On two-transitive parabolic ovals

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Abstract

We examine the state of knowledge on the following problem. Let $\pi$ be a finite projective plane of odd order $n$ with an oval $\Omega$ and let $G$ be a collineation group of $\pi$ fixing $\Omega$. Assume $G$ fixes a point $P$ on $\Omega$ and acts 2-transitively on $\Omega - \{P\}$. The usual basic question is: what can be said about $\pi$, $\Omega$ and $G$?

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1. Introduction

Let $\pi$ be a finite projective plane of odd order $n$. An oval $\Omega$ of $\pi$ is said to be a 2-transitive parabolic oval if there exists a collineation group $G$ of $\pi$ fixing $\Omega$ such that $G$ fixes a point $P$ on $\Omega$ and acts 2-transitively on $\Omega - \{P\}$. This situation has already been considered in some papers [6,3,4]. The major difficulty towards a final classification occurs when $G$ can

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be represented as a subgroup of the affine semilinear group $A\Gamma L(1, q)$. In this paper we gather some additional geometric information on this problem.

2. Point and line orbits

We denote by $\ell_\infty$ the tangent line to $\Omega$ at $P$ and observe first of all that the action of $G$ on $\Omega - \{P\}$ is equivalent to the action of $G$ on the tangent lines other than $\ell_\infty$ and consequently also to the action of $G$ on $\ell_\infty - \{P\}$.

Note that all points on $\ell_\infty$ other than $P$ are external. There are $\binom{n}{2}$ external points off $\ell_\infty$ forming a single $G$-orbit. Clearly the $n$ secants through $P$ form a single line orbit. The $\binom{n}{2}$ secants not through $P$ also form a single line orbit. There remain $\binom{n}{2}$ internal points and $\binom{n}{2}$ external lines. How are these partitioned into $G$-orbits? Certainly $G$ cannot fix internal points nor external lines. Observe that Enea and Korchmáros in [4] prove that one of the possibilities for a collineation group fixing $\Omega$ and acting transitively on internal points is that $G$ fixes a point $P$ on $\Omega$ and acts on $\Omega - \{P\}$ as an affine-type primitive permutation group, which is 2-transitive as soon as each involution in $G$ is a homology. With our assumptions is it always the case that internal points form a single $G$-orbit?

3. The involutions

Our group $G$ acts faithfully on $\Omega - \{P\}$ and so may be regarded as a 2-transitive permutation group of degree $n$, in particular $G$ contains involutions. We now prove that $G$ contains involutory homologies. The following property of transitive permutation groups of even degree can be found in [9, 15.4].

**Proposition 1.** Let $H$ be a finite permutation group acting transitively on the set $\Delta$ with $|\Delta|$ even. If $U$ is a Sylow 2-subgroup of $H$ and $u$ is in the center of $U$ then the number of elements left fixed by $u$ is different from $\sqrt{|\Delta|}$ and $\sqrt{|\Delta|} + 1 - 1$.

**Proposition 2.** The group $G$ contains some involutory homology. More precisely, each involution in the center of a Sylow 2-subgroup of $G$ must be an involutory homology.

**Proof.** We observe first of all that the Sylow 2-subgroups of $G$ are to be found in the one-point stabilizers of $G$. In fact we have $|G| = n \cdot |G_O|$, where $O$ is any point in $\Omega - \{P\}$. Now, $G_O$ is transitive and faithful on $\Omega - \{O, P\}$ which has even size. Assume all involutions in $G$ are Baer involutions. Let $S$ be a Sylow 2-subgroup of $G$, with $S \leq G_O$. The subgroup $S$ is cyclic, [2, Proposition 2.5]. A Baer involution $\sigma$ in $S$ fixes $\sqrt{n} - 1$ points on $\Omega - \{O, P\}$. Since $\Omega - \{O, P\}$ is even and the involution $\sigma$ lies in the center of $S$, it cannot fix $\sqrt{|\Omega - \{O, P\}|} + 1 - 1 = \sqrt{n} - 1$ points on $\Omega - \{O, P\}$ by Proposition 1. We get a contradiction. □

**Proposition 3.** The group $G$ has exactly $n$ involutory homologies. Each secant through $P$ is an axis and each point on $\ell_\infty - \{P\}$ is a center. All involutory homologies are conjugate in $G$. 


Proof. Let $h$ be an involutory homology in $G$. The tangent line $\ell_\infty$ is fixed by $h$ and so the axis of $h$ is a secant $PO$ for some point $O \in \Omega$ and the center of $h$ is an external point $C \in \ell_\infty$. The assertion follows from the transitivity of $G$ on $\Omega - \{P\}$. □

Proposition 4. If $c$ is a secant through $P$, then the stabilizer $G_c$ acts transitively on the set of external points of $c$.

Proof. Let $c=OP$ be a secant of $\Omega$ and let $X,Y$ be external points on $c$. As $G$ acts transitively on external points, there exists $g \in G$ such that $X^g = Y$, then $c^g = c$ and the assertion follows. □

In the terminology of Hering, [7], the group $G$ is not irreducible on $\pi$ as it fixes the point $P$. Nevertheless, whenever $n > 3$, $G$ fixes neither triangles nor subplanes. In fact, if $G$ fixes a triangle $I_1$, $I_2$, $I_3$, then two possibilities occur: either the points are on $\Omega$, against $n > 3$, or the three points are off $\Omega$. In this latter case the lines $I_1I_2$, $I_1I_3$, $I_2I_3$ determine an orbit on $\ell_\infty$ and we have a contradiction. Suppose now $G$ fixes a subplane of order $m$. Then $m$ lines of this subplane containing $P$ determine $m$ fixed points on $\Omega$, a contradiction.

If a Baer involution exists in $G$ at all, then it must fix $\sqrt{n}$ points of $\Omega$ other than $P$.

Proposition 5. Let $O$ be a point of $\Omega - \{P\}$. Let $C \in \ell_\infty$ be the center of the unique involutory homology with axis $PO$. The relation $G_O = G_{OC}$ holds.

Proof. Each collineation in $G_O$ fixes the tangent line to $\Omega$ at $O$ and this line intersects $\ell_\infty$ at $C$. □

Proposition 5 ensures that if $\sigma$ is a Baer involution in $G$, and if $O_1 \ldots O_\sqrt{n}$ are the fixed points of $\sigma$ on $\Omega - \{P\}$, then the fixed points of $\sigma$ on $\ell_\infty$ are exactly the centers of the involutory homologies with axes $PO_1, \ldots, PO_\sqrt{n}$.

Proposition 6. Let $H$ be a collineation group of a finite projective plane $\pi$ fixing an oval $\Omega$. Assume $H$ fixes at least three points on $\Omega$. Then $\text{Fix}(H)$ is a subplane $\pi_0$ of $\pi$ and the fixed points of $H$ on $\Omega$ form an oval $\Omega_0$ in $\pi_0$. If the order of $\pi$ is odd then so is the order of $\pi_0$.

Proof. This is Lemma 2 in [1]. □

Proposition 7. Let $O$ be a point of $\Omega - \{P\}$. Let $\tau$ be the unique involutory homology fixing $O$, and suppose there exists a Baer involution $\sigma$ that fixes $O$ itself. The group $\langle \sigma, \tau \rangle$ is elementary abelian of order 4.

Proof. Let $\Omega = \Omega \cap \text{Fix}(\sigma)$, $\Omega$ is an oval in $\text{Fix}(\sigma)$. Then the line $PO$ is a line of the Baer subplane $\text{Fix}(\sigma)$ which is secant to $\Omega$. Let $P \in PO$ be an internal point to $\Omega$. If $P$ is viewed as a point in $\pi$, then it is an external point to $\Omega$, [2, Proposition 2.2]. Let $A$ and $B$ be the points of contact of the tangents to $\Omega$ through $P$. Both $A$ and $B$ lie in $\Omega - \Omega$. Therefore we have $A^\sigma = B$ and $B^\sigma = A$. We also have $A^\tau = B$ and $B^\tau = A$ so that $\sigma \tau$ fixes both $A$ and $B$. There are exactly $(\sqrt{n} - 1)/2$ points on $PO$ which are internal to $\Omega$, this determines

\[ a \triangleleft G \triangleleft \Omega \]
at least $\sqrt{n} - 1$ points on $\Omega - \overline{\Omega}$ which are fixed by $\sigma\tau$. Let $H = \langle \sigma\tau \rangle$, by the previous Proposition $\text{Fix}(H)$ is a Baer subplane. Furthermore $|H| = 2$ [6, Lemma 7] and $\sigma = \tau\sigma$ is a Baer involution, the assertion now follows. \hfill \Box

4. Primitivity vs. 2-transitivity

We want to remark at this stage that if we only assume $G$ to be primitive on $\Omega - \{P\}$ then we cannot guarantee the existence of involutory homologies in $G$. We shall now show that counterexamples can be constructed in desarguesian planes of odd square order. For this purpose we need two facts.

Proposition 8. Set $q = p^{2h}$ where $p$ is an odd prime. The group $A\Gamma L(1, q)$ has exactly two conjugacy classes of involutions which are represented by the transformations $x \mapsto -x$ and $x \mapsto x^{\sqrt{q}}$ respectively. The involutions of the first class have no fixed point on $GF(q)$, while those of the second class have $\sqrt{q}$ fixed points.

Proposition 9. Let $q = p^{2h}$ where $p$ is an odd prime. Let $d$ be a divisor of $q - 1$ such that $d$ does not divide $p^k - 1$ for $k < 2h$ (the existence of such a divisor is assured by Zsigmondy’s Lemma [9]). Let $A$ be the subgroup of order $d$ of $GF(q)^*$ and let $G$ be the subgroup of $A\Gamma L(1, q)$ consisting of the transformations $x \mapsto ax^\sigma + \beta$ where $a \in A$, $\beta \in GF(q)$ and $\sigma \in \{1, \sqrt{q}\}$. Then the group $G$ acts primitively on $GF(q)$.

Proof. It is sufficient to prove that the subgroup $L = \{x \mapsto ax + \beta : a \in A, \beta \in GF(q)\}$ of $G$ acts primitively on $GF(q)$. We must show that the stabilizer $L_0 = \{x \mapsto ax : a \in A\}$ of zero is a maximal subgroup of $L$. We suppose that there exists a proper subgroup $L'$ of $L$ which contains $L_0$. Then $L'$ consists of all transformations $x \mapsto ax + \gamma$ where $\gamma$ runs over a proper additive subgroup of $GF(q)$. Thus $|L'| = |A|p^m = dp^m$ with $m < 2h$. But such a subgroup exists only when $d|(p^m - 1)$ [8], which is a contradiction. \hfill \Box

Let $\pi$ be a desarguesian plane of odd square order $q = p^{2h} > 3$ and let $\Omega$ be a conic in $\pi$. The setwise stabilizer $H$ of $\Omega$ in the full collineation group of $\pi$ acts on $\Omega$ as $P\Gamma L(2, q)$ in its natural permutation representation. Hence the stabilizer $H_P$ of one point $P \in \Omega$ is isomorphic to $A\Gamma L(1, q)$. If we take $G$ to be the subgroup of $H_P$ defined above, then we obtain from Proposition 9 that $G$ is primitive on $\Omega - \{P\}$. When $d$ can be chosen to be odd, a possibility which does occur, then no involution of the form $x \mapsto -x + b$ occurs in $G$, consequently each involution in $G$ fixes $P$ and $\sqrt{q}$ further points in $\Omega - \{P\}$ by Proposition 8 and so is necessarily a Baer involution.

It follows from the result in [2] that if a collineation group fixes an oval in a finite projective plane of odd order acting on it as $PGL(2, q)$ in its natural action, then the plane is desarguesian. The same conclusion cannot be drawn if one only assumes that the one–point–stabilizer acts on the remaining points of the oval as $AGL(1, q)$ (i.e. the one–point–stabilizer of $PGL(2, q)$) in its natural action. Namely, it has been proved in [3] that commutative semifield planes are the unique examples for translation planes of odd order admitting a 2-transitive parabolic oval. Examples are obtained when $s = m$ can be
chosen in such a way that $-1$ is not a $(p^t - 1)$th power in $GF(p^m)$. The affine points of the oval in the semifield plane are the pairs $(z, 2z^{p^t} + 1)$ as $z$ varies in $GF(p^m)$. The mappings:

$$
\sigma_{a,b} : \begin{cases} 
    x' = ax + b, \\
    y' = a^{p^t}x + 2a^{p^t}b^t + 2ab^{p^t}x + 2b^{p^t} + 1
\end{cases}
$$

for $a, b \in GF(p^m), a \neq 0$ yield collineations of the semifield plane leaving the oval invariant. It follows from $\sigma_{a,b} : (z, 2z^{p^t} + 1) \mapsto (az + b, 2(az + b)^{p^t} + 1)$ that they form a group acting on the affine points of the oval as $AGL(1, p^m)$ acts on $GF(p^m)$. That holds true in particular when the semifield plane is non-desarguesian.

If $\pi$ is a finite projective plane of odd order $n$ with an oval which is fixed by a collineation group $G$ acting 2-transitively on $\Omega - \{P\}$ for some given point on $\Omega$, then the possibilities for $G$ are listed in Theorem 9 in [6]. The conics in desarguesian planes and the ovals in commutative semifield planes which have just been described, both fall under case (1) of that result; namely, $G$ is a subgroup of $AFL(1, q)$ in these cases. Clearly if $\pi$ is desarguesian then any 2-transitive subgroup of $AFL(1, q)$ containing all transformations $x \mapsto -x + b, b \in GF(q)$ will do the job. The determination of the non-desarguesian projective planes with a 2-transitive parabolic oval is an open problem, and so is the problem of determining for these planes the 2-transitive subgroups of $AFL(1, q)$ which may give rise to the required situation.

5. Foulser’s description

A general description of the 2-transitive subgroups of $AFL(1, q)$ goes back to Foulser [5, Proposition 15.3]. Following those guidelines we give here some extra geometric information on the possible candidate subgroups in our special situation.

Let $p$ be an odd prime, $\omega$ a primitive root of $GF(p^t)$ and $\zeta$ be the automorphism of $GF(p^t)$ defined by $x^\omega = x^p$. The mappings $x \mapsto \omega \cdot x, x \mapsto x^\omega$, and $x \mapsto x + a$ (for $a \in GF(p^t)$) induce collineations of the affine line represented by $GF(p^t)$, which will be denoted by $\omega, \zeta$, and $\tau_a$, respectively. Let $T = \{\tau_a | a \in GF(p^t)\}$. If $G$ is a subgroup of $AFL(1, p^t)$ then it is the split extension of $T$ by $G_0$ [5, Lemma 15.2] and $G_0 = (\omega^d, \omega^e \cdot \zeta^s)$, for integers $d, e$ and $s$. By [5, Lemma 4.1], $d, e$ and $s$ can be chosen uniquely subject to the conditions:

(i) $d > 0$ and $p^t \equiv 1 \pmod{d}$;
(ii) $s > 0$ and $t \equiv 0 \pmod{s}$;
(iii) $0 \leq e < d$, and $(e(p^t - 1)/(p^s - 1)) \equiv 0 \pmod{d}$.

Furthermore, see [5, Proposition 15.3], the 2-transitivity of $G$ is equivalent to the following:

(a) the primes of $d$ divide $p^t - 1$;
(b) if $p^t \equiv 3 \pmod{4}$, then $d \equiv 0 \pmod{4}$;
(c) $t \equiv 0 \pmod{sd}$;
(d) $(e, d) = 1$. 


As an easy calculation shows, condition (iii) implies that \( \langle \omega^d \rangle \) is the linear part of \( G_0 \). Recalling that \( G \) contains involutory homologies, (Proposition 2), we obtain the following:

**Proposition 10.** The integer \((p^t - 1)/d\) is even.

The existence of Bear involutions in \( G \) is determined by the following:

**Proposition 11.** The group \( G \) contains Baer involutions if and only if \( t/sd \equiv 0 \mod 2 \). In this case \( t \) is even and there are exactly \( \sqrt{p^t} + 1 \) Baer involutions in \( G \).

**Proof.** Let \( O, R \in \Omega - \{P\} \). There exists a Baer involution in \( G \) if and only if a Baer involution lies in \( G_{OR} \). This occurs whenever the order of the stabilizer \( G_{OR} \) is even. Identify the points on \( \Omega - \{P\} \) with the elements of \( GF(q) \) so that \( O \) and \( R \) coincide with 0, 1 \( \in GF(q) \). By [5, Corollary 15.4], \( G_{01} = \langle \omega^d \rangle \) and condition (c) implies \( |G_{01}| = t/sd \). Now the first part of the assertion follows. For each \( R \in \Omega - \{P\} \) the group \( G_{OR} \) is cyclic, so that it contains a unique Baer involution which fixes \( \sqrt{p^t} \) points on \( \Omega - \{P\} \). Then two distinct Baer involutions in \( G_O \) have exactly two common fixed points on \( \Omega \), namely \( P, O, \) and the second part of the assertion follows. \( \square \)

Suppose now the group \( G \) is minimal, that is no proper subgroup of \( G \) is 2-transitive on \( \Omega - \{P\} \). Two possibilities occur according to [5, Proposition 15.5]: either \( G \) is sharply 2-transitive on \( \Omega - \{P\} \) or the following conditions are satisfied:

- \( d \) is even;
- \( p^x \equiv 3 \pmod{4} \);
- \( t/sd = 2^x \) for some \( x \geq 1 \).

In this latter case we obtain the following:

**Proposition 12.** If \( G \) is minimal and is not sharply 2-transitive on \( \Omega - \{P\} \), then

1. \( t \equiv 0 \mod 4 \);
2. a Sylow 2-subgroup of \( G \) has order at least \( 2^{x+3} \) and contains an elementary abelian subgroup of order 4;
3. if \( 2^u \) is the greatest power of 2 dividing \( t \), then \( x \leq u \) and \( \text{Fix}(G_{OR}) \) is a subplane of \( \pi \) of order \( p^{t/2^u} \).

**Proof.** The relation \( t \equiv 0 \mod 4 \) follows from \( t/sd = 2^x \) and \( d \) even. This relation implies also \( p^t - 1 \equiv 0 \pmod{8} \). Observe that \( |G| = p^t(p^t - 1)|G_{OR}| = p^t(p^t - 1)2^4 \), thus a Sylow 2-subgroup of \( G \) has order at least \( 2^{x+3} \). Let \( R \in \Omega - \{O, P\} \) and let \( T < G_0 \) be a Sylow 2-subgroup of \( G \) with \( G_{OR} \leq T \). Let \( \sigma \) be the Bear involution of \( G_{OR} \) and let \( \tau \) be the involutory homology in the center of \( T \). The group \( \langle \sigma, \tau \rangle \) is an elementary abelian subgroup of \( T \) of order 4. Let \( \beta \in G_{OR} \) be a collineation of order \( 2^x \) and let \( \sigma = \beta^{2^x - 1} \) be the Baer involution in \( G_{OR} \). Denote by \( G_i \) the subgroup of \( G_{OR} \) of order \( 2^i, i = 1, \ldots, x \). We see that \( \pi_1 = \text{Fix}(G_1) \) is a Baer subplane of \( \pi \); \( \pi_2 = \text{Fix}(G_2) \) is a Baer subplane of \( \pi_1 \);
we continue in this manner and see that \( \pi_x = \text{Fix}(G_x) = \text{Fix}(G_{OR}) \) is a Baer subplane of \( \pi_{x-1} \). The assertion now follows. \( \square \)

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