Equilogical spaces and filter spaces

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Introduction

The paper is about the comparison between (apparently) different cartesian closed extensions of the category of topological spaces. Since topological spaces do not in general allow formation of function spaces, the problem of determining suitable categories with such a property—and nicely related to that of topological spaces—was studied from many different perspectives: general topology, functional analysis, measure theory, computability.

From a computational perspective, the interest about topological properties of the function spaces arose after the discovery of topological models of the \( \lambda \)-calculus by D.S. Scott, see [22], and the work of Eršov on the partial continuous functionals, see [12]. At the same time, work on cartesian closed extensions of the category of topological spaces in general topology produced interesting quasitoposes of filter spaces where a notion of convergence replaced that of neighbourhood system, see [23] for a complete review.

The semantical analysis of computer behaviour requires that the mechanical black box which is the computer is simply a rule-executing grinder which determines values/outputs from given arguments/inputs. Although the mathematical concept of a Turing machine explains very precisely how the grinder operates, it is extremely useful to be able to have a more conceptual intuition about what a machine does. One of the most useful approximations to this intuition produced so far is the following: a computer executing a program evaluates a partial function. It follows that one of the most important things to know is whether, given a certain collection of inputs, the function evaluates on them all.

Since programs can be stored in the machine, and operated upon by the grinder, a direct consequence is that extensional models of programs are to be cartesian closed categories. By an extensional model, here we mean a mathematical structure where two maps are distinguished by their values on global elements, e.g. a category where the terminal object generates. This does not necessarily mean that a general notion of model of computation

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*Research partially supported by MURST.
has to be a cartesian closed category; it only ensues that the quotient along
the global section functor determines a cartesian closed category.

Natural categories modelling computation consist of topological spaces
obtained from directed-complete partial orders endowed with the topology of
directed sups (often called the Scott topology). These are cartesian closed,
but need not be closed under various useful constructions like subspaces
or quotients which are otherwise natural in the approach using (one of the
various notions of) filter spaces.

In an attempt to reunify the views, Dana Scott proposed the category
Equ of equilogical spaces: An equilogical space is a triple \( E = (|E|, \tau_E, \equiv_E) \)
where \((|E|, \tau_E)\) is a topological T_0-space and \( \equiv_E \subseteq |E| \times |E| \) is an equivalence
relation. A map \( f: E \to E' \) between equilogical spaces is a function
between the quotient sets \( f: |E|/\equiv_E \to |E'|/\equiv_{E'} \) which has a continuous
choice function \( g: |E| \to |E'| \) tracking it on the representatives. In other
words, for some appropriate continuous function \( g \),
\[
    f([x]_{\equiv_E}) = [g(x)]_{\equiv_{E'}}, \quad \text{all } x \in |E|.
\]

In [22], Scott had already noticed that these data provide us with a cartesian
closed category, and reaffirmed his proposal in a recent manuscript.

The construction has been compared to that of the category Mod of the
modest sets in the effective topos, see [6, 18]; in fact, in both situations,
one can explain local cartesian closure based on the same general facts, see
[3, 9]. And the crucial construction involved is that of the exact completion
of a category with finite limits.

In section 1 we review some of the work on the connection between exact
categories in general and the category of equilogical spaces in particular.
In section 2 we recall filter spaces and their relationship with topological
spaces. In section 3 we set up a general environment to compare equilogical
spaces and filter spaces. In section 4 we relate the two approaches and, in
section 5, we use the logic of the locally cartesian closed pretopos Top_ex,
the exact completion of the category of topological spaces, to describe the
other construction. Finally, in section 6 we propose some problems which
we think should be considered.

1 On exact categories and regular categories

There are many references on exact completion and its various descriptions
available in the literature, e.g. [5, 7, 15, 19]: here we shall confine ourselves
to recalling some basic facts. The exact completion \( C_{\text{ex}} \) of a category \( C \) with
finite limits consists of the following data: The objects are equivalence spans
\( C_1 \xrightarrow{r_1} C_0 \xleftarrow{r_2} C \). That requires that the pair of arrows satisfies abstract
conditions which assert
reflexivity: $\exists r \in C_0 \mapsto C_1$ symmetry: $\exists s \in C_1 \mapsto C_0$

and transitivity: $\exists t \in C_1 \mapsto C_0$

A map from $C_1 \xrightarrow{r_1} C_0$ into $C_1' \xrightarrow{r'_1} C_0'$ is an equivalence class $[f]$ of maps of $C$ such that

$$C_0 \xrightarrow{f} C_0' \xleftarrow{r'_1} C_1' \xrightarrow{r_2} C_0.$$

where $f \sim f'$ if

$$C_0 \xrightarrow{f} C_0' \xleftarrow{r_1'} C_1' \xrightarrow{r_2'} C_0'.$$

Composition of maps is defined using that of $C$ on representatives.

1.1 REMARK Often in the literature one finds the name pseudo-equivalence relation for what we call an equivalence span. The prefix “pseudo”, which must be read as modifying the whole concept of an “equivalence relation”, is inserted to remind one of the fact that the pair $(r_1, r_2)$ need not be jointly monic: for every $x,y: Z \rightarrow X_1$, if $r_1 x = r_1 y$ and $r_2 x = r_2 y$, then $x = y$.

It is a theorem that the data above define an exact category $C_{ex}$ (in the sense of Barr [1]), and the free one such, see [7]. That it is exact means that, beyond finite products and equalizers, there are quotients (=coequalizers) of equivalence relations, and these are effective and stable under pullbacks.
1.2 DEFINITION  Consider an equivalence relation $A_1 \xrightarrow{r_1} A_0 \xleftarrow{r_2} A$, i.e. a jointly monic, reflexive, symmetric, and transitive parallel pair. A quotient of the equivalence relation is a coequalizer $f: A_0 \xrightarrow{} A$ of the parallel pair. It is said to be effective if its kernel pair is isomorphic to the given equivalence relation. It is stable if any pullback of it is a quotient.

The notion of an exact category is algebraic in the language of categories and, for this reason, it is no surprise that there be the free exact completion of a category, or of a category with finite limits. What is important is that the description above is in finitary terms, see [5, 15].

The definition of exactness singles out properties of the standard construction of a set of equivalence classes: Given an equivalence relation $\sim$ on a set $S$, the quotient $S/ \sim$ is the smallest solution to the problem of identifying elements $s \sim s'$. The property (that one then checks in proving the factorization theorem for set-functions) that $[x]_\sim = [x']_\sim \iff x \sim x'$
can be restated as saying that the kernel equivalence relation induced by the canonical surjection $S \xrightarrow{} S/ \sim$ coincides with the given equivalence relation $\sim$. This makes a quotient of sets effective. Finally, it is a property of the logic that gives stability: any renaming of the equivalence classes $g: X \xrightarrow{} S/ \sim$ is (in bijection with) the classes for an equivalence relation on $\{(x, s) \mid g(x) = [s]_\sim\}$.

An exact category is always regular, because a regular category is a category with finite limits and stable quotients of kernel pairs.

As Aurelio Carboni explains it, the intuition about the elementary construction of the exact completion is that one wants to add quotients of equivalence relations, but, while doing that, one obtains a proper factorization system of regular epis and monos, and, as a byproduct of this, quotients of equivalence spans. For that reason, the formal addition of objects to $C$ must include general spans rather than just relations.

It is the same intuition which explains why the full subcategory of $C_{ex}$ consisting of those equivalence spans which are kernel pairs in $C$ gives the free regular completion $C_{reg}$ of $C$. And, by the same token, it is no surprise that they both are full subcategories of the category of presheaves on $C$

$$C_{reg} \subset_{full} C_{ex} \subset_{full} [C^{op}, Set],$$

see [19, 20], as the last is the free colimit completion of $C$. This provides a nice embedding into a topos when $C$ is small. When considering the appropriate completions to study $Equ$, we encounter a case where $C$ is not small.

Consider then the category of equilogical spaces: one can turn the original presentation to fit within the framework of the exact completion of
the category \textit{Top} of topological spaces and continuous functions: Given an equilogical space \( E = (|E|, \tau, \equiv) \), one can see the set \( \equiv \) of equivalent pairs topologized with the subspace topology on the product \(|E| \times |E|\), hence moving the equivalence relation into the category \textit{Top}\_0 of \( T_0 \)-spaces. This assignment obviously extends to a full embedding \( \text{Equ} \hookrightarrow (\text{Top}_0)_{\text{ex}} \) as maps are defined as equivalence classes with respect to the same equivalence relation as that for maps in an exact completion.

Moreover, the equivalence relation \( \equiv \hookrightarrow (|E|, \tau) \times (|E|, \tau) \) is clearly obtained as the kernel pair in \textit{Top} of the surjective continuous function \((|E|, \tau) \rightarrow \nabla(|E|/\equiv)\), the topological space on the set \(|E|/\equiv\) with only the trivial open sets. (Next to nothing is gained using the topological quotient \((|E|, \tau) \rightarrow (|E|/\equiv, \tau')\) as the quotient need not be \( T_0 \).) The assignment extends to a full embedding \( \text{Equ} \hookrightarrow \text{Top}_{\text{reg}} \). (Note that the same argument shows that relaxing the notion of equilogical space by dropping the requirement that the space be \( T_0 \) gives just another presentation of \textit{Top}_{\text{reg}}.)

Connecting with the previous general discussion about colimit completions, we have the following situation

\[
\begin{array}{ccc}
\text{Top}_0 & \perp & \text{Top} \\
\downarrow \text{full} & & \downarrow \text{full} \\
\text{Equ} & \perp & \text{Top}_{\text{reg}} \\
\downarrow \text{full} & & \downarrow \text{full} \\
(\text{Top}_0)_{\text{ex}} & \perp & \text{Top}_{\text{ex}} \\
\downarrow \text{full} & & \downarrow \text{full} \\
& & \text{[Top}^{\text{op}}, \text{Set}] \\
\end{array}
\]

It is easy to see that the square of embeddings commutes (up to a natural iso), and that each embedding in the square has a left adjoint. Those in the bottom square preserve products and commute with change of base in the codomain.\(^1\)

Since \textit{Top}_{\text{ex}} and \((\text{Top}_0)_{\text{ex}}\) are locally cartesian closed pretoposes (see [3, 9]), it follows that the other categories in the square are locally cartesian closed quasi-pretoposes as in [8]: a \textit{quasi-pretopos} is a regular category where strong equivalence relations are effective, finite coproducts exist and are stable under pullback, and epimorphisms and regular monomorphisms form a stable (proper) factorization system.

\(^1\)By this, we mean that, for \( B' \rightarrow B \leftarrow A \) with \( B \) and \( B' \) in the codomain of the left adjoint \( L \), the canonical map \( L(A \times_B B') \rightarrow L(A) \times_B B' \) is iso.
2 Extending the category of topological spaces

The category \( \text{Top} \) of topological spaces was extended by means of cartesian closed categories of spaces with a generalized notion of convergence, see [11, 23] for extensive references and [10, 17] for results more specifically related to the present discussion.

Many of these can be compared directly with those seen in the previous section via the abstract approach suggested by Brian Day: here we recall three notions of filter/limit spaces.

Recall a few notions about filters. Let \( S \) be a set; a filter on \( S \) is a collection \( \Phi \subseteq \mathcal{P}(S) \) of subsets of \( S \) which is

- **closed under finite intersections**: \( S \) is in \( \Phi \), and if \( A, B \) are in \( \Phi \), then \( A \cap B \) is in \( \Phi \),

- **closed under extensions**: if \( A \) is in \( \Phi \) and \( A \subseteq B \), then also \( B \) is in \( \Phi \).

A filter is **principal** if it is the filter generated by a subset of \( S \): if \( A \subseteq S \), we write \( A^\subseteq \) for the principal filter generated by \( A \). Hence maximal principal filters are those of the form \( \{s\}^\subseteq \) for \( s \in S \).

Following [17], we say that \( X = (|X|, \searrow) \) is a **filter space** if \( \searrow \) is a relation between filters on \( |X| \) and elements of \( |X| \)—suggested reading for \( \Phi \searrow x \) is “\( \Phi \) converges to \( x \)”—which satisfies

- \((\alpha)\) \( \{x\}^\subseteq \searrow x \)
- \((\beta)\) if \( \Phi \searrow x \) and \( \Phi \subseteq \Psi \), then \( \Psi \searrow x \).

Consider now the following additional conditions on a filter space:

- \((\gamma)\) if \( \Phi \searrow x \), then \( (\Phi \cap \{x\}^\subseteq) \searrow x \),
- \((\gamma')\) if \( \Phi \searrow x \searrow \Psi \), then \( (\Phi \cap \Psi) \searrow x \),
- \((\gamma'')\) if \( \Theta \searrow x \) for all maximal filters \( \Theta \) extending \( \Phi \), then \( \Phi \searrow x \).

A **continuous** map \( f: X \longrightarrow X' \) between two filter spaces is a function between the underlying sets which preserves convergence in the following sense:

- if \( \Phi \searrow x \), then \( f[\Phi] \searrow f(x) \),

where \( f[\Phi] \) is the filter generated by the collection of all the images along \( f \) of sets in \( \Phi \), i.e. the filter

\[ \{A' \mid f(A) \subseteq A' \text{ for some } A \in \Phi \} \]

The name “filter space” had been used in a variety of modifications: we strongly warn the reader to be aware of the different meanings, sometimes
even in a single paper. Other names had been found appropriate for the special notions which can be obtained by adding any of the conditions \((\gamma^{(n)})\), \(n = 0, 1, 2\): yet again their usage is not consistent throughout the literature. Thus we simply avoid using any more specific name.

The common motivation for (any of) these notions of convergence is that they are a natural extension of that of a topology: Given a topological space \(T = (|T|, \tau)\), consider the filter structure defined by stating that \(\Phi \rightarrow_{\gamma} p\) when \(\Phi \supseteq \mathcal{U}_p\), the filter of neighbourhoods of the point \(p\). The definition of continuity on filter structures induced by a topology is just one of the many equivalent ways of stating continuity between topological spaces:

1. \(\mathcal{U}_{f(p)} \subseteq f[\mathcal{U}_p]\) for all \(p\) in \(T\).

2.1 PROPOSITION  

(i) \((\gamma') \implies (\gamma') \implies (\gamma)\).

(ii) The filter structure defined on a topological space satisfies \((\gamma'')\).

If one defines a category \(\text{Filt}^{(n)}\) as consisting of those filter spaces which satisfy \((\gamma^{(n)})\), \(n = 0, 1, 2\), and continuous maps between them, then

\[
\begin{array}{cccc}
\text{Filt}'' & \hookrightarrow & \text{Filt}'/ & \hookrightarrow \text{Filt} \\
\text{[Top}^{op}, \text{Set}] & \downarrow & \text{full} & \downarrow \\
\text{[Top}^{op}, \text{Set}] & \downarrow & \text{full} & \downarrow \\
\end{array}
\]

where all left adjoints preserve finite products and commute with change of base in the codomain. Moreover, the assignment \(T \mapsto ([|T|, \rightarrow_\gamma])\) extends to a full embedding \(\text{Top} \hookrightarrow \text{Filt}'\).

2.2 WARNING  The category \([\text{Top}^{op}, \text{Set}]\) is not locally small, hence it is not cartesian closed, nor is it a topos. Yet, for some pair of objects \(R\) and \(R'\), we may happen to find that \(R^{R'}\) exists in \([\text{Top}^{op}, \text{Set}]\). In fact, we shall find very many such pairs.

The embedding \(\text{Filt} \hookrightarrow [\text{Top}^{op}, \text{Set}]\) is defined by taking a filter space \(X\) to the presheaf which maps a space \(T\) to the set of continuous functions from \((|T|, \rightarrow_\gamma)\) to \((X, \rightarrow_\gamma)\): \((\beta)\) makes the definition possible, \((\alpha)\) ensures faithfulness while both \((\alpha)\) and \((\gamma)\) are needed to prove fullness.

The assertions are rather straightforward to prove and can be all found in the existing literature. Below we prove a characterization of the image of the last embedding as the \(\neg\neg\)-separated objects of a category of sheaves on \(\text{Top}\).

Consider the following Grothendieck topology: For a topological space \(S\), a family of continuous maps \((f_i: T_i \rightarrow T)\) from topological spaces is a covering in \(\text{cov}(T)\) if

(a) the induced map \(f: \coprod_i T_i \rightarrow T\) from the coproduct of the spaces is a topological quotient,
(b) for every point \( p \) in \( T \) there is a point \( q \) in the coproduct and a neighbourhood \( V \) of \( q \) such that \( f(q) = p \) and \( f|_V \) maps neighbourhoods of \( q \) to neighbourhoods of \( p \).

One can see that the coverings are generated by inclusions into coproducts, and topological quotients satisfying condition (b).

Note that the coverings given above are stable effective epimorphic, in other words the notion of covering is subcanonical on \( \text{Top} \).

2.3 THEOREM \( \text{Flt} \subset \text{full sh(} \text{Top, cov}\text{)} \) is the inclusion of the full subcategory of \( \sqsubseteq \)-separated objects.

The proof of the theorem is inspired by the intuition behind Theorem 1.8 of [11], and it can be considered by now folkloric. The right adjoint to the global section functor (=evaluation at a one point space) is the composition

\[
\begin{align*}
\text{Set} \rightarrow \text{Top} \xrightarrow{\text{Yoneda}} \text{[Top}^\text{op}, \text{Set}].
\end{align*}
\]

It sends a set \( S \) to the presheaf represented by the trivial topology \( \nabla(S) \) on \( S \): more explicitly, it sends a topological space \( T \) to \( S[T] \), the “discrete” \( |T| \)-paths in \( S \), acted upon by precomposition of continuous functions. We shall write \( \nabla \) also for the right adjoint, as in what follows we shall always identify a category fully embedded into \( \text{[Top}^\text{op}, \text{Set}] \) with its image.

2.4 WARNING The statement that \( \nabla \) is a right adjoint does include the fact that, for every presheaf \( R \) on \( \text{Top} \) and every set \( S \), the class of natural transformations from \( R \) into \( \nabla(S) \) is indeed a set.

One shows that

(i) \( \nabla(S) \) is a sheaf for any subcanonical notion of coverings.

(ii) \( R' \rightarrow R \) is \( \sqsubseteq \)-closed in \( \text{sh(} \text{Top, cov}\text{)} \) if and only if for every \( f \in R(T) \)

\[
\forall p: 1 \rightarrow T[f p \in R'(1) \Rightarrow f \in R'(T)]
\]

if and only if

\[
\begin{align*}
R' \rightarrow \nabla(R'(1)) \\
\downarrow \quad \downarrow \\
R \rightarrow \nabla(R(1)) \quad \text{is a pullback.}
\end{align*}
\]
(In fact, this characterization holds in any full subcategory \( A \subseteq [\text{Top}^{\text{op}}, \text{Set}] \) which is closed under pullbacks and contains the representables with the empty space representing the initial object of \( A \).)

It follows immediately that \( R \) is \( \neg \neg \)-separated in \( \text{sh}(\text{Top}, \text{cov}) \) if and only if the unit \( R \to \nabla(R(1)) \) is monic. Every object in the image of \( \text{Flt} \to [\text{Top}^{\text{op}}, \text{Set}] \), is \( \neg \neg \)-separated, and it is a sheaf because the topology is subcanonical and condition (b) states that every neighbourhood filter in the quotient is the image of a neighbourhood filter from the source.

Suppose now \( R \) is \( \neg \neg \)-separated: with no loss of generality, we may assume \( R(T) \) consists of maps from \( |T| \) into \( R(1) \) for each topological space \( T \). Define the smallest convergence structure on \( R(1) \) which makes all maps in the various \( R(T) \)'s continuous: thus \( \Phi \downarrow x \) if there are a space \( T \), a map \( \alpha \in R(T) \), and a point \( p \) in \( T \) such that \( \alpha(p) = x \) and \( \alpha[U_p] \subseteq \Phi \).

To show:

- if \( f: (|T|, \nabla) \to (R(1), \nabla) \) is continuous, then \( f \in R(T) \).

To this aim, we cover \( T \) with a family \( (f_p: T_p \to T)_{p \in |T|} \) satisfying (a) and (b) such that \( f \circ f_p \in R(T_p) \) for all \( p \) in \( T \): Given \( p \) in \( T \), there are a space \( Q \), a point \( q \) in it, and a map \( g \in R(Q) \) such that \( g(q) = f(p) \) and \( g[U_q] \subseteq f[U_p] \). Hence, in the pullback

\[
\begin{array}{ccc}
P & \longrightarrow & Q \\
\downarrow f_p & & \downarrow g \\
T & \longrightarrow & R(1)
\end{array}
\]

the map \( f_p \) is continuous and maps neighbourhoods of \((p, q)\) to neighbourhoods of \( p \).

2.5 REMARK The object represented by the two point space \( \nabla(2) \) is a classifier for \( \neg \neg \)-closed subobjects. Moreover, one can see it as extending the notion of “subspace” because, for a topological space \( T \), one has

\[
\text{Subspaces}(T) \xrightarrow{\sim} \text{Sub}(|T|) \xrightarrow{\sim} \text{Set}(|T|, 2) \xrightarrow{\sim} \text{Top}_{\text{ex}}(1, (\nabla 2)^T). \quad \square
\]

Similar results are known for \( \text{Flt}' \) and \( \text{Flt}'' \): the Grothendieck topologies are described in [10, 23]. It is shown that \( \text{Flt}'' \) are the \( \neg \neg \)-separated objects in the category of sheaves for the canonical Grothendieck topology on \( \text{Top} \).

Note that from the characterization one can deduce a number of properties of filter spaces:

- the left adjoint to the inclusion \( \text{Flt} \to \text{sh}(\text{Top}, \text{cov}) \) exists, as it is defined by taking images of all the components of the unit of the adjunction

\[
R(T) \to R(1)^{|T|}.
\]
• Moreover, it preserves products and commutes with change of base in the codomain as these are computed pointwise in \([\mathbf{Top}^{op}, \mathbf{Set}]\).

• Filter spaces admit exponentiation: for filter spaces \(X\) and \(Y\), for any topological space \(T\), the class \(\text{Nat}(T \times X, Y)\) is in bijection with the subset of \(|Y|^{|T| \times |X|}\) of those maps which preserve the filter structure, because every natural transformation \(\nu: T \times X \rightarrow Y\) gives rise to a commutative square in \([\mathbf{Top}^{op}, \mathbf{Set}]\)

\[
\begin{array}{ccc}
T \times X & \xrightarrow{\nu} & Y \\
\downarrow & & \downarrow \\
\nabla|T| \times \nabla|X| & \xrightarrow{(\nu_1)} & \nabla|Y|
\end{array}
\]

• It follows that, for every presheaf \(R\) and every filter space \(X\), the exponentiation \(X^R\) exists and is a filter space.

3 Equilogical spaces

We have seen in the previous section that the situation in (1) is related to filter spaces. We saw that all existing limits are computed as in the category of presheaves on \(\mathbf{Top}\); more explicitly, the correlation between the two approaches can be summarized as follows:

Recall that all the categories in the rightmost column are not locally small, in particular they are not toposes. As we said, the description of the coverings in \(\text{cov}'(T)\) is given in [10], and \(\text{can}(T)\) denotes the covers in the canonical topology; these are described in \(\text{loc.cit.}\). Moreover we know that
(i) the categories appearing in the bottom row and in the rightmost column are exact in the sense of Barr,

(ii) each inclusion perpendicular to that row or that column, if not to the bottom right corner, is determined by the $\neg\neg$-topology on the larger category;

(iii) all left adjoints are strong and commute with change of base in the codomain,

(iv) all inclusions preserve limits and (local) exponentials: the proof follows from (iii) for all the inclusions but $\text{Top}_{\text{ex}} \hookrightarrow [\text{Top}^{\text{op}}, \text{Set}]$.

(v) Moreover, all squares of inclusions are pullbacks.

(vi) The leftmost inclusion on the bottom row is determined as consisting of the $\nabla 2$-discrete objects, similarly to [6, 18].

Explanations are in order only for the cases of $\text{Equ}$ and $\text{Top}_{\text{reg}}$ in (ii), for the exceptional case in (iv), and for (vi).

The exceptional case in (iv) is obvious from the results in [9] since every object in an exact completion is covered by a projective, and these are precisely the representable objects of $[\text{Top}^{\text{op}}, \text{Set}]$, see also Lemma 5.1 in [19].

Consider then the inclusion $\text{Top}_{\text{reg}} \hookrightarrow \text{Top}_{\text{ex}}$. Since we know that every object in an exact completion has a regular cover from a representable, the $\neg\neg$-separated objects in $\text{Top}_{\text{ex}}$ are precisely those which are covered by a representable and have a mono into a representable $T \twoheadrightarrow C \twoheadrightarrow T'$—in fact, a space of the form $T' = \nabla |T'|$ would do. This presents $C$ as a quotient of a kernel pair in $\text{Top}$, namely that of the composite $f: T \twoheadrightarrow C \twoheadrightarrow T'$. Hence these are precisely the object of $\text{Top}_{\text{reg}}$.

3.1 REMARK There is another presentation of $\text{Top}_{\text{reg}}$ in [2] which emphasizes how close the situation is to that of the effective topos and which is extremely useful for giving an explicit description of a splitting of the canonical fibration and to study properties of the indexed families.

Recall that an assembly on an algebraic lattice consists of a triple $A = (|A|, E_A, L_A)$ where $L$ is an algebraic lattice and $E_A \subseteq |A| \times |L_A|$ is a total relation between elements of the set $|A|$ and elements of (the underlying set of) $L_A$; the suggested reading is as an “existence” relation: $p \in E_Aa$ witnesses that $a$ exists in $A$.

A map $f: A \rightarrow B$ between assemblies is a function $f: |A| \rightarrow |B|$ for which there exists a map $g: L_A \rightarrow L_B$ preserving directed sups such that
the function $f \times g$ preserves the existence relation:

![Diagram](https://via.placeholder.com/150)

First projection from an existence relation is onto because of the requirement of totality. The unique fill-in marks the preservation of existence relations.

If one endows the set $E_A$ with the smallest topology $\sigma$ which makes the second projection to the algebraic lattice continuous, then one readily sees that the data for a map between assemblies correspond precisely to giving an equivalence class of continuous functions from the topological space $(E_A, \sigma)$ into the corresponding one for $B$ which preserve the kernel pairs induced by the surjections, where two continuous functions are declared equivalent if their pairing takes values into the kernel pair of $E_B \rightarrow |B|$. We now leave it to the conscientious reader to specify the details of the equivalence between the present category of assemblies and $\text{Top}_{\text{reg}}$. $\square$

Since a kernel pair in $\text{Top}$ is always a topological subspace of the product, it is obvious that $\text{Equ}$ is (equivalent to) the category of kernel pairs defined on $T_0$-spaces: in other words, it is the intersection of $\text{Top}_{\text{reg}}$ and $(\text{Top}_0)_{\text{ex}}$. As $(\text{Top}_0)_{\text{ex}} \subset \text{Top}_{\text{ex}}$ is closed under subobjects, it is then clear that $\text{Equ}$ consists of the $\sim$-separated objects of $(\text{Top}_0)_{\text{ex}}$. But this can be seen also as a consequence of the characterization of $(\text{Top}_0)_{\text{ex}} \subset \text{Top}_{\text{ex}}$ given as (vi), which we shall now prove.

Recall that, for a fixed object $I$, one says that $D$ is $I$-discrete if the constant map $I \rightarrow 1$ gives rise to an isomorphism on the function spaces into $D$

$$D \overset{\sim}{\longrightarrow} D^I.$$ 

In other words, it is internally the case that all maps from $I$ into $D$ are trivial.

In an exact category, the $I$-discrete objects form a reflective subcategory. The reflection of an object $A$ forces all elements of $A$ in the image of a function from $I$ to be equal: it is the quotient of the two evaluations $A^I \times I \times I \overset{\sim}{\longrightarrow} A$. In the case of an object like $\nabla 2$ with two global elements $1 \overset{w}{\rightarrow} \nabla 2$ which form a jointly epic pair, the quotient is simply that defined by the equivalence relation

$$A^{\nabla 2} \overset{A^v}{\underset{A^w}{\sim}} A,$$
It is useful to notice that the $T_0$-quotient of a space $T$ is of a special kind. The function space $T^\nabla$ always exists in $\text{Top}$: it is the set of pairs $(p_v, p_w)$ of elements of $T$ which belong to the same open sets of $T$—we write $p_v \sim p_w$ when this is the case. The topology is coarser than the product topology: sets $U \times U$ form a basis as $U$ varies among the open subsets of $T$. Moreover, the quotient

$$
\begin{array}{ccc}
T^\nabla & \xrightarrow{T^\omega} & T \\
\sim & \xrightarrow{T^\omega} & T/\sim \\
\end{array}
$$

is a split fork. From this, (vi) follows.

## 4 Filter spaces and equilogical spaces

There is another functor which relates the extensions of $\text{Top}$ we have considered so far: $\text{Flt} \xrightarrow{\text{full}} \text{Top}_{\text{reg}}$.

It is adapted from another construction considered in [16], p. 448, in the spirit of 3.1. First of all, notice that, for objects of $\text{Flt}$, the notion of convergence is completely determined by the restriction $\Phi \hookrightarrow x$ obtained by requesting that a filter $\Phi$ converging to a point $x$ consist only of subsets containing $x$: more explicitly, say that $\Phi \hookrightarrow x$ if $x \in \Phi \subseteq \{x\}^\circ$. Then $\Phi \hookrightarrow x$ if and only if $\Phi \not\hookrightarrow x$. Also the notion of continuous function can be rephrased in terms of the restricted convergence since any function maps a principal maximal filter into a principal maximal filter.

Consider then an object $X$ in $\text{Flt}$, and the restricted convergence relation as a topological space endowing it with the smallest topology making the first projection $\hookrightarrow X$ continuous. In $\text{Top}$, take the kernel pair $X_r = (\hookrightarrow X) \twoheadrightarrow (\hookrightarrow X)$ of the second projection $(\hookrightarrow X) \twoheadrightarrow \nabla X$: it defines an object of $\text{Top}_{\text{reg}}$. We want to show that this is isomorphic to the original filter space $X$ as objects in $[\text{Top}_{\text{op}}, \text{Set}]$. To do this, we shall prove the following

### 4.1 LEMMA

With the notation above, the second projection induces a continuous function $\pi: (\hookrightarrow X) \twoheadrightarrow X$ from a topological space to a filter space such that, for every topological space $T$, postcomposition with $\pi$ defines a natural bijection

$$
\pi \circ \cdot: \text{Top}_{\text{reg}}(T, X_r) \xrightarrow{\sim} \text{Flt}(T, X).
$$

Proof: To see that $\pi$ is continuous from the topological space $(\hookrightarrow X)$ into the filter space $X$, recall that the neighbourhood system of a point $\Phi \hookrightarrow x$ in it is the family $\{\Psi \hookrightarrow y : A \in \Psi \mid x \in A\}$. Clearly, $\pi$ takes this to $\{x\}^\circ \hookrightarrow x$. 
So composing with \( \pi \) determines a natural transformation as in the statement: it is obviously 1-1. To see that it is also surjective, let \( \alpha: T \to X \) be a continuous map into \( X \) from a topological space. Then the function

\[
t \mapsto (\alpha[U] \& \alpha(t)): T \to (\&)
\]

is continuous. Indeed, the inverse image of a neighbourhood of \( \alpha[U] \& \alpha(t) \), say determined by \( A \ni \alpha(t) \), is the inverse image \( \alpha^{-1}[\text{int}(A)] \) of the interior of \( A \). Clearly, composing that function with \( \pi \) gives \( \alpha \) back.

4.2 THEOREM There are full extensions of \( \text{Top} \)

\[
\begin{array}{ccc}
\text{Top} & \longrightarrow & \text{Top} \\
\text{Top}_0 & \longrightarrow & \text{Top} \\
\text{Equ} & \longrightarrow & \text{Top}_{\text{eq}} \\
(\text{Top}_0)_{\text{ex}} & \longrightarrow & \text{Top}_{\text{ex}} \\
\text{Filt} & \longrightarrow & \text{Filt}' & \longrightarrow & \text{Filt}''
\end{array}
\]

The categories in the middle row are all locally cartesian closed quasitoposes, those in the bottom row are locally cartesian closed pretoposes.

Note that, unlike in diagram (1), there are no non-locally small categories anymore. Since we have obtained locally cartesian closed pretoposes, which admit a canonical interpretation of intuitionistic logic with higher types, one can start using the logic of the category \( \text{Top}_{\text{ex}} \) to understand the kind of extension the others are.

Note also that there is an immediate characterization available for the inclusion \( \text{Top} \hookrightarrow \text{Top}_{\text{ex}} \), see [7, 5]: it consists of the internal projectives, \( i.e. \) those objects \( A \) on which choice is possible

\[
\forall a \in A \exists x \in C[f(x) = a] \longrightarrow \exists g \in C^A \forall a \in A[f(g(a)) = a].
\]

Moreover, the fact that the inclusion preserves any existing higher order structure yields that hierarchies of higher types defined in one category or another coincide as long as the basic types are in the intersection of the two categories: \( e.g. \) if one starts with \( T_0 \)-spaces like the flat natural numbers, the real line, or the one-point compactification of these, the higher types will all be isomorphic whether defined using partial continuous functionals, say within \( \text{Equ} \), or using filter spaces. The analysis of the density theorem in [2] adds some finer points to this, and it would be extremely important to be able to recast that within the present framework (see also the discussion in the next paragraph).
We saw that $\nabla^2$ classifies the notion of “subspace”, and it induces a topology on the pretopos which can be characterized as that of double negation. Moreover, $\text{Equ} \hookrightarrow \text{Top}_{\text{ex}}$ consists of the $\neg\neg$-separated, $\nabla^2$-discrete objects. So again, from the logic of $\text{Top}_{\text{ex}}$, we see that they are closed under the formation of local function spaces (=dependent products indexed by an equilogical space).

In fact, each of the two distinct notions of $\neg\neg$-separation and of $\nabla^2$-discreteness defines a cartesian closed extension of $(T_0)$-spaces—and we already know which they are.

I think it is interesting that the notion of separation connects with the notion due to Hausdorff. Recall that an object $A$ is $\neg\neg$-separated if the diagonal $A \supseteq A \times A$ is $\neg\neg$-closed, hence if (and only if) there is a pullback

$$
\begin{array}{ccc}
A & \rightarrow & 1 \\
\vee & & \vee \\
\downarrow & & \downarrow v \\
A \times A & \delta & \nabla^2
\end{array}
$$

Since such $\delta$ is unique, one can speak in the logic of $\text{Top}_{\text{ex}}$ of the $\neg\neg$-separated object $A$ as an object with the property that

$$
\forall a_1, a_2 \in A[a_1 = a_2 \leftarrow \delta(a_1, a_2) = v]
$$

for the appropriate $\delta: A \times A \rightarrow \nabla^2$. Then one deduces the closure properties of the $\neg\neg$-separated objects using the properties of the Heyting algebra on $\nabla^2$.

But there are two ways to define the reflection: One way goes by the logical characterization of $\nabla^2$ as a classifier of $\neg\neg$-closed subobjects, forcing an equality to be $\neg\neg$-closed. Thus, given an object $A$ one takes the $\neg\neg$-closure of the diagonal relation $A \Rightarrow A \times A$, which produces an equivalence relation on $A$. Since the category is exact, the quotient is the required reflection. The other way uses no properties of the object $\nabla^2$ and proceeds by factoring the canonical map $\eta: A \rightarrow (\nabla^2)^{(\nabla^2)^A}$, see also [4].

In fact, these two different ways are used in a similar context with another two point space $\top \supseteq \Sigma$, the Sierpinsky space. Note $\Sigma \supseteq \nabla^2$, the point $\top$ classifies open subspaces, and the point $\bot$ classifies closed subspaces: the corresponding notions of separation are, at the level of topological spaces, those of discrete spaces and Hausdorff spaces, respectively. But neither of the extensions to $\text{Top}_{\text{ex}}$ will produce a reflective subcategory because equality on $\Sigma$ is neither open nor closed. Yet, there is a weaker result.
5.1 DEFINITION  Fix an object $K$, and a global element $1 \xrightarrow{t} H$. We say that $A$ is $K$-separated if the canonical map $\eta: A \to K^{(K^A)}$ is monic.\footnote{Another name often found in the literature is that of a $K$-space. We used the present notion in \cite{21, 14} giving it no name: it seems suitable here to use a modification of separation.} We say that $A$ is $t$-separated if there is a pullback

$$
\begin{array}{ccc}
A & \xrightarrow{} & 1 \\
\downarrow & & \downarrow \\
\langle \text{id, id} \rangle & \xrightarrow{} & t \\
\downarrow & & \downarrow \\
A \times A & \xrightarrow{\delta} & H \\
\end{array}
$$

for an appropriate $\delta$.

5.2 THEOREM  Suppose $K$ is an object and $1 \xrightarrow{t} H$ is a pointed object in $\text{Top}_{\text{ex}}$:

(i) The full subcategory of $K$-separated objects is reflective in $\text{Top}_{\text{ex}}$.

(ii) If $H$ is a complete Heyting algebra, and $t$ is its top element, then the full subcategory of $t$-separated objects is locally cartesian closed.

5.3 EXAMPLES

- $\text{Top}_{\text{reg}} \subseteq \text{Top}_{\text{ex}}$ are the $\nabla^2$-separated objects, as well as the $v$-separated objects.
- $\text{Equ} \subseteq \text{Top}_{\text{ex}}$ are the $\nabla^2$-separated, $\nabla^2$-discrete objects.
- The category of $\bot$-separated objects is a locally cartesian closed extension of that of Hausdorff spaces.

The proof of 5.2 is easy. It is important to notice that (ii) rests on much more general hypotheses, as stated in the following lemma, taken almost directly from \cite{17}.

5.4 LEMMA  Let $C$ be a cartesian closed category with pullbacks, and let $1 \xrightarrow{t} H$ be a pointed object in $C$.

(i) Suppose that there is a map $m: H \times H \to H$ which pulls $t$ back to $1 \xrightarrow{\sim} 1 \times 1 \xrightarrow{t \times t} H \times H$. Then the category of $t$-separated objects is closed under binary products.

(ii) Suppose that, for a given object $I$, there is a map $m_I: H^I \to H$ which pulls $t$ back to $1 \xrightarrow{\sim} 1^I \xrightarrow{t^I} H^I$. Then the category of $t$-separated objects is closed under exponentiation by $I$. 

\footnote{Another name often found in the literature is that of a $K$-space. We used the present notion in \cite{21, 14} giving it no name: it seems suitable here to use a modification of separation.}
Proof: It is a simple pullback pasting: for instance, to prove (ii) consider a $t$-separated object $A$ and look at the diagram

$$
\begin{array}{c}
A^I \xrightarrow{\text{id}} A^I \xrightarrow{1^I} 1 \\
\langle \text{id}, \text{id} \rangle \downarrow \quad \langle \text{id}, \text{id} \rangle^I \downarrow \quad t^I \downarrow \\
A^I \times A^I \xrightarrow{\sim} (A \times A)^I \xrightarrow{\delta^I} H^I \xrightarrow{m_I} H
\end{array}
$$

where the middle square is a pullback because $(-)^I$ is a right adjoint. □

6 Problems

Lemma 4.1 provides us with a way to see the various notions of filter spaces as subcategories of $\text{Top}_{ex}$. We know they are $\neg \neg$-separated objects there, but how can we see each choice as determined by a precise logical property?

Is it possible to describe the notions of density and codensity in $\text{Top}_{ex}$ and prove the density theorem based on the logic of the pretopos?

Can one describe some game-theoretical models within $\text{Top}_{ex}$ by means of an appropriate monoidal closed structure?

Sobriety is external to $\text{Top}_{ex}$. By this I mean that the sober topological space whose points are the frame homomorphisms from $\Sigma^T$ into $\Sigma$ is not the one obtained by performing the same construction within $\text{Top}_{ex}$: one can take the equalizer

$$
[\Sigma^T, \Sigma] \xrightarrow{\eta_{\Sigma^T}} \Sigma^{(\Sigma^T)} \xrightarrow{\eta_{\Sigma^{(\Sigma^T)}}} \Sigma^{(\Sigma^{(\Sigma^T)})}
$$

of the two maps induced by the exponential adjunction, and this is not in general a topological space. How are the two related?

Is there any better hope of describing the $\Sigma$-replete objects in the category of directed-complete partial orders using the logic of $\text{Top}_{ex}$?

Can we bring the metric space approach closer to the domain approach by means of an appropriate notion of $t$-separation?

Acknowledgements

Discussions, direct or supported by electronic means, with Alex Simpson, Andrej Bauer, Anna Bucalo, Aurelio Carboni, Bernhard Reus, Dana Scott, Jaap van Oosten, John Longley, Lars Birkedal, Marcelo Fiore, Martin Escardo, Martin Hyland, Paul Taylor, Reinhold Heckmann, and Thomas Streicher, were extremely useful and stimulating.
References


