A note on the cardinality of certain classes of unlabeled multipartite tournaments

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Abstract

A multipartite tournament is an orientation of a complete multipartite graph. Simple derivations are obtained of the numbers of unlabeled acyclic and unicyclic multipartite tournaments, and unlabeled bipartite tournaments with exactly $k$ cycles, which are pairwise vertex-disjoint.

Keywords: Multipartite tournaments; Bipartite tournaments; Enumeration


In this note, we enumerate unlabeled acyclic and unicyclic multipartite tournaments. We partly generalize these results by counting unlabeled strictly $k$-cyclic bipartite tournaments, that is, bipartite tournaments with exactly $k$ cycles, which are pairwise vertex-disjoint. Our proofs are short and simple and based on certain bijections from classes of multipartite tournaments into sets of integral sequences or other classes of multipartite tournaments; unlike the proofs in [3] for the number of unlabeled acyclic bipartite tournaments, no calculations are required in the proofs of our results.

A $p$-partite (multipartite) tournament [2] $T$ is an orientation of a complete $p$-partite graph $G$. The colour classes of $T$ are the colour classes of $G$, i.e., the maximal independent sets of vertices in $G$. An unlabeled $p$-partite tournament is an ordered $(p + 1)$-tuple $(T, V_1, ..., V_p)$, where $T$ is a $p$-partite tournament and $(V_1, ..., V_p)$ an ordered $p$-tuple of its colour classes. (When $(V_1, ..., V_p)$ can be determined from the context we shall write $T$ rather than $(T, V_1, ..., V_p)$.) If the colour classes of $T$ are of order $n_1, ..., n_p$ respectively ($n_i > 0$, $i = 1, ..., p$), then $T$ is called an $(n_1, ..., n_p)$-tournament. We say that unlabeled $(n_1, ..., n_p)$-tournaments $(T, V_1, ..., V_p)$ and $(M, U_1, ..., U_p)$ are equivalent if there exists an isomorphism $f$ from $T$ to $M$ such that $f(V_i) = U_i$ for every $i = 1, ..., p$. Intuitively, this
means that vertices in the same colour class are interchangeable, but the colour classes themselves are not.

In what follows, \( n = n_1 + \ldots + n_p \). Let \( t_k(n_1, \ldots, n_p) \) denote the number of inequivalent unlabeled strictly \( k \)-cyclic \((n_1, \ldots, n_p)\)-tournaments \((k \geq 0)\). A sequence \( s_1, s_2, \ldots, s_n \) is called an \((n_1, \ldots, n_p)\)-sequence if it contains \( n_j \) elements equal to \( j \), for every \( j = 1, \ldots, p \), and no other elements. Clearly, the number of \((n_1, \ldots, n_p)\)-sequences equals the multinomial coefficient \((n \choose n_1, \ldots, n_p)\). The following result provides a graph-theoretical interpretation of multinomial coefficients.

**Theorem 1.** The number \( t_0(n_1, \ldots, n_p) \) of (inequivalent) unlabeled acyclic \((n_1, \ldots, n_p)\)-tournaments equals the number of \((n_1, \ldots, n_p)\)-sequences. Thus \( t_0(n_1, \ldots, n_p) = (n \choose n_1, \ldots, n_p) \).

**Proof:** Let \( T \) be an acyclic \((n_1, \ldots, n_p)\)-tournament with colour classes \( V_1, \ldots, V_p \). We can assign to \( T \) an \((n_1, \ldots, n_p)\)-sequence \( s(T) = s_1, s_2, \ldots, s_n \) as follows. The vertices of zero in-degree in \( T \) are all in the same colour class: let them be \( x_1, \ldots, x_{r_1} \), all in \( V_{j_1} \), and set \( s_1 = \ldots = s_{r_1} = j_1 \). Let the vertices of zero in-degree in \( T - \{x_1, \ldots, x_{r_1}\} \) be \( x_{r_1+1}, \ldots, x_{r_2} \), all in \( V_{j_2} \), and set \( s_{r_1+1} = \ldots = s_{r_2} = j_2 \). Continue in this way until all elements of \( s(T) = s_1, \ldots, s_n \) are defined.

Conversely, given an \((n_1, \ldots, n_p)\)-sequence \( s = s_1, s_2, \ldots, s_n \), we construct an acyclic \((n_1, \ldots, n_p)\)-tournament \( T(s) \) as follows. For every \( i = 1, 2, \ldots, n \), the \( i \)-th vertex \( x_i \) of \( T(s) \) belongs to \( V_{s_i} \), and it dominates (is dominated by) all vertices \( x_k \) not in \( V_{s_i} \) such that \( k > i \) \((i > k)\).

It is easy to see that these two constructions are inverses of each other, that is, \( T(s(T)) = T \) for each \( T \) and \( s(T(s)) = s \) for each \( s \). \( \square \)

It is easy to see that the formula in Theorem 1 is also valid when some of the cardinalities \( n_i \) are zero. This remark will be used in applications of Theorem 1.

Let \( T \) be a strictly \( k \)-cyclic multipartite tournament and let \( C_1, \ldots, C_k \) be its cycles. Contracting every cycle \( C_i \) into a single vertex \( w_i \) gives an acyclic digraph \( T' \). Let \( T^*(C_1, \ldots, C_k) \) denote the digraph obtained from \( T' \) by deleting all arcs between pairs of vertices in \( \{w_1, \ldots, w_k\} \).

Now we obtain a simple formula for \( t_k(n_1, n_2) \), \( k \geq 0 \). The problem to obtain a compact formula for \( t_k(n_1, \ldots, n_p) \) \((p \geq 3)\) for every \( k \geq 0 \) seems to be much more difficult. We prove a relatively compact formula for \( t_k(n_1, \ldots, n_p) \) in Theorem 3.

**Theorem 2.** For every integer \( k \) such that \( 0 \leq k \leq \frac{1}{2} \min\{n_1, n_2\} \), \( t_k(n_1, n_2) = \)
\[
\binom{n-3k}{(n_1-2k,n_2-2k,k)}.
\]

**Proof:** For \( k = 0 \), the formula follows from Theorem 1. Thus we may assume that \( k \geq 1 \). Let \( T \) be a strictly \( k \)-cyclic \((n_1,n_2)\)-tournament, and let \( C_1,\ldots,C_k \) be the cycles of \( T \). Every cycle \( C_i \) is of length four, since otherwise the chord joining two vertices distance 3 apart around \( C_i \) would complete another cycle. Thus, the cycles are ‘interchangeable’. Therefore, \( t_k(n_1,n_2) \) equals \( t_0(n_1-2k,n_2-2k,k) \), the number of unlabeled acyclic \((n_1-2k,n_2-2k,k)\)-tournaments of the form \( T^*(C_1,\ldots,C_k) \). The result now follows by Theorem 1. \( \square \)

Let \( S(p,k) \) denote the set of all unordered \( k \)-subsets of \( \{1,\ldots,p\} \). In what follows, we assume that \( \binom{m}{m_1,\ldots,m_p} = 0 \) if one of the integers \( m_i \) is negative. Note that
\[
\binom{m}{m_1,\ldots,m_p,1} = m\binom{m-1}{m_1,\ldots,m_p}
\]
if \( m_1 + \ldots + m_p = m-1 \).

**Theorem 3.** The number of unlabeled unicyclic \((n_1,\ldots,n_p)\)-tournaments \((p \geq 3)\) is
\[
t_1(n_1,\ldots,n_p) = (n-3) \sum_{\pi \in S(p,2)} \binom{n-4}{n_2^*\pi,n_3^*\pi} + 2(n-2) \sum_{\pi \in S(p,3)} \binom{n-3}{n_1^*\pi,n_2^*\pi,n_3^*\pi},
\]
where \( n_j^*\pi = n_j - c \) if \( j \in \pi \), and \( n_j^*\pi = n_j \) otherwise.

**Proof:** Let \( T \) be a unicyclic \((n_1,\ldots,n_p)\)-tournament with colour classes \( V_1,\ldots,V_p \) and let \( C \) be the unique cycle in \( T \). Two vertices of \( C \) that are not consecutive in \( C \) must be in the same colour class, since otherwise the chord between them would complete another cycle. Thus \( C \) is of length three, or of length four with vertices from two alternating colour classes.

Let us first assume that \( C \) has four vertices from \( V_i \) and \( V_j \), \( i < j \), and \( \pi = \{i,j\} \). Then the number of unlabeled unicyclic \((n_1,\ldots,n_p)\)-tournaments containing \( C \) equals the number of unlabeled acyclic \((n_1,\ldots,n_{i-1},n_i-2,n_{i+1},\ldots,n_{j-1},n_j-2,n_{j+1},\ldots,n_p,1)\)-tournaments of the form \( T^*(C) \), which is \( t_0(n_1^2(\pi),\ldots,n_p^2(\pi),1) \). By Theorem 1 and (1), this gives the first term in the formula for \( t_1(n_1,\ldots,n_p) \).

Now let \( C \) be a cycle with three vertices from classes \( V_i, V_j \) and \( V_k \), respectively, and in this order. Let also \( \pi = \{i,j,k\} \). Then the number of unlabeled unicyclic \((n_1,\ldots,n_p)\)-tournaments containing \( C \) equals \( t_0(n_1^2(\pi),\ldots,n_p^2(\pi),1) \). This fact and the possibility to have two unlabeled triangles \( C \) with vertices from classes \( V_i, V_j \) and \( V_k \) (in this order and in the opposite one) gives the second term in the formula for \( t_1(n_1,\ldots,n_p) \). \( \square \)
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References


