^{\mathrm{op}}$, $R = \operatorname{End}_A M$ and $\mathcal{S}_M = \{F \in \mathcal{C}_A \mid F(M) = 0\}$. By Corollary 4.11 there exists an equivalence of the categories $\mathcal{C}_A/\mathcal{S}_M \xrightarrow{h} \operatorname{Mod} R$ where $h(F_{\mathcal{S}_M}) = F_{\mathcal{S}_M}(M) = F(M)$ for any $F \in \mathcal{C}_A$. The quasi-inverse functor to $h$ is constructed as follows: $g : \operatorname{Mod} R \to \mathcal{C}_A/\mathcal{S}_M$, $E \overset{g}{\mapsto} ((M, E) \otimes_A -)_{\mathcal{S}_M}$. In particular, given $F \in \mathcal{C}_A$ there is an isomorphism $F_{\mathcal{S}_M} \approx ((M, F(M)) \otimes_A -)_{\mathcal{S}_M}$.

Suppose $\operatorname{Mod} \mathcal{A}$, $\mathcal{A} = \{P_i\}_{i \in I}$, is the category of right $\mathcal{A}$-modules. For any $P_i \in \mathcal{A}$ we put

$$\mathcal{S}_i = \{M \in \operatorname{Mod} \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(P_i, M) = 0\}.$$

By Corollary 4.11 $\operatorname{Mod} \mathcal{A}/\mathcal{S}_i \approx \operatorname{Mod} A_i$ with $A_i = \operatorname{End} P_i$. We consider this equivalence as identification. Then the following relation holds:

$$\bigcup_{P_i \in \mathcal{A}} \operatorname{Mod} A_i \subseteq \operatorname{Mod} \mathcal{A}.$$

Generally speaking, as it shows the next example, the categories of $A_i$-modules $\operatorname{Mod} A_i$ do not cover $\operatorname{Mod} \mathcal{A}$ intirely.

**Example.** Consider the category of generalized Abelian groups ${}_{\mathbb{Z}}\mathcal{C} = (\operatorname{mod} \mathbb{Z}, \mathrm{Ab})$. In section 1 we have said that the functor $M \mapsto - \otimes M$ identifies the pure-injective Abelian groups and the injective objects of ${}_{\mathbb{Z}}\mathcal{C}$. By the Kaplansky theorem [**25**] the indecomposable pure-injective Abelian groups $\operatorname{Zg} {}_{\mathbb{Z}}\mathcal{C}$ are precisely the Abelian groups of the form:

1. the injective modules $\mathbb{Q}$ and, for every prime $p$, the Prüfer groups $\mathbb{Z}_{p^\infty}$;
2. every cyclic group $\mathbb{Z}_{p^n}$ of order a prime power;
3. for every prime $p$, the $p$-adic completion $\overline{\mathbb{Z}}_p = \varprojlim \mathbb{Z}_{p^n}$ of the intergers.

Ziegler [**6**] has shown that a similar argument holds for Dedekind domains.

Let $\mathcal{S}_M = \{C \in {}_{\mathbb{Z}}\mathcal{C} \mid C(M) = 0\}$ with $M \in \operatorname{mod} \mathbb{Z}$. As any finitely generated Abelian group is isomorphic to $\mathbb{Z}^{\oplus m} \oplus \mathbb{Z}^{\oplus n}_{p_i^{k_i}}$ with $p_1, \ldots, p_n$ prime, it suffices to show that the relation

$$\operatorname{Mod} \mathbb{Z} \bigcup \Big[ \bigcup_{p \text{ is prime}} \operatorname{Mod} \mathbb{Z}_{p^n} \Big] \subsetneq {}_{\mathbb{Z}}\mathcal{C}$$

holds. According to [**10**, 2.4] $- \otimes Q \in \operatorname{Zg} {}_{\mathbb{Z}}\mathcal{C}$ belongs to ${}_{\mathbb{Z}}\mathcal{C}/\mathcal{S}_M \approx \operatorname{Mod} A_M$ with $A_M = \operatorname{End} M$ if and only if $Q \approx \operatorname{Hom}_{A_M}(M, E)$, where $E$ is an indecomposable injective $A_M$-module. In this case, $E \approx M \otimes Q$. Clearly, $\mathbb{Q}$, $\mathbb{Z}_{p^\infty} \in {}_{\mathbb{Z}}\mathcal{C}/\mathcal{S}_{\mathbb{Z}}$ and $\mathbb{Z}_{p^n} \in {}_{\mathbb{Z}}\mathcal{C}/\mathcal{S}_{\mathbb{Z}_{p^n}}$. However $\overline{\mathbb{Z}}_p$ does not satisfy the indicated condition. Indeed, it is not an injective Abelian group, and therefore $- \otimes \overline{\mathbb{Z}}_p \notin {}_{\mathbb{Z}}\mathcal{C}/\mathcal{S}_{\mathbb{Z}}$. Since for prime $p \neq q$ the object

$(\mathbb{Z}_p, -) \in \mathcal{S}_{\mathbb{Z}_{q^n}}$ and $\mathbb{Z}_p \otimes \overline{\mathbb{Z}}_p \approx \mathbb{Z}_p \neq 0$, it follows that $t_{\mathcal{S}_{\mathbb{Z}_{q^n}}}(- \otimes Q) \neq 0$ where $t_{\mathcal{S}_{\mathbb{Z}_{q^n}}}$ is the $\mathcal{S}_{\mathbb{Z}_{q^n}}$-torsion functor. Hence, in view of Lemma 1.3, $- \otimes \overline{\mathbb{Z}}_p \notin {}_{\mathbb{Z}}\mathcal{C}/\mathcal{S}_{\mathbb{Z}_{q^n}}$. Finally, $- \otimes \overline{\mathbb{Z}}_p \notin {}_{\mathbb{Z}}\mathcal{C}/\mathcal{S}_{\mathbb{Z}_{p^n}}$, because $(\mathbb{Z}_{p^n}, \mathbb{Z}_{p^n} \otimes \overline{\mathbb{Z}}_p) \approx \mathbb{Z}_{p^n} \not\approx \overline{\mathbb{Z}}_p$.

## 5. Finiteness conditions for localizing subcategories

Let $\mathcal{S}$ be a localizing subcategory of the Grothendieck category $\mathcal{C}$. In this section we show how some properties for the family of generators $\mathcal{U} = \{U_i\}_{i \in I}$ of the category $\mathcal{C}$ describe various finiteness conditions for $\mathcal{S}$.

Consider the inclusion functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$. We say that $\mathcal{S}$ is of *prefinite type* (respectively of *finite type*) provided that $i$ commutes with the direct unions (respectively with the direct limits), i.e., for any $\mathcal{S}$-closed object $C$ and any directed family $\{C_i\}_I$ of $\mathcal{S}$-closed subobjects of $C$ the relation

$$\sum\nolimits_{\mathcal{C}/\mathcal{S}} C_i = \sum\nolimits_{\mathcal{C}} C_i$$

holds (respectively $\varinjlim_{\mathcal{C}/\mathcal{S}} C_i = \varinjlim_{\mathcal{C}} C_i$).

It should be remarked that if $\mathcal{S}$ is of prefinite type, then, in particular, every direct sum of $\mathcal{S}$-closed objects is $\mathcal{S}$-closed. Recall also that a subcategory $\mathcal{B} \subseteq \mathcal{A}$ of an Abelian category $\mathcal{A}$ is *exact* provided that it is Abelian and the inclusion functor of $\mathcal{B}$ into $\mathcal{A}$ is exact.

**Proposition 5.1.** [**16**, 3.41] *A subcategory $\mathcal{B}$ of $\mathcal{A}$ is an exact subcategory if and only if the following two conditions hold:*

(1) *if $B_1$, $B_2 \in \mathcal{B}$ then the coproduct $B_1 \oplus B_2$ is an object of $\mathcal{B}$;*

(2) *If $\beta : B_1 \to B_2$ is a morphism in $\mathcal{B}$, then both the $\mathcal{A}$-kernel and $\mathcal{A}$-cokernel of $\beta$ are objects of $\mathcal{B}$.*

Finally, a subcategory $\mathcal{B}$ of an Abelian category $\mathcal{A}$ is said to be *coexact* provided that the perpendicular subcategory

$$\mathcal{B}^{\perp} = \{A \in \mathcal{A} \mid (B, A) = 0, \operatorname{Ext}^1(B, A) = 0 \text{ for all } B \in \mathcal{B}\}$$

is exact. For example, if $\mathcal{C}$ is a Grothendieck category, $\mathcal{S} \subseteq \mathcal{C}$ is localizing, then $\mathcal{S}$ is coexact if and only if the quotient category $\mathcal{C}/\mathcal{S}$ is exact, because $\mathcal{C}/\mathcal{S} = \mathcal{S}^{\perp}$.

**Proposition 5.2.** *If $\mathcal{S} \subseteq \mathcal{C}$ is a coexact localizing subcategory, then the localization $P_{\mathcal{S}}$ of any projective object $P$ of $\mathcal{C}$ is $\mathcal{C}/\mathcal{S}$-projective. If $\mathcal{C}$ has a family of projective generarors $\mathcal{A} = \{P_i\}_{i \in I}$ and each $(P_i)_{\mathcal{S}}$ is $\mathcal{C}/\mathcal{S}$-projective, then $\mathcal{S}$ is coexact.*

*Proof.* If $\mathcal{S}$ is coexact, then any short exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\beta} C \longrightarrow 0 \tag{5.1}$$

in $\mathcal{C}/\mathcal{S}$ is also exact in $\mathcal{C}$. Suppose $P \in \mathcal{C}$ is projective; then we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_{\mathcal{C}}(P, A) & \longrightarrow & {}_{\mathcal{C}}(P, B) & \xrightarrow{(P,\beta)} & {}_{\mathcal{C}}(P, C) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & {}_{\mathcal{C}/\mathcal{S}}(P_{\mathcal{S}}, A) & \longrightarrow & {}_{\mathcal{C}/\mathcal{S}}(P_{\mathcal{S}}, B) & \xrightarrow[(P_{\mathcal{S}},\beta)]{} & {}_{\mathcal{C}/\mathcal{S}}(P_{\mathcal{S}}, C) & \longrightarrow & 0
\end{array}$$

with exact rows and the vertical arrows isomorphisms. Hence $P_{\mathcal{S}}$ is $\mathcal{C}/\mathcal{S}$-projective.

Conversely, we must show that any $\mathcal{C}/\mathcal{S}$-epimorphism is a $\mathcal{C}$-epimorphism. Consider the exact sequence (5.1). By Lemma 1.4 $C/\operatorname{Im}_{\mathcal{C}}\beta \in \mathcal{S}$. By assumption, the morphism $(P_{\mathcal{S}}, \beta)$ is an epimorphism, where $P \in \mathcal{A}$, hence $(P, \beta)$ is an epimorphism. Therefore $(P, C/\operatorname{Im}_{\mathcal{C}}\beta) = 0$ for every $P \in \mathcal{A}$. Since $\mathcal{A}$ is a family of generators, we conclude that $C/\operatorname{Im}_{\mathcal{C}}\beta = 0$. □

The following statement charecterizes the coexact localizing subcategories of (pre)finite type.

**Proposition 5.3.** *For a coexact localizing subcategory $\mathcal{S}$ of a Grothendieck category $\mathcal{C}$ the following statements are equivalent:*
(1) *$\mathcal{S}$ is of finite type;*
(2) *$\mathcal{S}$ is of prefinite type;*
(3) *the inclusion functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ commutes with coproducts.*

*Proof.* (1) ⇒ (2) ⇒ (3). Obvious.

(3) ⇒ (1). Let $\varinjlim_{\mathcal{C}/\mathcal{S}} C_i$ be the $\mathcal{C}/\mathcal{S}$-direct limit of $C_i \in \mathcal{C}/\mathcal{S}$, $i \in I$. Denote by $\Lambda$ the subset of $I \times I$ that consists of the pairs $(i, j)$ with $i \leq j$. For any $\lambda \in \Lambda$ we put $s(\lambda) = i$, $t(\lambda) = j$. By [**3**, IV.8.4]

$$\varinjlim{}_{\mathcal{C}/\mathcal{S}} C_i = \operatorname{Coker}_{\mathcal{C}/\mathcal{S}} \big[ \oplus_{\lambda\in\Lambda} C_{s(\lambda)} \xrightarrow{\varphi} \oplus_{i\in I} C_i \big]$$

with $\varphi$ induced by $\varphi_\lambda = u_j \gamma_{ij} - u_i : C_{s(\lambda)} \to \oplus_{i\in I} C_i$, $\lambda = (i, j)$ and $\gamma_{ij} : C_{s(\lambda)} \to C_{t(\lambda)}$ the canonical morphism. Since, by assumption, the inclusion functor $\mathcal{C}/\mathcal{S} \to \mathcal{C}$ is exact and commutes with coproducts, we obtain

$$\varinjlim{}_{\mathcal{C}/\mathcal{S}} C_i = \operatorname{Coker}_{\mathcal{C}} \big[ \oplus_{\lambda\in\Lambda} C_{s(\lambda)} \xrightarrow{\varphi} \oplus_{i\in I} C_i \big] = \varinjlim{}_{\mathcal{C}} C_i$$

that implies the claim. □

We shall say that a localizing subcategory $\mathcal{S}$ of the Grothendieck category $\mathcal{C}$ is *strongly coexact* provided that the $\mathcal{S}$-torsion functor $t_{\mathcal{S}}$ is exact.

**Proposition 5.4.** *For a localizing subcategory $\mathcal{S}$ of the Grothendieck category $\mathcal{C}$ the following statements are equivalent:*
(1) *$\mathcal{S}$ is strongly coexact;*
(2) *every quotient object of an $\mathcal{S}$-torsionfree object is $\mathcal{S}$-torsionfree;*
(3) *every $\mathcal{S}$-torsionfree object is $\mathcal{S}$-closed;*
(4) *every subobject of an $\mathcal{S}$-closed object is $\mathcal{S}$-closed.*

*Proof.* The implications (1) ⇒ (2) and (3) ⇔ (4) are obvious.

(2) ⇒ (3). Let $t_{\mathcal{S}}(X) = 0$. Consider the short exact sequence

$$0 \longrightarrow X \longrightarrow X_{\mathcal{S}} \longrightarrow A \longrightarrow 0$$

with $A \in \mathcal{S}$. Since $t_{\mathcal{S}}(X_{\mathcal{S}}) = 0$, we get that $A = 0$.

(3), (4) ⇒ (1). Any short exact sequence in $\mathcal{C}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

can be embedded in the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t(A) & \longrightarrow & t(B) & \xrightarrow{\alpha} & t(C) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \varphi\downarrow & & \psi\downarrow & & \downarrow\gamma & & \\
0 & \longrightarrow & A_{\mathcal{S}} & \longrightarrow & B_{\mathcal{S}} & \underset{\delta}{\longrightarrow} & C_{\mathcal{S}}. & &
\end{array}$$

Since, by assumption, every $\mathcal{S}$-torsionfree object is $\mathcal{S}$-closed, the morphisms $\varphi, \psi, \gamma, \delta$ are epimorphisms. By $3 \times 3$-lemma the morphism $\alpha$ is an epimorphism. □

**Corollary 5.5.** *$\mathcal{S}$ is strongly coexact if and only if $\mathcal{C}/\mathcal{S} = \{X/t_{\mathcal{S}}(X) \mid X \in \mathcal{C}\}$.*

**Corollary 5.6.** *Every strongly coexact localizing subcategory is a coexact subcategory of finite type.*

*Proof.* Since a $\mathcal{C}/\mathcal{S}$-morphism $\alpha : X \to Y$ is a $\mathcal{C}/\mathcal{S}$-epimorphism if and only if $Y/\operatorname{Im}_{\mathcal{C}} \alpha \in \mathcal{S}$, it is easy to see that the strongly coexact subcategories are coexact.

Now, if $X = \sum_{\mathcal{C}/\mathcal{S}} X_i$, $X_i \in \mathcal{C}/\mathcal{S}$, obviously, $\sum_{\mathcal{C}} X_i$ is a subobject of $X$. By Proposition 5.4 $\sum_{\mathcal{C}} X_i$ is an $\mathcal{S}$-closed object and by Proposition 5.3(2) $\mathcal{S}$ is of finite type. □

**Lemma 5.7.** *Let $\mathcal{S}$ be a localizing subcategory of $\mathcal{C}$. The $\mathcal{S}$-torsion functor $t = t_{\mathcal{S}}$ preserves direct limits if and only if every $\mathcal{C}$-direct limit of $\mathcal{S}$-closed objects is $\mathcal{S}$-torsionfree.*

*Proof.* Suppose that $t$ preserves direct limits. If $\{C_i\}_I$ is a directed system of $\mathcal{S}$-closed objects, then

$$t(\varinjlim{}_{\mathcal{C}} C_i) = \varinjlim t(C_i) = 0$$

since $t(C_i) = 0$. Therefore $\varinjlim_{\mathcal{C}} C_i$ is $\mathcal{S}$-torsionfree.

Conversely, there is the exact sequence

$$0 \longrightarrow t(C_i) \longrightarrow C_i \xrightarrow{\lambda_{C_i}} (C_i)_{\mathcal{S}}.$$

As the direct limit functor is exact, we get the exact sequence

$$0 \longrightarrow \varinjlim t(C_i) \longrightarrow \varinjlim C_i \longrightarrow \varinjlim (C_i)_{\mathcal{S}}. \tag{5.2}$$

Since $\mathcal{S}$ is closed under taking direct limits, $\varinjlim t(C_i) \in \mathcal{S}$. Now, if we apply $t$ to (5.2), we obtain that

$$\varinjlim t(C_i) = t(\varinjlim t(C_i)) = t(\varinjlim C_i),$$

because $t(\varinjlim (C_i)_{\mathcal{S}}) = 0$. □

Let $\mathfrak{G}$ be a Gabriel topology for $\mathcal{U}$ that corresponds to $\mathcal{S}$. By a *basis* for the topology $\mathfrak{G}$ we mean a subset $\mathfrak{B}$ of $\mathfrak{G}$ such that every object in $\mathfrak{G}$ contains some $\mathfrak{b} \in \mathfrak{B}$.

**Theorem 5.8.** *Let $\mathcal{C}$ be a locally finitely generated Grothendieck category with a family of finitely generated generators $\mathcal{U} \subseteq \operatorname{fg}\mathcal{C}$ and suppose that $\mathcal{S}$ is a localizing subcategory of $\mathcal{C}$. Then the following conditions are equivalent:*

(1) *$\mathcal{S}$ is of prefinite type;*
(2) *for any $U \in \mathcal{U}$ the natural morphism $\varinjlim {}_{\mathcal{C}}(U, C_i) \to {}_{\mathcal{C}}(U, \sum_{\mathcal{C}/\mathcal{S}} C_i)$ induced by the $\mathcal{S}$-envelope $\lambda$ of $\sum_{\mathcal{C}} C_i$ is an isomorphism;*
(3) *$\mathcal{U}_{\mathcal{S}} = \{U_{\mathcal{S}}\}_{U \in \mathcal{U}}$ is the family of $\mathcal{C}/\mathcal{S}$-finitely generated generators for $\mathcal{C}/\mathcal{S}$;*
(4) *if $C$ is $\mathcal{C}$-finitely generated, then $C_{\mathcal{S}}$ is $\mathcal{C}/\mathcal{S}$-finitely generated;*
(5) *the torsion functor $t$ commutes with the direct limits;*
(6) *any $\mathcal{C}$-direct limit of $\mathcal{S}$-closed object is $\mathcal{S}$-torsionfree;*
(7) *$\mathfrak{G}$ has a basis of finitely generated objects.*

*Thus, if the conditions $(1) - (7)$ hold, then $\mathcal{C}/\mathcal{S}$ is a locally finitely generated Grothendieck category, and every $\mathcal{C}/\mathcal{S}$-finitely generated object $D$ is the localization $C_{\mathcal{S}}$ of some $C \in \operatorname{fg}\mathcal{C}$.*

*Proof.* The equivalence (5) $\Leftrightarrow$ (6) follows from Lemma 5.7.

(1) $\Rightarrow$ (2). By assumption, $\sum_{\mathcal{C}/\mathcal{S}} C_i = \sum_{\mathcal{C}} C_i$. Now, our statement follows from Theorem 1.8.

(2) $\Rightarrow$ (3). Let $\lambda : \sum_{\mathcal{C}} C_i \to \sum_{\mathcal{C}/\mathcal{S}} C_i$ be the $\mathcal{S}$-envelope for $\sum_{\mathcal{C}} C_i$. By assumption, the composed map

$$\varinjlim {}_{\mathcal{C}}(U, C_i) \xrightarrow{\Phi} {}_{\mathcal{C}}(U, \sum\nolimits_{\mathcal{C}} C_i) \xrightarrow{(U,\lambda)} {}_{\mathcal{C}}(U, \sum\nolimits_{\mathcal{C}/\mathcal{S}} C_i),$$

where $\Phi$ is the canonical morphism, is an isomorphism. Hence

$$\varinjlim {}_{\mathcal{C}/\mathcal{S}}(U_{\mathcal{S}}, C_i) \approx \varinjlim {}_{\mathcal{C}}(U, C_i) \approx {}_{\mathcal{C}}(U, \sum\nolimits_{\mathcal{C}/\mathcal{S}} C_i) \approx {}_{\mathcal{C}/\mathcal{S}}(U_{\mathcal{S}}, \sum\nolimits_{\mathcal{C}/\mathcal{S}} C_i).$$

By Theorem 1.8 $U_{\mathcal{S}} \in \operatorname{fg}\mathcal{C}/\mathcal{S}$.

(3) $\Rightarrow$ (4). If $C \in \operatorname{fg}\mathcal{C}$, there is an epimorphism $\eta : \oplus_{i=1}^n U_i \to C$ for some $U_1, \ldots, U_n \in \mathcal{U}$. Since $\oplus_{i=1}^n (U_i)_{\mathcal{S}} \in \operatorname{fg}\mathcal{C}/\mathcal{S}$, we get that $C_{\mathcal{S}} \in \operatorname{fg}\mathcal{C}/\mathcal{S}$.

(4) $\Rightarrow$ (7). Suppose $\mathfrak{a} \in \mathfrak{G}$, i.e., $\mathfrak{a}_{\mathcal{S}} = U_{\mathcal{S}}$ for some $U \in \mathcal{U}$. We write $\mathfrak{a} = \sum_{\mathcal{C}} \mathfrak{a}_i$ as a directed sum of $\mathcal{C}$-finitely generated subobjects $\mathfrak{a}_i$. Then $U_{\mathcal{S}} = \sum_{\mathcal{C}/\mathcal{S}} (\mathfrak{a}_i)_{\mathcal{S}}$. By assumption, there exists $i_0$ such that $U_{\mathcal{S}} = (\mathfrak{a}_{i_0})_{\mathcal{S}}$ whence $\mathfrak{a}_{i_0} \in \mathfrak{G}$.

(7) $\Rightarrow$ (1). First, we show that $U_{\mathcal{S}} \in \operatorname{fg}\mathcal{C}/\mathcal{S}$ for $U$ from $\mathcal{U}$. Indeed, let $U_{\mathcal{S}} = \sum_{\mathcal{C}/\mathcal{S}} \mathfrak{a}_i$. Then $\mathfrak{a} = \lambda_U^{-1}(\sum_{\mathcal{C}} \mathfrak{a}_i) = \sum_{\mathcal{C}} \lambda_U^{-1}(\mathfrak{a}_i) \in \mathfrak{G}$, where $\lambda_U$ is the $\mathcal{S}$-envelope for $U$. By assumption, there is a finitely generated subobject $\mathfrak{b}$ of the object $\mathfrak{a}$ such that $\mathfrak{b} \in \mathfrak{G}$. Then there exists $i_0$ such that $\mathfrak{b} \subseteq \lambda_U^{-1}(\mathfrak{a}_{i_0})$. One has

$$U_{\mathcal{S}} = \mathfrak{b}_{\mathcal{S}} \subseteq (\lambda_U^{-1}(\mathfrak{a}_{i_0}))_{\mathcal{S}} \subseteq \mathfrak{a}_{i_0} \subseteq U_{\mathcal{S}}.$$

So $\mathfrak{a}_{i_0} = U_{\mathcal{S}}$, and, hence, $U_{\mathcal{S}} \in \operatorname{fg}\mathcal{C}/\mathcal{S}$.

Further, the isomorphism

$$\begin{aligned} {}_{\mathcal{C}}(U, \sum\nolimits_{\mathcal{C}/\mathcal{S}} C_i) &\approx {}_{\mathcal{C}/\mathcal{S}}(U_{\mathcal{S}}, \sum\nolimits_{\mathcal{C}/\mathcal{S}} C_i) \\ &\approx \varinjlim {}_{\mathcal{C}/\mathcal{S}}(U_{\mathcal{S}}, C_i) \approx \varinjlim {}_{\mathcal{C}}(U, C_i) \approx {}_{\mathcal{C}}(U, \sum\nolimits_{\mathcal{C}} C_i) \end{aligned}$$

is functorial in $U$, and so $\sum_{\mathcal{C}/\mathcal{S}} C_i = \sum_{\mathcal{C}} C_i$.

(1) $\Rightarrow$ (6). The direct limit $\varinjlim_I C_i$ can be written as a quotient object of a coproduct $\oplus_I C_i$. To be precise, let $R$ be the subset of $I \times I$ that consists of the pairs $(i,j)$ such that $i \leq j$. For every $S \subseteq R$ we put

$$C_S = \sum_{(i,j)\in S} \operatorname{Im}(u_i - u_j\gamma_{ij}) \subseteq \oplus C_i,$$

where $u_i : C_i \to \oplus C_i$ is the canonical monomorphism for $i \in I$, $\gamma_{ij} : C_i \to C_j$ is the canonical monomorphism for $i \leq j$. Then $\varinjlim C_i = \oplus C_i / C_R = \oplus C_i / \sum C_S$, where $S$ runs over all finite subsets of $R$. By assumption, both $\oplus C_i$ and $\sum C_i$ are $\mathcal{S}$-closed, and, hence, $\varinjlim C_i$, being a quotient object of $\mathcal{S}$-closed objects, is $\mathcal{S}$-torsionfree.

(5) $\Rightarrow$ (1). Suppose $X$ is $\mathcal{S}$-closed. Write $X = \sum_{\mathcal{C}/\mathcal{S}} X_i$ as a direct union of $\mathcal{S}$-closed subobjects. Then $X/\sum_{\mathcal{C}} X_i = t(X/\sum_{\mathcal{C}} X_i) = t(\varinjlim_{\mathcal{C}} X/X_i) = \varinjlim t(X/X_i) = 0$, because $t(X/X_i) = 0$. Consequently, $\sum_{\mathcal{C}/\mathcal{S}} X_i = \sum_{\mathcal{C}} X_i$.

In turn, let $D \in \operatorname{fg}\mathcal{C}/\mathcal{S}$. Write $D = \sum_{\mathcal{C}} D_i$ as a direct union of $D_i \in \operatorname{fg}\mathcal{C}$. Then $D = D_{\mathcal{S}} = \sum_{\mathcal{C}/\mathcal{S}} (D_i)_{\mathcal{S}}$, whence $D = (D_{i_0})_{\mathcal{S}}$ for some $i_0$. $\square$

**Proposition 5.9.** *Let $\mathcal{C}$ be a locally finitely presented Grothendieck category with a family of finitely presented generators $\mathcal{U} \subseteq \operatorname{fp}\mathcal{C}$ and suppose that $\mathcal{S}$ is a localizing subcategory of $\mathcal{C}$. Then the following conditions are equivalent:*

1. *$\mathcal{S}$ is of finite type;*
2. *for every $U \in \mathcal{U}$ the canonical morphism $\varinjlim {}_{\mathcal{C}}(U, C_i) \to {}_{\mathcal{C}}(U, \varinjlim_{\mathcal{C}/\mathcal{S}} C_i)$ induced by the $\mathcal{S}$-envelope $\lambda$ of the object $\varinjlim_{\mathcal{C}} C_i$ is an isomorphism;*
3. *$\mathcal{U}_{\mathcal{S}} = \{U_{\mathcal{S}}\}_{U \in \mathcal{U}}$ is the family of $\mathcal{C}/\mathcal{S}$-finitely presented generators for $\mathcal{C}/\mathcal{S}$;*
4. *if $C \in \operatorname{fp}\mathcal{C}$, then $C_{\mathcal{S}} \in \operatorname{fp}\mathcal{C}/\mathcal{S}$.*

*Thus, if these conditions hold, then $\mathcal{C}/\mathcal{S}$ is a locally finitely presented Grothendieck category, and every $\mathcal{C}/\mathcal{S}$-finitely presented object $D$ is the localization $C_{\mathcal{S}}$ of some $C \in \operatorname{fp}\mathcal{C}$.*

*Proof.* It suffices to observe that for every $C \in \operatorname{fp}\mathcal{C}$ the representable functor $(C, -)$ commutes with the direct limits (see Theorm 1.9) and there is a presentation

$$\oplus_{i=1}^n U_i \longrightarrow \oplus_{j=1}^m U_j \longrightarrow C \longrightarrow 0$$

of $C$ by objects from $\mathcal{U}$. Now, our proof literally repeats that of Theorem 5.8.

In turn, if $D \in \operatorname{fp}\mathcal{C}/\mathcal{S}$, there is an epimorphism $\eta : B_{\mathcal{S}} \to D$ with $B \in \operatorname{fp}\mathcal{C}$. Then $\operatorname{Ker}\eta \in \operatorname{fg}\mathcal{C}/\mathcal{S}$. According to [**14**, 2.13] we can choose $C \subseteq B$ such that $C \in \operatorname{fg}\mathcal{C}$ and $C_{\mathcal{S}} = \operatorname{Ker}\eta$. Hence $D = (B/C)_{\mathcal{S}}$. $\square$

Now, we consider a localizing subcategory $\mathcal{S}$ of the module category $\operatorname{Mod}\mathcal{A}$ with $\mathcal{A} = \{P_i\}_{i \in I}$ some ring. Then the family $\mathcal{A}_{\mathcal{S}} = \{P_{\mathcal{S}}\}_{P \in \mathcal{A}}$ generates $\operatorname{Mod}\mathcal{A}/\mathcal{S}$ and we call it the *ring of quotients* of $\mathcal{A}$ with respect to $\mathcal{S}$.

There is the naturally defined functor $j : \operatorname{Mod}\mathcal{A}/\mathcal{S} \to \operatorname{Mod}\mathcal{A}_{\mathcal{S}} = (\mathcal{A}_{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab})$:

$$\operatorname{Hom}_{\mathcal{A}_{\mathcal{S}}}(P_{\mathcal{S}}, j(M)) := \operatorname{Hom}_{\mathcal{A}}(P, M)$$

for every $P \in \mathcal{A}$ and $M \in \operatorname{Mod}\mathcal{A}/\mathcal{S}$. The next statement is an analog of the Walker and Walker theorem (see [**3**, XI.3.4]).

**Proposition 5.10.** *The functor* $j : \operatorname{Mod}\mathcal{A}/\mathcal{S} \to \operatorname{Mod}\mathcal{A}_{\mathcal{S}}$ *is an equivalence if and only if the localizing subcategory* $\mathcal{S}$ *is of finite type and coexact.*

*Proof.* Suppose $j$ is an equivalence. Then every $P_{\mathcal{S}}$ is finitely generated and projective in $\operatorname{Mod}\mathcal{A}/\mathcal{S}$. By Proposition 5.2 $\mathcal{S}$ is coexact, and by Proposition 5.9 it is of finite type.

Conversely, let $\mathcal{S}$ be of finite type and coexact. By Proposition 5.2 every $P_{\mathcal{S}}$ is projective in $\operatorname{Mod}\mathcal{A}/\mathcal{S}$, and by Proposition 5.9 it is finitely generated in $\operatorname{Mod}\mathcal{A}/\mathcal{S}$. Now, our statement follows from Proposition 2.2. □

5.1. **Left exact functors.** Let $\mathcal{C}$ be a locally finitely presented Grothendieck category. By Theorem 4.1 the functor $T : \mathcal{C} \to \operatorname{Mod}\mathcal{A}$ with $\mathcal{A} = \{h_X\}_{X \in \operatorname{fp}\mathcal{C}}$ induces an equivalence of $\mathcal{C}$ and $\operatorname{Mod}\mathcal{A}/\mathcal{S}$, where $\mathcal{S}$ is some localizing subcategory of $\operatorname{Mod}\mathcal{A}$.

We denote by $\mathcal{L}$ the subcategory of $\operatorname{Mod}\mathcal{A}$ consisting of $L \in \operatorname{Mod}\mathcal{A}$ such that for every $x \in L(X)$, $X \in \operatorname{fp}\mathcal{C}$, there exists an epimorphism $f : Y \to X$ such that $L(f)(x) = 0$. We leave to the reader to check that $\mathcal{L}$ is a localizing subcategory. Denote by $\mathfrak{F} = \cup_{X \in \operatorname{fp}\mathcal{C}} \mathfrak{F}^X$ the corresponding Gabriel topology for $\mathcal{A}$.

**Lemma 5.11.** *If* $X' \xrightarrow{p} X \xrightarrow{f} X'' \to 0$ *is an exact sequence in* $\operatorname{fp}\mathcal{C}$*, then* $\operatorname{Coker}(Tf) \in \mathcal{L}$.

*Proof.* Let $Y \in \operatorname{fp}\mathcal{C}$ and $y \in \operatorname{Coker}(Tf)(Y)$; then there is a morphism $g \in {}_{\mathcal{C}}(Y, X'')$ such that $u_y = r \circ Tg$, where $r : h_{X''} \to \operatorname{Coker}(Tf)$ is the canonical epimorphism and $u_{y,Y}(1_Y) = y$. In $\mathcal{C}$, consider the following commutative diagram:

$$\begin{array}{ccc} X \Pi_{X''} Y & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & X'', \end{array}$$

in which $f'$ is an epimorphism. Since $Y \in \operatorname{fp}\mathcal{C}$, there is a finitely generated subobject $Z'$ of $X\Pi_{X''}Y$ such that $f'(Z') = Y$. There exists an epimorphism $h : Z \to Z'$ with $Z \in \operatorname{fp}\mathcal{C}$. So, $f'h$ is an epimorphism. It is easy to show that $\operatorname{Coker}(Tf)(f'h)(y) = 0$, hence $\operatorname{Coker}(Tf) \in \mathcal{L}$. □

Recall that a functor $M \in \operatorname{Mod}\mathcal{A}$, $\mathcal{A} = \{h_X\}_{X \in \operatorname{fp}\mathcal{C}}$, is *left exact* provided that for any exact sequence in $\operatorname{fp}\mathcal{C}$

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

the sequence of Abelian groups

$$0 \longrightarrow M(X'') \longrightarrow M(X) \longrightarrow M(X')$$

is exact.

**Proposition 5.12.** *Every left exact functor* $M \in \operatorname{Mod}\mathcal{A}$ *is* $\mathcal{L}$*-closed.*

*Proof.* Let $M : \operatorname{fp}\mathcal{C} \to \mathrm{Ab}$ be a contravariant left exact functor, $X \in \operatorname{fp}\mathcal{C}$ and $x \in M(X)$ such that $\operatorname{Ker} u_x \in \mathfrak{F}^X$. Here, $u_x$ denotes the unique morphism such that $u_{x,X}(1_X) = x$. Then there exists an epimorphism $f : Y \to X$ in $\operatorname{fp}\mathcal{C}$ such that

$M(f)(x) = 0$. But $M(f)$ is a monomorphism of Abelian groups and thus $x = 0$. So, $M$ is $\mathcal{L}$-torsionfree.

Now, let $\mathfrak{a} \in \mathfrak{F}^X$ and $g : \mathfrak{a} \to M$ a morphism in $\operatorname{Mod}\mathcal{A}$. There exists an epimorphism $f : Y \to X$ such that $\operatorname{Im}(Tf) \subseteq \mathfrak{a}$. By Lemma 5.11 $\operatorname{Im}(Tf) \in \mathfrak{F}^X$. Consider the morphism $g \circ Tf = t : h_Y \to M$. Since $\operatorname{Ker} f \in \operatorname{fg}\mathcal{C}$, there exists an epimorphism $Z \to \operatorname{Ker} f$ and thus we result in the exact sequence

$$Z \xrightarrow{p} Y \xrightarrow{f} X \longrightarrow 0.$$

We have the following commutative diagram of Abelian groups:

$$\begin{array}{ccccccc} 0 & \longrightarrow & h_Y(X) & \xrightarrow{h_Y(f)} & h_Y(Y) & \xrightarrow{h_Y(p)} & h_Y(Z) \\ & & \downarrow{\scriptstyle t_X} & & \downarrow{\scriptstyle t_Y} & & \downarrow{\scriptstyle t_Z} \\ 0 & \longrightarrow & M(X) & \xrightarrow[M(f)]{} & M(Y) & \xrightarrow[M(p)]{} & M(Z). \end{array}$$

Thus,

$$(M(p)t_Y)(1_Y) = (t_Z h_Y(p))(1_Y) = (g_Z(Tf)_Z h_Y(p))(1_Y) = g_Z(fp) = 0.$$

Then there exists an element $x \in M(X)$ such that $M(f)(x) = t_Y(1_Y) = (gTf)_Y(1_Y) = g_Y(f)$. Hence $u_x|_{\operatorname{Im}(Tf)} = g|_{\operatorname{Im}(Tf)}$. Since $\operatorname{Im}(Tf) \in \mathfrak{F}^X$ and $M$ is $\mathcal{L}$-torsionfree, it follows that $u_x|_{\mathfrak{a}} = g$. Thus $u_x : h_X \to M$ is a morphism prolonging $g$. The uniqueness of $u_x$ follows from the fact that $M$ is $\mathcal{L}$-torsionfree. So $M$ is $\mathcal{L}$-closed. $\square$

An object $C$ of the Grothendieck category $\mathcal{C}$ with a family of generators $\mathcal{U}$ is said to be *$\mathcal{U}$-finitely presented* provided that there is an exact sequence

$$\oplus_{k=1}^n Y_k \longrightarrow \oplus_{j=1}^m X_j \longrightarrow C \longrightarrow 0$$

with $Y_k$, $X_j \in \mathcal{U}$. The corresponding subcategory of $\mathcal{U}$-finitely presented objects we denote by $\operatorname{fp}_{\mathcal{U}}\mathcal{C}$.

Define $\operatorname{Lex}((\operatorname{fp}\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ to be the category of contravariant left exact functors from $\operatorname{fp}\mathcal{C}$ to Ab. We are now in a position to prove the following result.

**Theorem 5.13** (Breitsprecher [**26**])**.** *Let $\mathcal{C}$ be a Grothendieck category with a family of generators $\mathcal{U} = \{U_i\}_{i\in I}$. Then the representation functor $T = (-, ?) : \mathcal{C} \to ((\operatorname{fp}_{\mathcal{U}}\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ determines an equivalence between $\mathcal{C}$ and $((\operatorname{fp}_{\mathcal{U}}\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})/\mathcal{S}$, where $\mathcal{S}$ is some localizing subcategory of $((\operatorname{fp}_{\mathcal{U}}\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$. Moreover, $\mathcal{S}$ is of finite type if and only if $\operatorname{fp}_{\mathcal{U}}\mathcal{C} = \operatorname{fp}\mathcal{C}$. In this case, $\mathcal{C}$ is equivalent to the category $\operatorname{Lex}((\operatorname{fp}\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$.*

*Proof.* The first part of the theorem easily follows from Corollary 4.3 and Proposition 5.9. So, let $\operatorname{fp}_{\mathcal{U}}\mathcal{C} = \operatorname{fp}\mathcal{C}$. Our statement would be proved, if we showed that $\mathcal{S} = \mathcal{L}$. Since for every $X \in \mathcal{C}$ the functor $T(X)$ is left exact, Proposition 5.12 implies that $T(X)$ is $\mathcal{L}$-closed, and, hence, $\mathcal{L} \subseteq \mathcal{S}$. Conversely, let $M \in \mathcal{S}$. We can choose a projective presentation for $M$

$$\oplus h_{Y_j} \xrightarrow{g} \oplus h_{X_i} \longrightarrow M \longrightarrow 0.$$

By Proposition 5.9 $\mathcal{S}$ is of finite type. Therefore, any coproduct of $\mathcal{S}$-closed objects is an $\mathcal{S}$-closed object. Therefore there exists a morphism $f \in \operatorname{Mor}\mathcal{C}$ such that $Tf = g$.

Furthermore, $f$ is an epimorphism in $\mathcal{C}$ since $M_{\mathcal{S}} = 0$. Thus, without loss of generality, we may assume that for every $M \in \mathcal{S}$ there is an exact sequence

$$TY \xrightarrow{Tf} TX \longrightarrow M \longrightarrow 0,$$

where $f$ is a $\mathcal{C}$-epimorphism.

According to [**27**, 5.7] $f$ is a direct limit $f_\alpha : Y_\alpha \to X_\alpha$ of epimorphisms $f_\alpha$ in $\mathrm{fp}\,\mathcal{C}$, and, hence, $M \approx \varinjlim \mathrm{Coker}(Tf_\alpha)$. By Lemma 5.11 every $\mathrm{Coker}(Tf_\alpha) \in \mathcal{L}$ whence $M \in \mathcal{L}$. □

The next result characterizes the localizing subcategories of finite type for locally coherent Grothendieck categories.

**Theorem 5.14.** *Let $\mathcal{C}$ be a locally coherent Grothendieck category with a family of coherent generators $\mathcal{U} \subseteq \mathrm{coh}\,\mathcal{C}$. Then the following conditions are equivalent:*

(1) *$\mathcal{S}$ is of finite type;*
(2) *$\mathcal{S}$ is of prefinite type;*
(3) *the torsion functor $t$ commutes with the direct limits;*
(4) *any $\mathcal{C}$-direct limit of $\mathcal{S}$-closed objects is $\mathcal{S}$-torsionfree;*
(5) *for any $U \in \mathcal{U}$ the natural morphism $\varinjlim_{\mathcal{C}}(U, C_i) \to {}_{\mathcal{C}}(U, \varinjlim_{\mathcal{C}/\mathcal{S}} C_i)$ induced by the $\mathcal{S}$-envelope $\lambda$ of $\varinjlim_{\mathcal{C}} C_i$ is an isomorphism;*
(6) *$\mathcal{U}_{\mathcal{S}} = \{U_{\mathcal{S}}\}_{U \in \mathcal{U}}$ is the family of $\mathcal{C}/\mathcal{S}$-coherent generators for $\mathcal{C}/\mathcal{S}$;*
(7) *if $C$ is $\mathcal{C}$-coherent, then $C_{\mathcal{S}}$ is $\mathcal{C}/\mathcal{S}$-coherent;*
(8) *$\mathfrak{G}$ has a basis of coherent objects;*

*Thus, if these conditions hold, then the category $\mathcal{C}/\mathcal{S}$ is locally coherent, and every $\mathcal{C}/\mathcal{S}$-coherent object $D$ is the localization $C_{\mathcal{S}}$ of some $C \in \mathrm{coh}\,\mathcal{C}$.*

*Proof.* (1) ⇒ (5). This follows from Proposition 5.9.

(5) ⇒ (6). By Proposition 5.9 $\mathcal{U}_{\mathcal{S}} \subseteq \mathrm{fp}\,\mathcal{C}/\mathcal{S}$. Now, we show that for any $U \in \mathcal{U}$ the object $U_{\mathcal{S}}$ is coherent. Let $C \subseteq U_{\mathcal{S}}$ be a $\mathcal{C}/\mathcal{S}$-finitely generated subobject of $U_{\mathcal{S}}$. From [**14**, 2.13] it follows that there exists a $\mathcal{C}$-finitely generated subobject $A \subseteq U$ such that $A_{\mathcal{S}} = C$. Since $U$ is coherent, the object $A \in \mathrm{coh}\,\mathcal{C}$. By Proposition 5.9 $A_{\mathcal{S}} \in \mathrm{fp}\,\mathcal{C}/\mathcal{S}$.

(6) ⇒ (7). Obvious.

(7) ⇒ (1). $\mathcal{S}$ is of finite type by Proposition 5.9.

(2) ⇔ (3) ⇔ (4). This follows from Theorem 5.8.

(2) ⇔ (8). Since every finitely generated subobject of a coherent object is coherent, our assertion follows from Theorem 5.8.

(1) ⇒ (2). Obvious.

(3) ⇒ (1). This is a consequence of [**15**, 2.4]. □

As in [**14**] for an arbitrary subcategory $\mathcal{X}$ of $\mathcal{C}$, denote by $\vec{\mathcal{X}}$ the subcategory of $\mathcal{C}$ consisting of the direct limits of objects in $\mathcal{X}$. Herzog [**14**] and Krause [**15**] have observed that the localizing subcategories of finite type in the locally coherent categories are determined by the Serre subcategories of the Abelian subcategory $\mathrm{coh}\,\mathcal{C}$. Namely, the following theorem holds.

**Theorem 5.15** (Herzog, Krause)**.** *Let $\mathcal{C}$ be a locally coherent Grothendieck category. There is a bijective correspondence between the Serre subcategories $\mathcal{P}$ of* $\operatorname{coh}\mathcal{C}$ *and the localizing subcategories $\mathcal{S}$ of $\mathcal{C}$ of finite type. This correspondence is given by the functions*

$$\mathcal{P} \longmapsto \vec{\mathcal{P}}$$
$$\mathcal{S} \longmapsto \operatorname{coh}\mathcal{S} = \mathcal{S} \cap \operatorname{coh}\mathcal{C}$$

*which are mutual inverses.*

Later, given a Serre subcategory $\mathcal{P}$ of $\operatorname{coh}\mathcal{C}$, the $\vec{\mathcal{P}}$-torsion functor will be denoted by $t_{\mathcal{P}}$.

Recall that an object $C \in \mathcal{C}$ of the Grothendieck category $\mathcal{C}$ is *Noetherian* provided that any subobject of $C$ is finitely generated. $\mathcal{C}$ is said to be *locally Noetherian* if it has a family of Noetherian generators. In that case, the following relations hold:

$$\operatorname{fg}\mathcal{C} = \operatorname{fp}\mathcal{C} = \operatorname{coh}\mathcal{C}.$$

Moreover, every localizing subcategory $\mathcal{S}$ of $\mathcal{C}$ is of finite type. Indeed, every $X \in \mathcal{S}$ is a direct union $\sum_{i\in I} X_i$ of objects $X_i \in \operatorname{fg}\mathcal{C}\cap\mathcal{S} = \operatorname{coh}\mathcal{C}\cap\mathcal{S} = \operatorname{coh}\mathcal{S}$. Consequently, any quotient category $\mathcal{C}/\mathcal{S}$ is locally Noetherian with the family of Noetherian generators $\{C_{\mathcal{S}}\}_{C\in\operatorname{coh}\mathcal{C}}$.

5.2. **The Ziegler topology.** The study of pure-injective (= algebraically compact) modules over different classes of rings plays the important role in the theory of rings and modules. Since the pure-injective modules can be defined, using some condition of solvability (in this module) for linear equations systems, many problems of (algebraic!) structure for pure-injective modules admit reformulations using concepts of the model theory. It is such an approach led Ziegler [**6**] to construction of a topological space ("The Ziegler spectrum") whose points are the indecomposable pure-injective modules. Recently Herzog [**14**] and Krause [**15**] have proposed the algebraic definition of the Ziegler spectrum.

Let $\mathcal{C}$ be a Grothendieck category with a family of generators $\mathcal{U}$. We denote by $\operatorname{Zg}\mathcal{C}$ the set of isomorphism classes for indecomposable injective objects in $\mathcal{C}$ and call $\operatorname{Zg}\mathcal{C}$ the *Ziegler spectrum* of $\mathcal{C}$. The fact that $\operatorname{Zg}\mathcal{C}$ forms a set follows from that any indecomposable injective object in $\mathcal{C}$ occurs as the injective envelope of some $\mathcal{U}$-finitely generated object $X \in \operatorname{fg}_{\mathcal{U}}\mathcal{C}$ and $\operatorname{fg}_{\mathcal{U}}\mathcal{C}$ is skeletally small. It will be convenient to identify each isomorphism class with a representative belonging to it. If $\mathcal{S}$ is a localizing subcategory in $\mathcal{C}$, the assignment $X \mapsto E(X)$ induces injective maps $\operatorname{Zg}\mathcal{S} \to \operatorname{Zg}\mathcal{C}$ and $\operatorname{Zg}\mathcal{C}/\mathcal{S} \to \operatorname{Zg}\mathcal{C}$. We consider both maps as identifications. They satisfy $\operatorname{Zg}\mathcal{S} \cup \operatorname{Zg}\mathcal{C}/\mathcal{S} = \operatorname{Zg}\mathcal{C}$ and $\operatorname{Zg}\mathcal{S} \cap \operatorname{Zg}\mathcal{C}/\mathcal{S} = \emptyset$ [**2**, III.3.2].

Now, let $\mathcal{C}$ be a locally coherent category, i.e., we may suppose that $\mathcal{U} \subseteq \operatorname{coh}\mathcal{C}$. To an arbitrary subcategory $\mathcal{X} \subset \operatorname{coh}\mathcal{C}$, we associate the subset of $\operatorname{Zg}\mathcal{C}$

$$\mathcal{O}(\mathcal{X}) = \{E \in \operatorname{Zg}\mathcal{C} \mid {}_{\mathcal{C}}(C,E) \neq 0 \text{ for some } C \in \mathcal{X}\}.$$

If $\mathcal{X} = \{C\}$ is singleton, we abbreviate $\mathcal{O}(\mathcal{X})$ to $\mathcal{O}(C)$; thus $\mathcal{O}(\mathcal{X}) = \cup_{C\in\mathcal{X}}\mathcal{O}(C)$.

We restrict the discussion to subcategories of the form $\mathcal{O}(\mathcal{S})$ where $\mathcal{S} \subseteq \operatorname{coh} \mathcal{C}$ is a Serre subcategory. In that case,

$$\mathcal{O}(\mathcal{S}) = \{E \in \operatorname{Zg} \mathcal{C} \mid t_{\mathcal{S}}(E) \neq 0\}.$$

**Theorem 5.16** (Herzog [**14**], Krause [**15**]). *For a locally coherent Grothendieck category $\mathcal{C}$ the following statements hold:*

(1) *The collection of subsets of* $\operatorname{Zg} \mathcal{C}$

$$\{\mathcal{O}(\mathcal{S}) \mid \mathcal{S} \text{ is a Serre subcategory}\}$$

*satisfies the axioms for the open sets of a topological space on* $\operatorname{Zg} \mathcal{C}$*. This topological space we also call the Ziegler spectrum of* $\mathcal{C}$*.*

(2) *There is a bijective inclusion preserving correspondence between the Serre subcategories $\mathcal{S}$ of* $\operatorname{coh} \mathcal{C}$ *and the open subsets $\mathcal{O}$ of* $\operatorname{Zg} \mathcal{C}$*. This correspondence is given by the functions*

$$\mathcal{S} \mapsto \mathcal{O}(\mathcal{S})$$
$$\mathcal{O} \mapsto \mathcal{S}_{\mathcal{O}} = \{C \in \operatorname{coh} \mathcal{C} \mid \mathcal{O}(C) \subseteq \mathcal{O}\}$$

*which are mutual inverses.*

Recall that a topological space $\mathcal{X}$ is *quasi-compact* provided that for every family $\{\mathcal{O}_i\}_{i \in I}$ of open subsets the equality $\mathcal{X} = \cup_{i \in I} \mathcal{O}_i$ implies $\mathcal{X} = \cup_{i \in J} \mathcal{O}_i$ for some finite subset $J$ of $I$. A subset of $\mathcal{X}$ is quasi-compact if it is quasi-compact with respect to the induced topology.

By [**14**, 3.9] and [**15**, 4.6] an open subset $\mathcal{O}$ of $\operatorname{Zg} \mathcal{C}$ is quasi-compact if and only if it is one of the basic open subsets $\mathcal{O}(C)$ with $C \in \operatorname{coh} \mathcal{C}$.

The Serre subcategories $\mathcal{S}$ of $\operatorname{coh} \mathcal{C}$ arise in the following natural way. An object $M \in \mathcal{C}$ is said to be *coh-injective* provided that $\operatorname{Ext}^1_{\mathcal{C}}(C, M) = 0$ for every $C \in \operatorname{coh} \mathcal{C}$. Then the subcategory

$$\mathcal{S}_M = \{C \in \operatorname{coh} \mathcal{C} \mid {}_{\mathcal{C}}(C, M) = 0\}$$

generated by $M$ is Serre. Moreover, every Serre subcategory of $\operatorname{coh} \mathcal{C}$ arises in this fashion [**14**, 3.11].

**Examples.** Here are some examples of the Ziegler-closed subsets:

(1). Let $\mathcal{C}$ be a locally coherent Grothendieck category; then the functor category $((\operatorname{coh} \mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ is locally coherent. By Theorem 5.13 the category $\mathcal{C}$ is equivalent to $\operatorname{Lex}((\operatorname{coh} \mathcal{C})^{\mathrm{op}}, \mathrm{Ab}) = ((\operatorname{coh} \mathcal{C})^{\mathrm{op}}, \mathrm{Ab})/\vec{\mathcal{L}}$, where $\mathcal{L}$ is the Serre subcategory of $\operatorname{coh}((\operatorname{coh} \mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ consisting of objects isomorphic to $\operatorname{Coker}(-, \mu)$ for some epimorphism $\mu : A \to B$ in $\operatorname{coh} \mathcal{C}$. Thus $\operatorname{Zg} \mathcal{C}$ is closed in $\operatorname{Zg}((\operatorname{coh} \mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$.

(2). If $\mathcal{C} = \mathcal{C}_A$, the Ziegler spectrum of $\mathcal{C}_A$

$$\operatorname{Zg} \mathcal{C}_A = \{Q \otimes_A - \mid Q_A \text{ is an indecomposable pure-injective module}\}.$$

We notice that $\operatorname{Zg} \mathcal{C}_A$ is quasi-compact since $\operatorname{Zg} \mathcal{C}_A = \mathcal{O}(A \otimes_A -)$. Prest, Rothmaler and Ziegler have shown [**28**, 4.4] (see also [**10**, 2.5]) that a ring $A$ is right coherent if and only if the set

$$I_{\mathrm{inj}} = \{Q \otimes_A - \in \operatorname{Zg} \mathcal{C}_A \mid Q \text{ is an injective module}\}$$

is closed in $\operatorname{Zg} \mathcal{C}_A$.

(3). If $A$ is left coherent, from [**11**, 2.4] it follows that

$$I_{\text{flat}} = \{Q \otimes_A - \in \operatorname{Zg} \mathcal{C}_A \mid Q \text{ is a flat module}\}$$

is closed in $\operatorname{Zg} \mathcal{C}_A$.

Recall that a ring $A$ is *weakly quasi-Frobenius* provided that the functor $\operatorname{Hom}_A(-, A)$ puts the categories $\operatorname{mod} A^{\text{op}}$ and $\operatorname{mod} A$ of finitely presented modules in duality. The weakly quasi-Frobenius rings can be characterized as (two-sided) coherent $FP$-injective rings [**11**, 2.11]. Over the weakly quasi-Frobenius rings $I_{\text{inj}} = I_{\text{flat}}$. Conversely, if $A$ is a (two-sided) coherent ring and $I_{\text{inj}} = I_{\text{flat}}$, then $A$ is a weakly quasi-Frobenius ring [**11**, 2.12].

(4). Let $\rho : A \to B$ be a ring homomorphism; then $\rho$ induces the exact functor $\rho^* : \operatorname{coh} \mathcal{C}_A \to \operatorname{coh} \mathcal{C}_B$. If $\rho$ is an epimorphism of rings, then the map $M_B \mapsto M_A$ induces a homeomorphism $\operatorname{Zg} \mathcal{C}_B \to \operatorname{Zg} \mathcal{C}_A \backslash \mathcal{O}(\mathcal{S})$ with $\mathcal{S} = \operatorname{Ker} \rho^*$. Thus $\operatorname{Zg} \mathcal{C}_B$ is closed in $\operatorname{Zg} \mathcal{C}_A$ [**29**].

## 6. The categories of generalized modules

In this section, similar to the category of generalized $A$-modules $\mathcal{C}_A$, we construct the category $\mathcal{C}_{\mathcal{A}}$ with $\mathcal{A} = \{P_i\}_{i \in I}$ the ring of finitely generated projective objects of the category $\operatorname{Mod} \mathcal{A}$. The model-theoretic constructions the reader can find in [**30, 31**]. For the most part, we shall adhere to the work of Herzog [**14**].

6.1. **Tensor products.** Let $\operatorname{Mod} \mathcal{A}$ be the category of right $\mathcal{A}$-modules with the family of finitely generated projective generators $\mathcal{A} = \{P_i\}_{i \in I}$. By Proposition 2.2 $\operatorname{Mod} \mathcal{A} \approx (\mathcal{A}^{\text{op}}, \text{Ab})$. We refer to the functor category $(\mathcal{A}, \text{Ab})$ as the *category of left $\mathcal{A}$-modules* and denote it by $\operatorname{Mod} \mathcal{A}^{\text{op}}$. It is a locally finitely presented Grothendieck category with the family of finitely generated projective generators $\mathcal{A}^{\text{op}} = \{h^P = \operatorname{Hom}_{\mathcal{A}}(P, -)\}_{P \in \mathcal{A}}$.

**Proposition 6.1.** [**5**, III.6.3] *Let $M : \mathcal{A}^{\text{op}} \to \text{Ab}$ ($N : \mathcal{A} \to \text{Ab}$) be a right (left) $\mathcal{A}$-module. Then, the unique functor $M \otimes_{\mathcal{A}} - : \operatorname{Mod} \mathcal{A}^{\text{op}} \to \text{Ab}$ ($- \otimes_{\mathcal{A}} N : \operatorname{Mod} \mathcal{A} \to \text{Ab}$) exists such that:*

(1) *there are functorial isomorphisms $M \otimes_{\mathcal{A}} h^P \approx M(P)$ and $h_P \otimes_{\mathcal{A}} N \approx N(P)$ for $P \in \mathcal{A}$;*

(2) *$M \otimes_{\mathcal{A}} -$ and $- \otimes_{\mathcal{A}} N$ have right adjoints;*

Note also that the *tensor product functor* $M \otimes_{\mathcal{A}} -$ is right exact and commutes with the direct limits.

Let $\operatorname{Mod} \mathcal{A} = (\mathcal{A}^{\text{op}}, \text{Ab})$ ($\operatorname{Mod} \mathcal{A}^{\text{op}} = (\mathcal{A}, \text{Ab})$) be the category of right (left) $\mathcal{A}$-modules with $\mathcal{A} = \{P_i\}_{i \in I}$, $\operatorname{mod} \mathcal{A}^{\text{op}}$ ($\operatorname{mod} \mathcal{A}$) be the category of finitely presented left (right) $\mathcal{A}$-modules. By definition, every $M \in \operatorname{mod} \mathcal{A}$ has a projective presentation in $\operatorname{Mod} \mathcal{A}$

$$\oplus_{j=1}^{n} h_{P_j} \longrightarrow \oplus_{k=1}^{m} h_{P_k} \longrightarrow M \longrightarrow 0. \tag{6.1}$$

Similarly, every $M \in \operatorname{mod} \mathcal{A}^{\text{op}}$ has a presentation in $\operatorname{Mod} \mathcal{A}^{\text{op}}$

$$\oplus_{j=1}^{n} h^{P_j} \longrightarrow \oplus_{k=1}^{m} h^{P_k} \longrightarrow M \longrightarrow 0.$$

Denote by $\mathcal{C}_{\mathcal{A}} = (\operatorname{mod} \mathcal{A}^{\text{op}}, \text{Ab})$ (${}_{\mathcal{A}}\mathcal{C} = (\operatorname{mod} \mathcal{A}, \text{Ab})$) the category of additive covariant functors defined on $\operatorname{mod} \mathcal{A}^{\text{op}}$ ($\operatorname{mod} \mathcal{A}$) and call $\mathcal{C}_{\mathcal{A}}$ (${}_{\mathcal{A}}\mathcal{C}$) the *category of generalized* *right (left) $\mathcal{A}$-modules.* They naturally generalize the corresponding categories of generalized $A$-modules with $\mathcal{A} = \{A\}$. Since $\operatorname{mod}\mathcal{A}^{\mathrm{op}}$ is closed under cokernels, $\mathcal{C}_{\mathcal{A}}$ is a locally coherent Grothendieck category. The finitely generated projective objects of $\mathcal{C}_{\mathcal{A}}$ are precisely the objects of the form $\{(M,-)\}_{M\in\operatorname{mod}\mathcal{A}^{\mathrm{op}}}$ and this family generates $\mathcal{C}_{\mathcal{A}}$. The tensor product functor $?\otimes_{\mathcal{A}} - : \operatorname{Mod}\mathcal{A} \to \mathcal{C}_{\mathcal{A}}$ defined by the rule $M_{\mathcal{A}} \mapsto M\otimes_{\mathcal{A}} -$ is fully faithful and right exact.

Every finitely presented (= coherent) generalized module $C \in \mathcal{C}_{\mathcal{A}}$ has a projective presentation

$$(K,-) \xrightarrow{(f,-)} (L,-) \longrightarrow C \longrightarrow 0$$

with $K, L \in \operatorname{mod}\mathcal{A}^{\mathrm{op}}$. Moreover, $M \in \operatorname{mod}\mathcal{A}$ if and only if $M\otimes_{\mathcal{A}} - \in \operatorname{coh}\mathcal{C}_{\mathcal{A}}$. Indeed, it suffices to observe that the functor $(M\otimes_{\mathcal{A}} -,-)$ preserves the direct limits if and only if $M\otimes_{\mathcal{A}} - \in \operatorname{coh}\mathcal{C}_{\mathcal{A}}$ and make use of Theorem 1.9.

**Lemma 6.2.** *An object $E \in \mathcal{C}_{\mathcal{A}}$ is coh-injective if and only if it is isomorphic to one of the functors $M\otimes_{\mathcal{A}} -$ where $M$ is a right $\mathcal{A}$-module.*

*Proof.* Herzog [**14**, 2.2] has shown that $E$ is coh-injective if and only if it is right exact. Therefore the functor $M\otimes_{\mathcal{A}} -$ is coh-injective.

Suppose $E$ is coh-injective. We define $M_{\mathcal{A}}$ by the rule $\operatorname{Hom}_{\mathcal{A}}(P,M) = E(h^P)$ for each $P \in \mathcal{A}$. Now, the proof of our assertion is similar to that of [**3**, IV.10.1]. □

Thus the category $\operatorname{Mod}\mathcal{A}$ of right $\mathcal{A}$-modules can be considered as the subcategory of the coh-injective objects of the category $\mathcal{C}_{\mathcal{A}}$.

In order to describe the points of the Ziegler spectrum $\operatorname{Zg}\mathcal{C}_{\mathcal{A}}$ of $\mathcal{C}_{\mathcal{A}}$, recall that a short exact sequence

$$0 \longrightarrow L \xrightarrow{\mu} M \xrightarrow{\delta} N \longrightarrow 0$$

of a locally finitely presented Grothendieck category $\mathcal{C}$ is *pure-exact* provided that the sequence

$$0 \longrightarrow {}_{\mathcal{C}}(X,L) \longrightarrow {}_{\mathcal{C}}(X,M) \longrightarrow {}_{\mathcal{C}}(X,N) \longrightarrow 0$$

is exact for all $X \in \operatorname{fp}\mathcal{C}$. In this case, $\mu$ is called a *pure-monomorphism.* An object $Q \in \mathcal{C}$ is said to be *pure-injective* provided that every pure-exact sequence with the first term $Q$ splits.

**Proposition 6.3.** *Let $\varepsilon : 0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathcal{C}$. Then the following statements are equivalent:*

(1) *$\varepsilon$ is pure in $\mathcal{C}$;*

(2) *$\varepsilon$ is a direct limit of split exact sequences $0 \to L_i \to M_i \to N_i \to 0$ in $\mathcal{C}$.*

*Proof.* Write $N = \varinjlim_I N_i$ as a direct limit of finitely presented objects $N_i$. For every $i \in I$ consider the following commutative diagram

$$\begin{array}{ccccccccc}
\varepsilon_i : 0 & \longrightarrow & L & \longrightarrow & M_i & \xrightarrow{\delta_i} & N_i & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \varphi_i} & & \\
0 & \longrightarrow & L & \longrightarrow & M & \underset{\delta}{\longrightarrow} & N & \longrightarrow & 0
\end{array}$$

in which the right square is pullback and $\varphi_i$ the canonical morphism. Since for $i \le j$ the relations

$$\delta\psi_i = \varphi_i\delta_i = \varphi_j(\varphi_{ij}\delta_i)$$

hold, there exists the unique $\psi_{ij} : M_i \to M_j$ such that $\psi_i = \psi_j\psi_{ij}$. Clearly, the system $\{M_i, \psi_{ij}\}_I$ is direct and $\varinjlim \varepsilon_i = \varepsilon$. By assumption, there exists a morphism $f : N_i \to M$ such that $\delta f = \varphi_i$, and, hence, there exists $g : N_i \to M_i$ such that $\delta_i g = 1_{M_i}$, i.e., each sequence $\varepsilon_i$ splits.

Conversely, if each $\varepsilon_i$ splits, then the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, L_i) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, M_i) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, N_i) \longrightarrow 0$$

is exact. The fact that $\mathrm{Hom}_{\mathcal{A}}(X, -)$, $X \in \mathrm{fp}\,\mathcal{C}$, commutes with the direct limits implies (1). $\square$

**Corollary 6.4.** *The sequence* $\varepsilon : 0 \to L \to M \to N \to 0$ *of right* $\mathcal{A}$*-modules is pure-exact if and only if the* $\mathcal{C}_{\mathcal{A}}$*-sequence* $\varepsilon \otimes_{\mathcal{A}} - : 0 \to L \otimes_{\mathcal{A}} - \to M \otimes_{\mathcal{A}} - \to N \otimes_{\mathcal{A}} - \to 0$ *is exact.*

*Proof.* As tensoring commutes with the direct limits, the necessary condition follows from the preceding proposition. Conversely, if $\varepsilon \otimes_{\mathcal{A}} -$ is exact, then for any $X \in \mathrm{mod}\,\mathcal{A}$ one has the following exact sequence

$$0 \longrightarrow (X \otimes_{\mathcal{A}} -, L \otimes_{\mathcal{A}} -) \longrightarrow (X \otimes_{\mathcal{A}} -, M \otimes_{\mathcal{A}} -) \longrightarrow (X \otimes_{\mathcal{A}} -, N \otimes_{\mathcal{A}} -) \longrightarrow \mathrm{Ext}^1(X \otimes_{\mathcal{A}} -, L \otimes_{\mathcal{A}} -).$$

Since $X \otimes_{\mathcal{A}} - \in \mathrm{coh}\,\mathcal{C}_{\mathcal{A}}$ and the object $L \otimes_{\mathcal{A}} -$ is coh-injective, it follows that $\mathrm{Ext}^1(X \otimes_{\mathcal{A}} -, L \otimes_{\mathcal{A}} -) = 0$. $\square$

As tensoring commutes with the direct limits, we note that a monomorphism $\mu : L \to M$ is pure if and only if $\mu \otimes_{\mathcal{A}} X$ is a monomorphism for any left $\mathcal{A}$-module $X$.

The proof of the following result is similar to that of [**14**, 4.1].

**Proposition 6.5** (Gruson, Jensen)**.** *An object* $E \in \mathcal{C}_{\mathcal{A}}$ *is an injective object if and only if it is isomorphic to one of the functors* $Q \otimes_{\mathcal{A}} -$ *where* $Q$ *is a pure-injective right* $\mathcal{A}$*-module.*

Thus the points of the Ziegler spectrum $\mathrm{Zg}\,\mathcal{C}_{\mathcal{A}}$ of $\mathcal{C}_{\mathcal{A}}$ are represented by the pure-injective indecomposable right $\mathcal{A}$-modules. Note that every indecomposable injective right $\mathcal{A}$-module $E_{\mathcal{A}}$ is pure-injective, and, hence, $E \otimes_{\mathcal{A}} -$ is a point of $\mathrm{Zg}\,\mathcal{C}_{\mathcal{A}}$.

It is easy to see that $\mathrm{Zg}\,\mathcal{C}_{\mathcal{A}} = \cup_{P \in \mathcal{A}} \mathcal{O}(P \otimes_{\mathcal{A}} -)$ is a union of basic open quasi-compact subsets $\mathcal{O}(P \otimes_{\mathcal{A}} -)$. Prest has shown [**31**] that $\mathrm{Zg}\,\mathcal{C}_{\mathcal{A}}$ need not be compact, in contrast to the case $\mathcal{A} = \{A\}$ where $A$ is a ring, the whole spase need not be basic open.

6.2. **The Auslander-Gruson-Jensen duality.** Let $A$ be an arbitrary ring. Gruson and Jensen [**20**] and Auslander [**32**] proved that there is a duality $D : (\mathrm{coh}\,{}_A\mathcal{C})^{\mathrm{op}} \approx \mathrm{coh}\,\mathcal{C}_A$ between the corresponding subcategories of the coherent objects of ${}_A\mathcal{C}$ and $\mathcal{C}_A$. A similar duality can easily be constructed for the categories ${}_{\mathcal{A}}\mathcal{C}$ and $\mathcal{C}_{\mathcal{A}}$ with $\mathcal{A} = \{P_i\}_{i \in I}$.

Namely, let the functor $D : (\mathrm{coh}\,{}_{\mathcal{A}}\mathcal{C})^{\mathrm{op}} \to \mathrm{coh}\,\mathcal{C}_{\mathcal{A}}$ be given by

$$(DC)({}_{\mathcal{A}}N) = {}_{\mathcal{A}}c(C, - \otimes_{\mathcal{A}} N)$$

with $C \in \operatorname{coh}{}_{\mathcal{A}}\mathcal{C}$ and $N \in \operatorname{mod}\mathcal{A}^{\mathrm{op}}$. If $\delta : B \to C$ is a morphism in $\operatorname{coh}{}_{\mathcal{A}}\mathcal{C}$, we put

$$D(\delta)_N : (DC)({}_{\mathcal{A}}N) \longrightarrow (DB)({}_{\mathcal{A}}N)$$

to be equal to ${}_{\mathcal{A}\mathcal{C}}(\delta, - \otimes_{\mathcal{A}} N)$.

**Theorem 6.6** (Auslander, Gruson, Jensen)**.** *The functor $D : (\operatorname{coh}{}_{\mathcal{A}}\mathcal{C})^{\mathrm{op}} \to \operatorname{coh}\mathcal{C}_{\mathcal{A}}$ defined above puts the categories $\operatorname{coh}{}_{\mathcal{A}}\mathcal{C}$ and $\operatorname{coh}\mathcal{C}_{\mathcal{A}}$ in duality. Furthermore, for $M_{\mathcal{A}} \in \operatorname{mod}\mathcal{A}$ and ${}_{\mathcal{A}}N \in \operatorname{mod}\mathcal{A}^{\mathrm{op}}$ we have that*

$$D(M_{\mathcal{A}}, -) \approx M \otimes_{\mathcal{A}} - \quad \text{and} \quad D(- \otimes_{\mathcal{A}} N) \approx ({}_{\mathcal{A}}N, -).$$

*Proof.* The proof is similar to the case when $\mathcal{A} = \{A\}$ (see [**14**]). □

Since the category $\operatorname{coh}{}_{\mathcal{A}}\mathcal{C}$ has enough projectives, the duality gives the following.

**Proposition 6.7** (Auslander)**.** *The category $\operatorname{coh}\mathcal{C}_{\mathcal{A}}$ has enough injectives and they are precisely the objects of the form $M \otimes_{\mathcal{A}} -$ where $M_{\mathcal{A}} \in \operatorname{mod}\mathcal{A}$.*

Thus every coherent object $C \in \operatorname{coh}\mathcal{C}_{\mathcal{A}}$ has both a projective presentation in $\mathcal{C}_{\mathcal{A}}$

$$({}_{\mathcal{A}}K, -) \longrightarrow ({}_{\mathcal{A}}L, -) \longrightarrow C \longrightarrow 0 \tag{6.2}$$

and an injective presentation in $\operatorname{coh}\mathcal{C}_{\mathcal{A}}$

$$0 \longrightarrow C \longrightarrow M \otimes_{\mathcal{A}} - \longrightarrow N \otimes_{\mathcal{A}} - \tag{6.3}$$

where $K$, $L \in \operatorname{mod}\mathcal{A}^{\mathrm{op}}$ and $M$, $N \in \operatorname{mod}\mathcal{A}$.

We conclude the section by the Herzog theorem. Let $\mathcal{S} \subseteq \operatorname{coh}{}_{\mathcal{A}}\mathcal{C}$; then the subcategory

$$D\mathcal{S} = \{DC \mid C \in \mathcal{S}\}$$

is Serre in $\operatorname{coh}\mathcal{C}_{\mathcal{A}}$ and the restriction to $\mathcal{S}$ of the duality $D$ gives a duality $D : \mathcal{S}^{\mathrm{op}} \to D\mathcal{S}$. By Theorem 5.16 the map $\mathcal{O}(\mathcal{S}) \mapsto \mathcal{O}(D\mathcal{S})$ induced on the open subsets of the Ziegler spectrum is an iclusion-preserving bijection. Similar to [**14**], it is easy to show that the functor $D$ induces an isomorphism of Abelian groups

$${}_{\mathcal{A}\mathcal{C}/\vec{\mathcal{S}}}(A, B) \approx {}_{\mathcal{C}_{\mathcal{A}}/\overrightarrow{D\mathcal{S}}}(DB, DA)$$

where $A$ and $B \in \operatorname{coh}{}_{\mathcal{A}}\mathcal{C}$ and the assignment given by $A_{\mathcal{S}} \mapsto (DA)_{D\mathcal{S}}$ is functorial. Thus we have the following.

**Theorem 6.8** (Herzog)**.** *Let $\mathcal{A}$ be a ring. There is an inclusion-preserving bijective correspondence between the Serre subcategories of $\operatorname{coh}{}_{\mathcal{A}}\mathcal{C}$ and those of $\operatorname{coh}\mathcal{C}_{\mathcal{A}}$ given by*

$$\mathcal{S} \longmapsto D\mathcal{S}.$$

*The induced map $\mathcal{O}(\mathcal{S}) \mapsto \mathcal{O}(D\mathcal{S})$ is an isomorphism between the topologies, that is, the respective algebras of open sets, of the Ziegler spectra of ${}_{\mathcal{A}}\mathcal{C}$ and $\mathcal{C}_{\mathcal{A}}$. Furthermore, the duality $D$ induces dualities between the respective subcategories $D : \mathcal{S}^{\mathrm{op}} \to D\mathcal{S}$ and $D : (\operatorname{coh}{}_{\mathcal{A}}\mathcal{C}/\vec{\mathcal{S}})^{\mathrm{op}} \to \operatorname{coh}\mathcal{C}_{\mathcal{A}}/\overrightarrow{D\mathcal{S}}$ as given by the following commutative diagram of Abelian categories:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S} & \longrightarrow & \operatorname{coh}{}_{\mathcal{A}}\mathcal{C} & \longrightarrow & \operatorname{coh}{}_{\mathcal{A}}\mathcal{C}/\vec{\mathcal{S}} & \longrightarrow & 0 \\
 & & {\scriptstyle D}\big\downarrow & & {\scriptstyle D}\big\downarrow & & \big\downarrow{\scriptstyle D} & & \\
0 & \longrightarrow & D\mathcal{S} & \longrightarrow & \operatorname{coh}\mathcal{C}_{\mathcal{A}} & \longrightarrow & \operatorname{coh}\mathcal{C}_{\mathcal{A}}/\overrightarrow{D\mathcal{S}} & \longrightarrow & 0.
\end{array}$$

## 7. Grothendieck categories as quotient categories of $(\operatorname{mod}\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$

In this section we give another representation of the Grothendieck category $\mathcal{C}$ as a quotient category of $\mathcal{C}_{\mathcal{A}}$, where $\mathcal{A} = \{P_i\}_{i\in I}$ is a ring. Given an arbitrary functor $F \in \mathcal{C}_{\mathcal{A}}$, denote by $F(\mathcal{A})$ the right $\mathcal{A}$-module defined as follows. If $P \in \mathcal{A}$, we put $F(\mathcal{A})(P) = F(h^P)$. It is directly verified that $F(\mathcal{A}) \in \operatorname{Mod}\mathcal{A}$.

**Proposition 7.1.** *Let* $\operatorname{Mod}\mathcal{A}$ *be the category of right* $\mathcal{A}$*-modules, where* $\mathcal{A} = \{P_i\}_{i\in I}$ *is a ring, and let* $\mathcal{C}_{\mathcal{A}}$ *be the corresponding category of generalized right* $\mathcal{A}$*-modules. Then the category* $\operatorname{Mod}\mathcal{A}$ *is equivalent to the quotient category* $\mathcal{C}_{\mathcal{A}}$ *with respect to the localizing subcategory* $\mathcal{P}^{\mathcal{A}} = \{F \in \mathcal{C}_{\mathcal{A}} \mid F(h^P) = 0 \text{ for all } P \in \mathcal{A}\}$.

*Proof.* Theorem 4.10 implies that the functor $\Phi : \mathcal{C}_{\mathcal{A}} \to \operatorname{Mod}\mathcal{A}$, $F \mapsto F(\mathcal{A})$, gives an equivalence for the categories $\operatorname{Mod}\mathcal{A}$ and $\mathcal{C}_{\mathcal{A}}/\operatorname{Ker}\Phi$. Clearly, $\mathcal{P}^{\mathcal{A}} = \operatorname{Ker}\Phi$. □

Since the functors $h_P \otimes_{\mathcal{A}} -$ and $(h^P, -)$ are isomorphic, Theorem 4.10 implies that the quotient category $\mathcal{C}_{\mathcal{A}}/\mathcal{P}^{\mathcal{A}}$ is equivalent to the subcategory $\{M \otimes_{\mathcal{A}} - \mid M \in \operatorname{Mod}\mathcal{A}\}$ generated by the set $\{P \otimes_{\mathcal{A}} -\}_{P\in\mathcal{A}}$.

**Remark.** Obviously,

$$\mathcal{P}^{\mathcal{A}} = \{\operatorname{Ker}(\mu \otimes_{\mathcal{A}} -) \mid \mu \text{ is a monomorphism in } \operatorname{Mod}\mathcal{A}\}.$$

It should also be remarked that the $\mathcal{C}_{\mathcal{A}}/\mathcal{P}^{\mathcal{A}}$-injective objects, in view of Lemma 1.3, are precisely the objects of the form $E \otimes_{\mathcal{A}} -$, where $E_{\mathcal{A}}$ is an injective right $\mathcal{A}$-module.

Next, we consider a Grothendieck category $\mathcal{C}$ with a family of generators $\mathcal{U}$. As usual, let $\operatorname{Mod}\mathcal{A}$ be the category of right $\mathcal{A}$-modules with $\mathcal{A} = \{h_U\}_{U\in\mathcal{U}}$. We are now in a position to prove the following statement.

**Theorem 7.2.** *Every Grothendieck category* $\mathcal{C}$ *with a family of generators* $\mathcal{U}$ *is equivalent to a quotient category of* $\mathcal{C}_{\mathcal{A}}$ *with respect to some localizing subcategory* $\mathcal{S}$ *of* $\mathcal{C}_{\mathcal{A}}$.

*Proof.* By Theorem 4.1 there is the pair of functors $(s, q)$, $s : \mathcal{C} \to \operatorname{Mod}\mathcal{A}$ and $q : \operatorname{Mod}\mathcal{A} \to \mathcal{C}$ that determines $\mathcal{C}$ as a quotient category of $\operatorname{Mod}\mathcal{A}$. In turn, by Proposition 7.1 there is the pair of functors $(g, h)$, $g : \operatorname{Mod}\mathcal{A} \to \mathcal{C}_{\mathcal{A}}$ and $h : \mathcal{C}_{\mathcal{A}} \to \operatorname{Mod}\mathcal{A}$ that determines $\operatorname{Mod}\mathcal{A}$ as a quotient category of $\mathcal{C}_{\mathcal{A}}$. It thus suffices to show that $gs$ is a fully faithful functor, the functor $qh$ is exact and left adjoint to $gs$. Indeed, the composition $gs$ of the fully faithful functors $g$ and $s$ is again a fully faithful functor and the composition $qh$ of the exact functors $q$ and $h$ is an exact functor. The fact that $gs$ is right adjoint to $qh$ follows from the following isomorphisms:

$${}_{\mathcal{C}_{\mathcal{A}}}(F, gs(C)) \approx \operatorname{Hom}_{\mathcal{A}}(h(F), s(C)) \approx {}_{\mathcal{C}}(qh(F), C).$$

Hence $\mathcal{C}$ is equivalent to the quotient category of $\mathcal{C}_{\mathcal{A}}$ with respect to the localizing subcategory $\mathcal{S} = \operatorname{Ker}(qh)$. □

**Corollary 7.3.** [**10**, 2.3] *Every Grothendieck category* $\mathcal{C}$ *with a generator* $U$ *is equivalent to a quotient category of* $\mathcal{C}_A$, $A = \operatorname{End}_{\mathcal{C}} U$, *with respect to some localizing subcategory* $\mathcal{S}$ *of* $\mathcal{C}_A$.

*Proof.* This is a consequence of Corollary 4.4 and Theorem 7.2. □

A ring $\mathcal{A}$ is said to be *right coherent* provided that each object $P \in \mathcal{A}$ is coherent. Now, let $\mathcal{C}$ be a locally coherent, i.e., we may suppose that $\mathcal{U} \subseteq \operatorname{coh}\mathcal{C}$ and let $\operatorname{Mod}\mathcal{A}$ be the corresponding category of modules with $\mathcal{A} = \{h_U\}_{U\in\mathcal{U}}$. One easily verifies that $\mathcal{A}$ is right coherent.

**Theorem 7.4.** *Let $\mathcal{C}$ be a Grothendieck category. Consider the following conditions:*

(1) *$\mathcal{C}$ is locally coherent, i.e., there is a family of generators $\mathcal{U} \subseteq \operatorname{coh}\mathcal{C}$;*

(2) *the localizing subcategory $\mathcal{S}$ of the preceding theorem is of finite type;*

(3) *$\mathcal{S}$ is of prefinite type;*

(4) $\operatorname{Zg}\mathcal{C} = \{E \otimes_{\mathcal{A}} - \mid E$ *is $\mathcal{C}$-injective$\}$ is closed in* $\operatorname{Zg}\mathcal{C}_{\mathcal{A}}$.

*Then conditions* (1), (2) *and* (3) *are equivalent and* (1), (2), (3) *imply* (4). *If $\mathcal{C}$ is a locally finitely generated category, then also* (4) *implies* (1), (2), (3).

*Proof.* The equivalence (2) $\Leftrightarrow$ (3) follows from Theorem 5.14.

(1) $\Rightarrow$ (4). By assumption, the ring $\mathcal{A}$ is right coherent. Therefore the category of right $\mathcal{A}$-modules $\operatorname{Mod}\mathcal{A}$ is locally coherent. By Theorem 5.14 $\operatorname{Zg}\mathcal{C}$ is closed in $\operatorname{Zg}(\operatorname{Mod}\mathcal{A})$. Our assertion would be proved if we showed that $\operatorname{Zg}(\operatorname{Mod}\mathcal{A})$ is a closed subset of $\operatorname{Zg}\mathcal{C}_{\mathcal{A}}$. By Theorem 5.16 and Theorem 5.15 it suffices to show that the localizing subcategory $\mathcal{P}^{\mathcal{A}}$ of $\mathcal{C}_{\mathcal{A}}$ is of finite type. Thus we must show that $\mathcal{P}^{\mathcal{A}} = \vec{\mathcal{S}}^{\mathcal{A}}$ where $\mathcal{S}^{\mathcal{A}} = \mathcal{P}^{\mathcal{A}} \cap \operatorname{coh}\mathcal{C}_{\mathcal{A}}$ is a Serre subcategory of $\operatorname{coh}\mathcal{C}_{\mathcal{A}}$. Clearly, $\vec{\mathcal{S}}^{\mathcal{A}} \subseteq \mathcal{P}^{\mathcal{A}}$. One verifies the inverse inclusion.

Let $F \in \mathcal{P}^{\mathcal{A}}$; then there exists the exact sequence

$$0 \longrightarrow F \xrightarrow{\iota} M \otimes_{\mathcal{A}} - \xrightarrow{\mu\otimes -} N \otimes_{\mathcal{A}} - \xrightarrow{\nu\otimes -} L \otimes_{\mathcal{A}} - \longrightarrow 0,$$

where $M \otimes_{\mathcal{A}} - = E(F)$, $N \otimes_{\mathcal{A}} - = E(\operatorname{coker}\iota)$, $\mu : M \to N$ is a monomorphism. By assumption, $\mathcal{A}$ is right coherent, and, hence, the exact sequence $0 \to M \xrightarrow{\mu} N \xrightarrow{\nu} L \to 0$ is the direct limit of exact sequences $0 \to M_i \xrightarrow{\mu_i} N_i \xrightarrow{\nu_i} L_i \to 0$ with $M_i, N_i, L_i \in \operatorname{mod}\mathcal{A}$ [**27**, 5.7]. If $C_i = \operatorname{Ker}(\mu_i \otimes -)$, then $C_i \in \mathcal{S}^{\mathcal{A}}$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_i & \xrightarrow{\rho_i} & M_i \otimes_{\mathcal{A}} - & \xrightarrow{\mu_i\otimes -} & N_i \otimes - \\ & & & & \big\downarrow{\scriptstyle \alpha_i\otimes -} & & \big\downarrow{\scriptstyle \beta_i\otimes -} \\ 0 & \longrightarrow & F & \xrightarrow{\iota} & M \otimes - & \xrightarrow{\mu\otimes -} & N \otimes - \end{array} \tag{7.1}$$

with $\alpha_i$ and $\beta_i$ the canonical homomorphisms. Since $(\mu\alpha_i \otimes -)\rho_i = (\beta_i\mu_i \otimes -)\rho_i = 0$, it follows that there exists $\gamma_i : C_i \to F$ such that $\iota\gamma_i = (\alpha_i \otimes -)\rho_i$. Similarly, given $i \leq j$ we can construct the commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_i & \xrightarrow{\rho_i} & M_i \otimes_{\mathcal{A}} - & \xrightarrow{\mu_i\otimes -} & N_i \otimes_{\mathcal{A}} - \\ & & {\scriptstyle \gamma_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_{ij}\otimes -} & & \big\downarrow{\scriptstyle \beta_{ij}\otimes -} \\ 0 & \longrightarrow & C_j & \xrightarrow{\rho_j} & M_j \otimes_{\mathcal{A}} - & \xrightarrow{\mu_j\otimes -} & N_j \otimes_{\mathcal{A}} -. \end{array}$$

It is directly verified that the family $\{C_i, \gamma_{ij}\}$ is direct. Now, taking the direct limit on the upper row in diagram (7.1), we obtain $F = \varinjlim C_i$. Thus $F \in \vec{\mathcal{S}}^{\mathcal{A}}$.

(2) $\Rightarrow$ (1). By Theorem 5.14 the $\mathcal{S}$-localization $(h_U \otimes_{\mathcal{A}} -)_{\mathcal{S}} \approx U$ of the $\mathcal{C}_{\mathcal{A}}$-coherent object $h_U \otimes_{\mathcal{A}} -$ is $\mathcal{C}$-coherent.

(4) $\Rightarrow$ (2). Suppose $\mathcal{C}$ is a locally finitely generated category and $\operatorname{Zg}\mathcal{C} = \operatorname{Zg}\mathcal{C}_{\mathcal{A}}/\mathcal{S}$ is a closed subset of $\operatorname{Zg}\mathcal{C}_{\mathcal{A}}$. By Theorem 5.16 there is the Serre subcategory $\mathcal{P}$ of $\operatorname{coh}\mathcal{C}_{\mathcal{A}}$ such that $\operatorname{Zg}\mathcal{C} = \operatorname{Zg}\mathcal{C}_{\mathcal{A}}/\vec{\mathcal{P}}$. From [**10**, 2.7] it follows that $\mathcal{S} = \vec{\mathcal{P}}$. Now, our assertion follows from Theorem 5.15. □

**Corollary 7.5.** *Let* $\operatorname{Mod}\mathcal{A}$ *be the module category with* $\mathcal{A} = \{P_i\}_{i\in I}$. *Then the following statements are equivalent:*
(1) *the ring* $\mathcal{A}$ *is right coherent;*
(2) *the localizing subcategory* $\mathcal{P}^{\mathcal{A}}$ *is of finite type;*
(3) *the localizing subcategory* $\mathcal{P}^{\mathcal{A}}$ *is of prefinite type;*
(4) $\operatorname{Zg}(\operatorname{Mod}\mathcal{A}) = \{E \otimes_{\mathcal{A}} - \mid E_{\mathcal{A}}$ *is an injective right* $\mathcal{A}$*-module*$\}$ *is closed in* $\operatorname{Zg}\mathcal{C}_{\mathcal{A}}$.

The next statement extends the list of properties characterizing the coherent rings (see also [**33**]):

**Proposition 7.6.** *For a ring* $\mathcal{A} = \{P_i\}_{i\in I}$ *the following assertions are equivalent:*
(1) $\mathcal{A}$ *is right coherent;*
(2) *for any finitely presented left* $\mathcal{A}$*-module* $M$ *the right* $\mathcal{A}$*-module* $M^* = \operatorname{Hom}_{\mathcal{A}}(M, \mathcal{A})$ *is finitely presented;*
(3) *for any finitely presented left* $\mathcal{A}$*-module* $M$ *the right* $\mathcal{A}$*-module* $M^* = \operatorname{Hom}_{\mathcal{A}}(M, \mathcal{A})$ *is finitely generated;*
(4) *for any coherent object* $C \in \operatorname{coh}\mathcal{C}_{\mathcal{A}}$ *the right* $\mathcal{A}$*-module* $C(\mathcal{A})$ *is finitely presented;*
(5) *for any coherent object* $C \in \operatorname{coh}\mathcal{C}_{\mathcal{A}}$ *the right* $\mathcal{A}$*-module* $C(\mathcal{A})$ *is finitely generated.*

*Proof.* By Proposition 7.1 the functor $\mathcal{C}_{\mathcal{A}} \to \operatorname{Mod}\mathcal{A}$, $F \mapsto F(\mathcal{A})$, yields an equivalence of the categories $\mathcal{C}_{\mathcal{A}}/\mathcal{P}^{\mathcal{A}}$ and $\operatorname{Mod}\mathcal{A}$. By the preceding corollary, $\mathcal{A}$ is right coherent if and only if $\mathcal{P}^{\mathcal{A}}$ is of finite type (= of prefinite type). Since the family $\{({}_{\mathcal{A}}M, -)\}_{M\in \operatorname{mod}\mathcal{A}^{\mathrm{op}}}$ is a family of generators for $\mathcal{C}_{\mathcal{A}}$, our assertion immediately follows from Theorem 5.8 and Proposition 5.9. □

**Corollary 7.7.** *For a ring* $\mathcal{A} = \{P_i\}_{i\in I}$ *the following statements are equivalent:*
(1) $\mathcal{P}^{\mathcal{A}}$ *is coexact;*
(2) *for any finitely presented left* $\mathcal{A}$*-module* $M$ *the right* $\mathcal{A}$*-module* $M^* = \operatorname{Hom}_{\mathcal{A}}(M, \mathcal{A})$ *is projective.*

*Proof.* This follows from Proposition 5.2. □

As an example, we describe rings for which the subcategory $\mathcal{P}^{A}$ of the category $\mathcal{C}_{A}$ is coexact.

**Proposition 7.8.** *For a right coherent ring* $A$ *the following statements are equivalent:*
(1) $\operatorname{r.w.dim} A \le 2$*;*
(2) $\mathcal{P}^{A}$ *is coexact;*
(3) *for every finitely presented left* $A$*-module* $M$ *the right* $A$*-module* $M^* = \operatorname{Hom}(M, A)$ *is projective.*

*Proof.* Since $A$ is right coherent, it is easy to see that

$$\operatorname{r.w.dim} A = \sup\{\operatorname{pd} M \mid M \in \operatorname{mod} A\}.$$

By the preceding corollary the conditions (2) and (3) are equivalent.

(1) ⇒ (3). For $M \in \operatorname{mod} A^{\mathrm{op}}$ we consider an exact sequence

$$P_1 \xrightarrow{\alpha} P_0 \longrightarrow M \longrightarrow 0,$$

where $P_1$ and $P_0$ are finitely generated projective modules. Then there is an exact sequence

$$0 \longrightarrow M^* \longrightarrow P_0^* \xrightarrow{\alpha^*} P_1^* \longrightarrow N \longrightarrow 0$$

with $N = \operatorname{Coker} \alpha$. By assumption $M^*$ is projective.

(3) ⇒ (1). For $N \in \operatorname{mod} A$ consider an exact sequenc

$$0 \longrightarrow K \longrightarrow P_1 \xrightarrow{\alpha} P_0 \longrightarrow N \longrightarrow 0.$$

We must prove that the module $K$ is projective. Indeed, let $M = \operatorname{Coker} \alpha^*$; then $K$ is isomorphic to $M^*$. By assumption, $M^*$ is projective, and, hence, $K$ is projective. □

**Corollary 7.9** (Bass [**34**])**.** *If $A$ is a two-sided Noetherian ring, then* $\mathrm{r.dim}\, A \le 2$ *if and only if for every finitely generated left $A$-module $M$ the right $A$-module* $M^* = \operatorname{Hom}(M, A)$ *is projective.*

*Proof.* Over the Noetherian rings the categories of finitely generated and finitely presented modules coincide. □

Now, we want to describe the localizing subcategories of prefinite type in the locally finitely generated Grothendieck categories in terms of localizing subcategories of finite type of the category $\mathcal{C}_{\mathcal{A}}$. Let $\mathcal{S}^{\mathcal{A}}$ be a Serre subcategory $\mathcal{P}^{\mathcal{A}} \cap \operatorname{coh} \mathcal{C}_{\mathcal{A}}$ of $\mathcal{C}_{\mathcal{A}}$. If $\mathcal{P}$ is a subcategory of $\mathcal{C}_{\mathcal{A}}$, by $\mathcal{P}(\mathcal{A})$ we denote the subcategory of $\operatorname{Mod} \mathcal{A}$ consisting of the modules $F(\mathcal{A})$ with $F \in \mathcal{P}$.

**Proposition 7.10.** *For a localizing subcategory $\mathcal{S}$ of* $\operatorname{Mod} \mathcal{A}$*, $\mathcal{A} = \{P_i\}_{i \in I}$, the following statements are equivalent:*

(1) *$\mathcal{S}$ is of prefinite type;*

(2) *there is a Serre subcategory $\mathcal{T}$ of* $\operatorname{coh} \mathcal{C}_{\mathcal{A}}$ *such that $\mathcal{T} \supseteq \mathcal{S}^{\mathcal{A}}$ and $\vec{\mathcal{T}}(\mathcal{A}) = \mathcal{S}$.*

*Proof.* Denote by $\mathcal{P}$ the localizing subcategory of $\mathcal{C}_{\mathcal{A}}$ such that $\mathcal{P} \supseteq \mathcal{P}^{\mathcal{A}}$ and $\mathcal{P}(\mathcal{A}) = \mathcal{S}$ (see Proposition 1.6). Let $\mathcal{T} = \mathcal{P} \cap \operatorname{coh} \mathcal{C}_{\mathcal{A}}$; then $\mathcal{T}$ is a Serre subcategory of $\operatorname{coh} \mathcal{C}_{\mathcal{A}}$. Clearly, $\mathcal{T} \supseteq \mathcal{S}^{\mathcal{A}}$ and $\vec{\mathcal{T}} \subseteq \mathcal{P}$. So, $\vec{\mathcal{T}}(\mathcal{A}) \subseteq \mathcal{S}$. Now, we prove the inverse inclusion.

Let $\mathfrak{F} = \cup_{P \in \mathcal{A}} \mathfrak{F}^P$ be the Gabriel topology that corresponds to $\mathcal{S}$. Our statement would be proved, if we showed that $\mathfrak{F}$ has the basis consisting of the ideals $\mathfrak{c} \subseteq P$ such that $\mathfrak{c} = \mathfrak{b}(\mathcal{A})$, where $\mathfrak{b}$ is a coherent object of $P \otimes_{\mathcal{A}} -$ such that $(P \otimes_{\mathcal{A}} -)/\mathfrak{b} \in \mathcal{T}$.

So, let $\mathfrak{a} \in \mathfrak{F}^P$. Consider the following exact sequence:

$$0 \longrightarrow \operatorname{Ker}(\alpha \otimes -) \longrightarrow \mathfrak{a} \otimes_{\mathcal{A}} - \xrightarrow{\alpha \otimes -} P \otimes_{\mathcal{A}} -.$$

Since $\alpha$ is a monomorphism, $\operatorname{Ker}(\alpha \otimes -) \in \mathcal{P}^{\mathcal{A}}$. Let $\widetilde{\mathfrak{a}} = \operatorname{Im}(\alpha \otimes -)$; then $\widetilde{\mathfrak{a}}(\mathcal{A}) = \mathfrak{a}$ and $(P \otimes_{\mathcal{A}} -)/\widetilde{\mathfrak{a}} \in \mathcal{P}$. We write $\widetilde{\mathfrak{a}} = \sum_{i \in I} \mathfrak{a}_i$ as a sum of finitely generated subobjects $\mathfrak{a}_i \subseteq \widetilde{\mathfrak{a}}$. Since every $\mathfrak{a}_i$ is a subobject of the $\mathcal{C}_{\mathcal{A}}$-coherent object $P \otimes_{\mathcal{A}} -$, it follows that $\mathfrak{a}_i$ is coherent. We have

$$P_{\mathcal{S}} = \mathfrak{a}_{\mathcal{S}} = (\widetilde{\mathfrak{a}}(\mathcal{A}))_{\mathcal{S}} = \sum\nolimits_{i \in I} (\mathfrak{a}_i(\mathcal{A}))_{\mathcal{S}}.$$

By Theorem 5.8 the object $P_{\mathcal{S}} \in \operatorname{fg}(\operatorname{Mod} \mathcal{A}/\mathcal{S})$, and, hence, there is a finite subset $J$ of $I$ such that $P_{\mathcal{S}} = \sum_{i \in J} (\mathfrak{a}_i(\mathcal{A}))_{\mathcal{S}}$. Let $\mathfrak{b} = \sum_{i \in J} \mathfrak{a}_i$; then $(P \otimes_{\mathcal{A}} -)/\mathfrak{b} \in \mathcal{P}$. Since

$(P \otimes_{\mathcal{A}} -)/\mathfrak{b}$ is coherent, $(P \otimes_{\mathcal{A}} -)/\mathfrak{b} \in \mathcal{T}$. Consequently, $\mathfrak{b}(\mathcal{A}) \subseteq \mathfrak{a}$ and $(\mathfrak{b}(\mathcal{A}))_{\mathcal{S}} = P_{\mathcal{S}}$. This implies the claim. □

Now, we consider a locally finitely generated Grothendieck category $\mathcal{C}$. By Theorem 7.2 there is a localizing subcategory $\mathcal{S}$ of $\mathcal{C}_{\mathcal{A}}$ such that $\mathcal{C}$ is equivalent to $\mathcal{C}_{\mathcal{A}}/\mathcal{S}$ with $\mathcal{A} = \{h_U\}_{U\in\mathcal{U}}$. If $\mathcal{Q}$ and $\mathcal{P}$ are localizing subcategories of $\mathcal{C}_{\mathcal{A}}$, by $\mathcal{Q}_{\mathcal{P}}$ denote the subcategory of $\mathcal{C}_{\mathcal{A}}/\mathcal{P}$ consisting of $\{Q_{\mathcal{P}} \mid Q \in \mathcal{Q}\}$. Also, denote by $\mathcal{L}$ the Serre subcategory $\mathcal{S} \cap \operatorname{coh}\mathcal{C}_{\mathcal{A}}$ of $\operatorname{coh}\mathcal{C}_{\mathcal{A}}$.

**Proposition 7.11.** *Let $\mathcal{Q}$ be a localizing subcategory of a locally finitely generated Grothendieck category $\mathcal{C}$ with a family of generators $\mathcal{U} \subseteq \operatorname{fg}\mathcal{C}$. If $\mathcal{A} = \{h_U\}_{U\in\mathcal{U}}$ is the ring generated by $\mathcal{U}$, then the following statements are equivalent:*

(1) *$\mathcal{Q}$ is of prefinite type;*

(2) *there is the Serre subcategory $\mathcal{T}$ of $\operatorname{coh}\mathcal{C}_{\mathcal{A}}$ such that $\mathcal{T} \supseteq \mathcal{L}$ and $\vec{\mathcal{T}}_{\mathcal{S}} = \mathcal{Q}$.*

*Proof.* By Theorem 4.1 there is a localizing subcategory $\mathcal{P}$ of $\operatorname{Mod}\mathcal{A}$ such that $\mathcal{C}$ is equivalent to $\operatorname{Mod}\mathcal{A}/\mathcal{P}$ and by Proposition 1.6 there is a localizing subcategory $\mathcal{V}$ of $\operatorname{Mod}\mathcal{A}$ such that $\mathcal{V} \supseteq \mathcal{P}$ and $\mathcal{V}/\mathcal{P} = \mathcal{Q}$. Since both $\mathcal{P}$ and $\mathcal{Q}$ are of prefinite type, it follows that $\mathcal{V}$ is of prefinite type. The preceding proposition implies that $\mathcal{V} = \vec{\mathcal{T}}(\mathcal{A})$ for some localizing subcategory of finite type $\vec{\mathcal{T}}$ of $\mathcal{C}_{\mathcal{A}}$. Then $\mathcal{Q} = \mathcal{V}_{\mathcal{P}} = (\vec{\mathcal{T}}(\mathcal{A}))_{\mathcal{P}} = \vec{\mathcal{T}}_{\mathcal{S}}$. □

**Question.** Is it true that if $\vec{\mathcal{T}}$ is a localizing subcategory of finite type of $\mathcal{C}_{\mathcal{A}}$ containing the subcategory $\vec{\mathcal{S}}^{\mathcal{A}}$, then the subcategory $\vec{\mathcal{T}}(\mathcal{A})$ is localizing and of prefinite type in $\operatorname{Mod}\mathcal{A}$? If this was true, we could construct the Ziegler topology for an arbitrary locally finitely generated Grothendieck category.

## 8. $FP$-injective and flat modules

In this section we sketch how the classes of $FP$-injective and flat $\mathcal{A}$-modules can be studied with the help of some torsion/localizing functors of the category $\mathcal{C}_{\mathcal{A}}$.

Let $\mathcal{C}$ be a locally finitely presented Grothendieck category. An object $C \in \mathcal{C}$ is said to be *$FP$-injective* (or *absolutely pure*) if $\operatorname{Ext}^1_{\mathcal{C}}(X, C) = 0$ for any $X \in \operatorname{fp}\mathcal{C}$. An object $C \in \mathcal{C}$ is said to be *$fp$-injective* if for any monomorphism $\mu : X \to Y$ in $\operatorname{fp}\mathcal{C}$ the morphism ${}_{\mathcal{C}}(\mu, C)$ is an epimorphism. Evidently, every $FP$-injective object is $fp$-injective, and every $fp$-injective finitely presented object is $FP$-injective. The ring $\mathcal{A} = \{P_i\}_{i\in I}$ is *right $FP$-injective* if every right $\mathcal{A}$-module $P \in \mathcal{A}$ is $FP$-injective.

Let $\operatorname{Mod}\mathcal{A}$, $\mathcal{A} = \{P_i\}_{i\in I}$, be the category of right $\mathcal{A}$-modules. A module $M \in \operatorname{Mod}\mathcal{A}$ is *flat* if the tensor functor $M \otimes_{\mathcal{A}} -$ is exact. We refer to $M$ as an *$fp$-flat module* if for any monomorphism $\mu : {}_{\mathcal{A}}K \to {}_{\mathcal{A}}L$ in $\operatorname{mod}\mathcal{A}^{\mathrm{op}}$ the morphism $M \otimes_{\mathcal{A}} \mu$ is a monomorphism. Evidently, every flat module is $fp$-flat.

One easily verifies:

**Lemma 8.1.** *Let $\mathcal{C}$ be a locally finitely presented Grothendieck category. The following statements are equivalent for an object $C \in \mathcal{C}$:*

(1) *$C$ is $FP$-injective;*

(2) *every exact sequence $0 \to C \to C' \to C'' \to 0$ is pure;*

(3) *there exists a pure-exact sequence $0 \to C \to C' \to C'' \to 0$ with $C'$ being $FP$-injective.*

In the sequel, we use the following notation:

$$\mathcal{P}^{\mathcal{A}} = \{F \in \mathcal{C}_{\mathcal{A}} \mid F(\mathcal{A}) = 0\},$$
$$\mathcal{S}^{\mathcal{A}} = \{C \in \operatorname{coh} \mathcal{C}_{\mathcal{A}} \mid C(\mathcal{A}) = 0\},$$
$$\mathcal{S}_{\mathcal{A}} = \{C \in \operatorname{coh} \mathcal{C}_{\mathcal{A}} \mid (C, P \otimes_{\mathcal{A}} -) = 0 \text{ for all } P \in \mathcal{A}\}.$$

The definition of the subcategories ${}^{\mathcal{A}}\mathcal{S}$ and ${}_{\mathcal{A}}\mathcal{S}$ of $\operatorname{coh}{}_{\mathcal{A}}\mathcal{C}$ is similar to that of $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{A}}$, respectively. By Theorem 5.15 $\vec{\mathcal{S}}^{\mathcal{A}}$ and $\vec{\mathcal{S}}_{\mathcal{A}}$ are localizing subcategories of finite type. By Corollary 7.5 $\mathcal{P}^{\mathcal{A}} = \vec{\mathcal{S}}^{\mathcal{A}}$ if and only if the ring $\mathcal{A}$ is right coherent. From the presentation (6.3), it easily follows that

$$\mathcal{S}^{\mathcal{A}} = \{\operatorname{Ker}(\mu \otimes -) \mid \mu : M \to N \text{ is a monomorphism in } \operatorname{mod} \mathcal{A}\}.$$

In a similar way

$$\mathcal{S}_{\mathcal{A}} = \{\operatorname{Coker}(\mu, -) \mid \mu : L \to K \text{ is a monomorphism in } \operatorname{mod} \mathcal{A}^{\mathrm{op}}\}.$$

**Proposition 8.2.** *If $K \in \operatorname{Mod} \mathcal{A}$, then:*
(1) *$K$ is $FP$-injective if and only if the functor $K \otimes_{\mathcal{A}} -$ is $\mathcal{P}^{\mathcal{A}}$-torsionfree;*
(2) *$K$ is fp-injective if and only if the functor $K \otimes_{\mathcal{A}} -$ is $\vec{\mathcal{S}}^{\mathcal{A}}$-torsionfree;*
(3) *$K$ is fp-flat if and only if the functor $K \otimes_{\mathcal{A}} -$ is $\vec{\mathcal{S}}_{\mathcal{A}}$-torsionfree.*

*Proof.* Adapt the proof for modules over a ring $\mathcal{A} = \{A\}$ [**11**, 2.2]. □

**Corollary 8.3.** *The set of the pure-injective fp-injective (fp-flat) modules is closed in $\operatorname{Zg} \mathcal{C}_{\mathcal{A}}$.*

**Proposition 8.4.** *For a ring $A$ the following statements are equivalent:*
(1) *$A$ is right semihereditary;*
(2) *the subcategory $\mathcal{P}^{A}$ is strongly coexact;*
(3) *any quotient module of an $FP$-injective right $A$-module is $FP$-injective;*
(4) *any quotient module of an injective right $A$-module is $FP$-injective.*

*Proof.* The proof of the equivalences (1) ⇔ (3) ⇔ (4) is similar to that of [**3**, I.9.5].

(2) ⇒ (3). By assumption, the functor $t_{\mathcal{P}^A}$ is exact. Therefore our assertion follows from the preceding proposition.

(3) ⇒ (2). Clearly, every $\mathcal{P}^{A}$-torsionfree object $X$ is of the form $X \approx (M \otimes_A -)/Y$ with $M = X(A)$, $Y = t_{\mathcal{P}^A}(M \otimes_A -)$. If $E \otimes_A - = E(X)$ is the injective envelope of $X$, there is the exact sequence

$$0 \longrightarrow X \longrightarrow E \otimes_A - \longrightarrow (E/M) \otimes_A - \longrightarrow 0.$$

Obviously, the module $E$ is injective. By assumption, $t_{\mathcal{P}^A}((E/M) \otimes_A -) = 0$. If we consider the exact sequence

$$0 = t_{\mathcal{P}^A}((E/M) \otimes_A -) \longrightarrow t^1_{\mathcal{P}^A}(X) \longrightarrow t^1_{\mathcal{P}^A}(E \otimes_A -) = 0,$$

we get that $t^1_{\mathcal{P}^A}(X) = 0$, i.e., $X$ is $\mathcal{P}^A$-closed. Now, Proposition 5.4(3) completes our proof. □

**Corollary 8.5.** *A ring $A$ is right semihereditary if and only if the functor*

$$M_A \longmapsto M \otimes_A -/t_{\mathcal{P}^A}(M \otimes_A -)$$

*yields an equivalence between the category* Mod $A$ *and the category consisting of the* $\mathcal{P}^A$*-torsionfree objects.*

*Proof.* It suffices to apply Proposition 7.1 and Corollary 5.5. □

It is well-known (see e.g. [**3**]) that for a ring $A$ a right $A$-module $M$ is finitely presented (finitely generated) if and only if the natural map $M \otimes_A (\prod_{i\in I} N_i) \to \prod_{i\in I}(M \otimes_A N_i)$ is an isomorphism (epimorphism) for every family $\{N_i\}_{i\in I}$ of left $A$-modules. This generalises to arbitrary module categories $\operatorname{Mod}\mathcal{A}$ as follows.

**Lemma 8.6.** [**27**, 7.1] *Let $\mathcal{A}$ be a ring. For $M \in \operatorname{Mod}\mathcal{A}$ the following statements are equivalent:*

(1) *$M$ is finitely presented (finitely generated);*

(2) *the natural morphism $M \otimes_{\mathcal{A}} (\prod_{i\in I} N_i) \to \prod_{i\in I}(M \otimes_{\mathcal{A}} N_i)$ is an isomorphism (epimorphism) for every family $\{N_i\}_{i\in I}$ in $\operatorname{Mod}\mathcal{A}^{\mathrm{op}}$;*

(3) *the natural morphism $M \otimes_{\mathcal{A}} (\prod_{i\in I} P_i) \to \prod_{i\in I}(M \otimes_{\mathcal{A}} P_i) = \prod_{i\in I} M(P_i)$ is an isomorphism (epimorphism) for every family $\{P_i\}_{i\in I}$ in $\mathcal{A}$.*

Now, the proof of the next result is similar to that of [**11**, 2.3].

**Proposition 8.7.** *Let $\mathcal{A} = \{P_i\}_{i\in I}$ be a ring; then:*

(1) *for every family of right $\mathcal{A}$-modules $\{M_i\}_I$ the module $\prod_I M_i$ is $FP$-injective (respectively $fp$-injective, $fp$-flat) if and only if every $M_i$ is $FP$-injective (respectively $fp$-injective, $fp$-flat);*

(2) *the direct limit $\varinjlim M_i$ of $fp$-injective (respectively $fp$-flat) right $\mathcal{A}$-modules $M_i$ is an $fp$-injective (respectively $fp$-flat) module.*

Now, we consider a locally finitely presented Grothendieck category $\mathcal{C}$ with a family of generators $\mathcal{U} \subseteq \operatorname{fp}\mathcal{C}$. As usual, let $\operatorname{Mod}\mathcal{A}$ be the category of modules with $\mathcal{A} = \{h_U\}_{U\in\mathcal{U}}$. By Theorem 4.1 $\mathcal{C}$ is equivalent to a quotient category $\operatorname{Mod}\mathcal{A}/\mathcal{S}$. Furthermore, by Proposition 5.9 $\mathcal{S}$ is of finite type. By Theorem 7.2 there is a localizing subcategory $\mathcal{P}$ of $\mathcal{C}_{\mathcal{A}}$ such that $\mathcal{C}$ is equivalent to the quotient category $\mathcal{C}_{\mathcal{A}}/\mathcal{P}$. Similar to the category of modules, $FP$-/$fp$-injective objects of $\mathcal{C}$ can be described in terms of torsion functors of $\mathcal{C}_{\mathcal{A}}$. To begin, we prove the following.

**Proposition 8.8.** *For an object $C \in \mathcal{C}$ the following statements hold:*

(1) *$C$ is $FP$-injective if and only if it is $FP$-injective as a right $\mathcal{A}$-module.*

(2) *$C$ is $fp$-injective if and only if it is $fp$-injective as a right $\mathcal{A}$-module.*

*Proof.* (1). Let $C$ be an $FP$-injective object of $\mathcal{C}$ and $M \in \operatorname{mod}\mathcal{A}$. We must show that $\operatorname{Ext}^1_{\mathcal{A}}(M, C) = 0$. Equivalently, any short exact sequence

$$0 \longrightarrow C \xrightarrow{\alpha} X \longrightarrow M \longrightarrow 0$$

of right $\mathcal{A}$-modules splits. By Proposition 5.9 $M_{\mathcal{S}} \in \operatorname{fp}\mathcal{C}$. By assumption, the morphism $\alpha_{\mathcal{S}}$ splits, i.e., there exists $\beta : X_{\mathcal{S}} \to C$ such that $\beta\alpha_{\mathcal{S}} = 1_C$. Then $(\beta\lambda_X)\alpha = \beta\alpha_{\mathcal{S}} = 1_C$ where $\lambda_X$ is the $\mathcal{S}$-envelope for $X$. So $\alpha$ splits.

Conversely, let $C$ be an $FP$-injective right $\mathcal{A}$-module and let

$$\varepsilon : 0 \longrightarrow C \xrightarrow{\alpha} E \xrightarrow{\beta} X \longrightarrow 0$$

be the $\mathcal{C}$-exact sequence with $E = E(C)$ and $X = E/C$. By assumption, the short exact sequence

$$\bar{\varepsilon} : 0 \longrightarrow C \xrightarrow{\alpha} E \xrightarrow{\bar{\beta}} \operatorname{Im}\beta \longrightarrow 0$$

is pure-exact in $\operatorname{Mod}\mathcal{A}$. Clearly, $\bar{\varepsilon}_{\mathcal{S}} = \varepsilon$. From Proposition 6.3 it follows that $\bar{\varepsilon}$ is a direct limit of split exact sequences

$$\bar{\varepsilon}_i : 0 \longrightarrow C_i \longrightarrow E_i \longrightarrow M_i \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{A}$. Then $\varepsilon$ is a direct limit of split exact sequences $\varepsilon_i = (\bar{\varepsilon}_i)_{\mathcal{S}}$. Thus $C$ is an $FP$-injective object of $\mathcal{C}$.

(2). Suppose $C$ is an $fp$-injective object of $\mathcal{C}$ and $\mu : M \to N$ is a monomorphism in $\operatorname{mod}\mathcal{A}$. Since $\mathcal{S}$ is of finite type, the morphism $\mu_{\mathcal{S}}$ is a monomorphism in $\operatorname{fp}\mathcal{C}$. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{A}}(N,C) & \xrightarrow{(\mu,C)} & \operatorname{Hom}_{\mathcal{A}}(M,C) \\ \downarrow & & \downarrow \\ {}_{\mathcal{C}}(N_{\mathcal{S}},C) & \xrightarrow[(\mu_{\mathcal{S}},C)]{} & {}_{\mathcal{C}}(M_{\mathcal{S}},C) \end{array}$$

where the vertical arrows are isomorphisms. Since $(\mu_{\mathcal{S}}, C)$ is an epimorphism, it follows that $(\mu, C)$ is an epimorphism.

Coversely, suppose $\mu : X \to Y$ is a monomorphism in $\operatorname{fp}\mathcal{C}$. Then there is a monomorphism $\gamma : M \to N$ in $\operatorname{mod}\mathcal{A}$ such that $\gamma_{\mathcal{S}} = \mu$. Indeed, we can embed $\mu$ into the commutative diagram in $\mathcal{C}$ with exact rows:

$$\begin{array}{ccccccc} (-,\oplus_{i=1}^{n}U_i) & \xrightarrow{\psi} & (-,\oplus_{j=1}^{m}U_j) & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mu} & & \\ (-,\oplus_{k=1}^{s}U_k) & \xrightarrow[\varphi]{} & (-,\oplus_{l=1}^{t}U_l) & \longrightarrow & Y & \longrightarrow & 0. \end{array}$$

Since each $\oplus h_{U_i}$ is $\mathcal{S}$-closed and finitely generated projective in $\operatorname{Mod}\mathcal{A}$, both $\operatorname{Coker}\psi$ and $\operatorname{Coker}\varphi$ are finitely presented right $\mathcal{A}$-modules. We put $M = \operatorname{Coker}\psi$ and $N = \operatorname{Coker}\varphi$. There is the unique morphism $\gamma : M \to N$. Since $M_{\mathcal{S}} = X$ and $N_{\mathcal{S}} = Y$, it follows that $\gamma_{\mathcal{S}} = \mu$. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{A}}(N,C) & \xrightarrow{(\gamma,C)} & \operatorname{Hom}_{\mathcal{A}}(M,C) \\ \downarrow & & \downarrow \\ {}_{\mathcal{C}}(Y,C) & \xrightarrow[(\mu,C)]{} & {}_{\mathcal{C}}(X,C) \end{array}$$

where the vertical arrows are isomorphisms. Since $(\gamma, C)$ is an epimorphism, it follows that $(\mu, C)$ is an epimorphism. So $C$ is $fp$-injective in $\mathcal{C}$. □

**Corollary 8.9.** *The ring $\mathcal{A} = \{h_U\}_{U\in\mathcal{U}}$ is $FP$-injective if and only if each $U \in \mathcal{U}$ is an $FP$-injective object of $\mathcal{C}$.*

*Proof.* It suffices to observe that each $h_U$ is $\mathcal{S}$-closed (see Theorem 4.1) and then apply the preceding proposition. □

Denote by $\mathcal{T} = \operatorname{coh}\mathcal{C} \cap \mathcal{P}$ and let $t_{\mathcal{P}}$ and $t_{\mathcal{T}}$ be the torsion functors corresponding to the localizing subcategories $\mathcal{P}$ and $\vec{\mathcal{T}}$ of $\mathcal{C}_{\mathcal{A}}$.

**Proposition 8.10.** *Let $C \in \mathcal{C}$; then the following statements hold:*
(1) *$C$ is an $FP$-injective object of $\mathcal{C}$ if and only if $t_{\mathcal{P}}(C \otimes_{\mathcal{A}} -) = 0$.*
(2) *$C$ is an $fp$-injective object of $\mathcal{C}$ if and only if $t_{\mathcal{T}}(C \otimes_{\mathcal{A}} -) = 0$.*

*Proof.* (1). Let $C$ be $FP$-injective. By the preceding proposition it is an $FP$-injective right $\mathcal{A}$-module. Now, let $E$ be an injective envelope for $C$. Then $C \otimes_{\mathcal{A}} -$ is a subobject of $E \otimes_{\mathcal{A}} -$. Since $E \otimes_{\mathcal{A}} -$ is $\mathcal{P}$-torsionfree, it follows that $C \otimes_{\mathcal{A}} -$ is $\mathcal{P}$-torsionfree. Conversely, since $\mathcal{P} \supseteq \mathcal{P}^{\mathcal{A}}$, our assertion follows from Propositions 8.2 and 8.8.

(2). Let $C$ be $fp$-injective and $T \in \mathcal{T}$. Consider the exact sequence (6.3)

$$0 \longrightarrow T \longrightarrow M \otimes_{\mathcal{A}} - \xrightarrow{\mu\otimes -} N \otimes_{\mathcal{A}} -$$

where $M,\ N \in \operatorname{mod}\mathcal{A}$. Since $0 = T_{\mathcal{P}} = T(\mathcal{A})_{\mathcal{S}}$, it follows that the morphism $\mu_{\mathcal{S}}$ is a monomorphism in $\operatorname{fp}\mathcal{C}$. Consequently, the morphism $(\mu_{\mathcal{S}}, C)$ is an epimorphism, and, hence, the morphism $(\mu, C)$ is also an epimorphism. As $C \otimes_{\mathcal{A}} -$ is a coh-injective object, one has an exact sequence

$$(N \otimes_{\mathcal{A}} -, C \otimes_{\mathcal{A}} -) \xrightarrow{(\mu\otimes -, C\otimes_{\mathcal{A}} -)} (M \otimes_{\mathcal{A}} -, C \otimes_{\mathcal{A}} -) \longrightarrow (T, C \otimes_{\mathcal{A}} -) \longrightarrow 0.$$

But $(\mu, C)$ is an epimorphism, hence $(T, C \otimes_{\mathcal{A}} -) = 0$. So $t_{\mathcal{T}}(C \otimes_{\mathcal{A}} -) = 0$. Since $\mathcal{T} \supseteq \mathcal{S}^{\mathcal{A}}$, the converse follows from the preceding proposition and Proposition 8.2. □

For a ring $A$ the Chase theorem states that $A$ is left coherent if and only if any direct product $\prod M_i$ of flat right $A$-modules $M_i$ is flat. This generalizes to an arbitrary ring $\mathcal{A} = \{P_i\}_{i \in I}$ as follows.

**Proposition 8.11** (Chase)**.** *Let $\mathcal{A} = \{P_i\}_{i \in I}$ be a ring. Then the following are a equivalent:*
(1) *$\mathcal{A}$ is left coherent;*
(2) *every product of flat right $\mathcal{A}$-modules is flat;*
(3) *every product $\prod_{j \in J} P_j$ of $P_j \in \mathcal{A}$ is a flat right $\mathcal{A}$-module for every set $J$.*

*Proof.* (1) $\Rightarrow$ (2). Let $\{M_j\}_{j \in J}$ be the family of flat right $\mathcal{A}$-modules. By Proposition 8.7 the module $\prod_{j \in J} M_j$ is $fp$-flat. Let $\varphi : K \to L$ be a monomorphism in $\operatorname{Mod}\mathcal{A}^{\mathrm{op}}$. As $\mathcal{A}$ is left coherent, it follows that $\varphi = \varinjlim \varphi_i$ is a direct limit of monomorphisms $\varphi_i$ in $\operatorname{mod}\mathcal{A}^{\mathrm{op}}$ [**27**, 5.9]. Then the morphism $\prod M_j \otimes \varphi = \varinjlim(\prod M_j \otimes \varphi_i)$ is a direct limit of monomorphisms $\prod M_j \otimes \varphi_i$. Therefore, it is also a monomorphism.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let ${}_{\mathcal{A}}K$ be a finitely generated submodule of a finitely presented module ${}_{\mathcal{A}}L$. For any index set $J$ we have a commutative diagram

$$\begin{array}{ccc} \prod P_j \otimes_{\mathcal{A}} K & \longrightarrow & \prod P_j \otimes_{\mathcal{A}} L \\ {\scriptstyle \varphi_K}\downarrow & & \downarrow{\scriptstyle \varphi_L} \\ \prod K(P_j) & \longrightarrow & \prod L(P_j) \end{array}$$

where the horizontal arrows are monomorphisms. Since $\varphi_L$ is a monomorphism by Lemma 8.6, also $\varphi_K$ is a monomorphism. Thus $K$ is finitely presented by Lemma 8.6. □

In contrast to the $FP$-injective right $\mathcal{A}$-modules the class of the flat right $\mathcal{A}$-modules can be realized in $\mathcal{C}_\mathcal{A}$ as the class of the functors $M \otimes_\mathcal{A} -$ that satisfy the condition $t_\mathcal{S}(M \otimes_\mathcal{A} -) = 0$ for some localizing subcategory $\mathcal{S}$ of $\mathcal{C}_\mathcal{A}$ if and only if $\mathcal{A}$ is left coherent.

**Theorem 8.12.** *For a ring $\mathcal{A}$ the following statements are equivalent:*
(1) *$\mathcal{A}$ is left coherent;*
(2) *there is a localizing subcategory $\mathcal{S}$ of $\mathcal{C}_\mathcal{A}$ such that any right $\mathcal{A}$-module $M$ is flat if and only if the functor $M \otimes_\mathcal{A} -$ is $\mathcal{S}$-torsionfree;*
(3) *every left $fp$-injective $\mathcal{A}$-module is $FP$-injective;*
(4) *every right $fp$-flat $\mathcal{A}$-module is flat;*
(5) *a direct limit of $FP$-injective left $\mathcal{A}$-modules is $FP$-injective.*

*Proof.* (1) $\Leftrightarrow$ (5). This follows from [**27**, 9.3].
The proof of the remaining statements is similar to that of [**11**, 2.4]. □

**Theorem 8.13.** *For a ring $\mathcal{A}$ the following statements are equivalent:*
(1) *$\mathcal{A}$ is right $FP$-injective;*
(2) *$\mathcal{S}^\mathcal{A} \subseteq \mathcal{S}_\mathcal{A}$;*
(3) *${}_\mathcal{A}\mathcal{S} \subseteq {}^\mathcal{A}\mathcal{S}$;*
(4) *every $fp$-flat right $\mathcal{A}$-module is $fp$-injective;*
(5) *every indecomposable pure-injective $fp$-flat right $\mathcal{A}$-module is $fp$-injective;*
(6) *every pure-injective $fp$-flat right $\mathcal{A}$-module is $fp$-injective;*
(7) *every $fp$-injective left $\mathcal{A}$-module is $fp$-flat;*
(8) *every indecomposable pure-injective $fp$-injective left $\mathcal{A}$-module is $fp$-flat;*
(9) *every pure-injective $fp$-injective left $\mathcal{A}$-module is $fp$-flat.*

*Proof.* Adapt the proof for the case when $\mathcal{A} = \{A\}$ (see [**11**, 2.5]). □

**Example.** Let $\mathcal{A} = \{A\}$ be a ring and $\mathcal{C}_A$ the category of generalized right $A$-modules. Then the ring $\mathcal{B} = \{(M, -)\}_{M \in \operatorname{mod} A^{\mathrm{op}}}$ is right $FP$-injective if and only if $A$ is (von Neumann) regular.

Indeed, let $\mathcal{B}$ be right $FP$-injective; then each $(K, -)$ with $K \in \operatorname{mod} A^{\mathrm{op}}$ is coh-injective. Therefore $(K, -)$ is isomorphic to the object $K^* \otimes_A -$ where $K^* = \operatorname{Hom}_A(K, A)$. Since every coherent object $C \in \operatorname{coh} \mathcal{C}_A$ is a cokernel

$$(K, -) \xrightarrow{(\alpha,-)} (L, -) \longrightarrow C \longrightarrow 0$$

of $(\alpha, -)$ (see the sequence (6.2)), it follows that $C$ is isomorphic to $(\operatorname{Coker} \alpha^*) \otimes_A -$. So, every $C \in \operatorname{coh} \mathcal{C}_A$ is coh-injective, and, hence, $A$ is a regular ring [**14**, 4.4]. The converse also follows from [**14**, 4.4].

To conclude, we give a criterion of duality for the categories of finitely presented left and right $\mathcal{A}$-modules. A ring $\mathcal{A}$ over which the functor $\operatorname{Hom}_\mathcal{A}(-, \mathcal{A})$ puts the indicated categories in duality we call *weakly quasi-Frobenius*. For the case $\mathcal{A} = \{A\}$ we refer the reader to [**11**].

**Theorem 8.14.** *For a ring $\mathcal{A} = \{P_i\}_{i\in I}$ the following statements are equivalent:*
(1) *$\mathcal{A}$ is weakly quasi-Frobenius;*
(2) *$\mathcal{A}$ is (left and right) $FP$-injective and (left and right) coherent;*
(3) *the classes of flat right $\mathcal{A}$-modules and right $FP$-injective $\mathcal{A}$-modules coincide;*
(4) *$\mathcal{A}$ is left $FP$-injective and left coherent, and any flat right $\mathcal{A}$-module is $FP$-injective;*
(5) *$\mathcal{A}$ is right $FP$-injective and right coherent, and any $FP$-injective right $\mathcal{A}$-module is flat.*

*Proof.* (1) $\Rightarrow$ (2). By assumption, given a finitely presented left $\mathcal{A}$-module $M$, the right $\mathcal{A}$-module $M^* = \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})$ is finitely presented. From Proposition 7.6 it follows that $\mathcal{A}$ is right coherent. By symmetry, $\mathcal{A}$ is left coherent. Since the functor $\mathrm{Hom}_{\mathcal{A}}(-, P)$ with $P \in \mathcal{A}$ is exact both on $\mathrm{mod}\,\mathcal{A}^{\mathrm{op}}$ and on $\mathrm{mod}\,\mathcal{A}$, it follows that $P$ is an $FP$-injective left and right $\mathcal{A}$-module. So $\mathcal{A}$ is an (two-sided) $FP$-injective ring.

(2) $\Rightarrow$ (1). Since $\mathcal{A}$ is a (two-sided) coherent ring, from Proposition 7.5 it follows that $\mathcal{P}^{\mathcal{A}} = \vec{\mathcal{S}}^{\mathcal{A}}$, and, hence, there is an equivalence of categories $\mathrm{mod}\,\mathcal{A}$ and $\mathrm{coh}\,\mathcal{C}_{\mathcal{A}}/\vec{\mathcal{S}}^{\mathcal{A}}$. In a similar way, there is an equivalence of categories $\mathrm{mod}\,\mathcal{A}^{\mathrm{op}}$ and $\mathrm{coh}\,{}_{\mathcal{A}}\mathcal{C}/{}^{\mathcal{A}}\vec{\mathcal{S}}$. If we apply Theorem 8.13, we get the following relations:

$$\mathcal{S}^{\mathcal{A}} = D({}_{\mathcal{A}}\mathcal{S}) = D({}^{\mathcal{A}}\mathcal{S})$$
$$\mathcal{S}_{\mathcal{A}} = D({}^{\mathcal{A}}\mathcal{S}) = D({}_{\mathcal{A}}\mathcal{S}).$$

Now, our implication follows from Theorem 6.8.

(2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4). It suffices to apply Theorems 8.12 and 8.13.

(3) $\Rightarrow$ (5). It suffices to show that $\mathcal{A}$ is right coherent. For this, consider a direct system of $FP$-injective right $\mathcal{A}$-modules $\{M_i\}_{i\in I}$. Since each $M_i$ is flat, by assumption, the module $\varinjlim M_i$ is flat, and, hence, $FP$-injective. Therefore $\mathcal{A}$ is right coherent by Theorem 8.12.

(4) $\Rightarrow$ (3). By Theorem 8.13, any $FP$-injective right $\mathcal{A}$-module is $fp$-flat; hence it is flat by Theorem 8.12.

(5) $\Rightarrow$ (2). Since the ring $\mathcal{A}$ is right $FP$-injective, then the module $\prod_J P_j$ is $FP$-injective, where $P_j \in \mathcal{A}$ and $J$ is some set of indices; the latter module is flat by assumption. By Proposition 8.11 $\mathcal{A}$ is left coherent. By Theorem 8.12, any $fp$-injective right $R$-module is $FP$-injective, and, hence, flat. From Theorem 8.13 it follows that $\mathcal{A}$ is left $FP$-injective. □

## The conclusion

Speaking about the further study of the classes of $FP$-injective and weakly quasi-Frobenius rings, we have not concerned here one important concept: $FP$-cogenerator (see [**12**]). Actually the condition of right $FP$-injectivity of a ring turns out equivalent to that it is a left $FP$-cogenerator [**12**].

In fact, carrying properties of the category of modules $\mathrm{Mod}\,\mathcal{A}$ over similar to them properties of the category of finitely presented modules $\mathrm{mod}\,\mathcal{A}$, many concepts arise with the pairs $FP$-property/$fp$-property (for example $FP$-injectivity/$fp$-injectivity, or $FP$-cogenerator/$fp$-cogenerator [**12**]). Here in full force properties of the category ${}_{\mathcal{A}}\mathcal{C}$ work, and torsion functors turn out the extremely convenient tool in this territory.

Going in this direction, it is difficult to keep track of all consequences and we invite the reader to the further study of the category mod $\mathcal{A}$.

## References


[1] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J. **9** (1957), 119-221.

[2] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323-448.

[3] B. Stenström, *Rings of quotients*, Grundlehren Math. Wiss., vol. 217, Springer-Verlag, New York–Heidelberg, 1975.

[4] J. Lambek, *Lectures on rings and modules*, Braisdal Publishing Co., Waltham, 1966.

[5] N. Popescu, *Abelian categories with applications to rings and modules*, Academic Press, London–New York, 1973.

[6] M. Ziegler, *Model theory of modules*, Ann. Pure Appl. Logic **26** (1984), 149-213.

[7] M. Prest, *Model theory and modules*, Cambridge University Press, Cambridge, 1988.

[8] C. U. Jensen, M. Lenzing, *Model theoretic algebra*, Logic and its Applications 2, Gordon and Breach, New York, 1989.

[9] M. Auslander, *Coherent functors*, in Proc. Conf. on Categorical Algebra (La Jolla, CA, 1965), Springer, New York, 1966, pp. 189-231.

[10] G. A. Garkusha, A. I. Generalov, *Grothendieck categories as quotient categories of* $(R-\text{mod},\text{Ab})$, Fund. i Prikl. Mat., to appear. (Russian)

[11] G. A. Garkusha, A. I. Generalov, *Duality for categories of finitely presented modules*, Algebra i Analiz (6) **11** (1999), 139-152. (Russian) English transl. in St. Petersburg Math. J. (6) **11** (2000).

[12] G. A. Garkusha, *FP-injective and weakly quasi-Frobenius rings*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **265** (1999), 110-129. (Russian) English transl. in J. Math. Sci. (New York).

[13] H. Krause, *The spectrum of a module category*, Habilitationsschrift, Universität Bielefeld, 1998.

[14] I. Herzog, *The Ziegler spectrum of a locally coherent Grothendieck category*, Proc. London Math. Soc. (3) **74** (1997), 503-558.

[15] H. Krause, *The spectrum of a locally coherent category*, J. Pure Appl. Algebra **114** (1997), 259-271.

[16] P. Freyd, *Abelian categories*, Harper and Row, New York, 1964.

[17] C. Faith, *Algebra: rings, modules and categories. I*, Grundlehren Math. Wiss., vol. 190, Springer-Verlag, New York–Heidelberg, 1973.

[18] I. Bucur, A. Deleanu, *Introduction to the theory of categories and functors*, Wiley, London, 1968.

[19] J.-E. Roos, *Locally noetherian categories*, Lecture Notes in Math., vol. 92, 1969, pp. 197-277.

[20] L. Gruson, C. U. Jensen, *Dimensions cohomologiques reliées aux foncteurs* $\varprojlim^{(i)}$, Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980), Lecture Notes in Math., vol. 867, Springer, Berlin, 1981, pp. 234-294.

[21] M. Prest, *Elementary torsion theories and locally finitely presented Abelian categories*, J. Pure Appl. Algebra **18** (1980), 205-212.

[22] N. Popescu, P. Gabriel, *Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes*, C. R. Acad. Sc. Paris **258** (1964), 4188-4190.

[23] A. Jeremías López, M. P. López López, E. Villanueva Nóvoa, *Gabriel filters in Grothendieck categories*, Publ. Math. **36** (1992), 673-683.

[24] A. Jeremías López, M. P. López López, E. Villanueva Nóvoa, *Localization in symmetric closed Grothendieck categories*, Bull. Soc. Math. Belgique Ser. A **45** (1993), 197-221.

[25] I. Kaplansky, *Infinite Abelian groups*, Ann Arbor, University of Michigan Press, 1969.

[26] S. Breitsprecher, *Lokal endlich präsentierbare Grothendieck-Kategorien*, Mitt. Math. Sem. Giessen **85** (1970), 1-25.

[27] H. Krause, *Functors on locally finitely presented additive categories*, Colloq. Math. **75** (1998), 105-132.

[28] M. Prest, Ph. Rothmaler, M. Ziegler, *Absolutely pure and flat modules and "indiscrete" rings*, J. Algebra **174** (1995), 349-372.

[29] M. Prest, *Epimorphisms of rings, interpretations of modules and strictly wild algebras*, Comm. Algebra **24** (1996), 517-531.

[30] K. Burke, *Some Model-Theoretic Properties of Functor Categories for Modules*, Doctoral Thesis, University of Manchester, 1994.

[31] M. Prest, *The Zariski spectrum of the category of finitely presented modules*, Preprint, University of Manchester, 1998.

[32] M. Auslander, *Isolated singularities and almost split sequences*, Representation Theory. II (Ottawa, Ont., 1984), Lecture Notes in Math., vol. 1178, Springer, Berlin–New York, 1986, pp. 194-242.

[33] R. R. Colby, *Rings which have flat injective modules*, J. Algebra **35** (1975), 239-252.

[34] H. Bass, *Finistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466-488.

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