Observability and reachability of simple
grid and torus graphs *

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Abstract: In this paper we investigate the observability and reachability properties of a network system, running a Laplacian based average consensus algorithm, when the communication graph is a grid or a torus. More in detail, under suitable conditions on the eigenvalue multiplicity, we provide necessary and sufficient conditions, based on simple algebraic rules from number theory, to characterize all and only the nodes from which the network system is observable (reachable). For any set of observation (leader) nodes, we provide a closed form expression for the unobservable (unreachable) eigenvalues and for the eigenvectors of the unobservable (unreachable) subsystem.

Keywords: Laplacian eigenvectors, consensus, controllability, observability

1. INTRODUCTION

Distributed computation in network (control) systems has received great attention in the last years. One of the most studied problems is the consensus problem. Given a network of processors, the task of reaching consensus consists of computing a common desired value by performing local computation and exchanging local information. A variety of distributed algorithms for diverse system dynamics and consensus objectives has been proposed in the literature. We are interested in two problems that may arise in a network running a consensus algorithm when only a subset of nodes is controlled by an external input or measured by an external sensor. Namely, is it possible to reconstruct the entire network state just knowing the state of a limited number of nodes? Respectively, is it possible to reach all the node configurations just controlling a limited number of nodes?

In this paper we will concentrate on a network system with fixed communication graph topology running a Laplacian based average consensus algorithm. Average consensus has been widely studied in the last years. Several distributed feedback laws have been proposed. A survey on these algorithms and their performance may be found e.g. in Olfati-Saber et al. [2007] and references therein. The dynamical system arising from a consensus network with fixed topology is a linear time-invariant system and the problem of understanding if the network state may be reconstructed is an observability problem. Observability for a network system running an average consensus algorithm has been studied for the first time in Ji and Egerstedt [2007]. In that paper the authors provide a necessary condition for observability. The condition is based on algebraic (graph) tools based on the notion of equitable partitions of a graph. Equitable partitions have been also used in Rahmani et al. [2009] and Martini et al. [2010] in order to study the dual controllability problem for a leader-follower network. In Mesbahi and Egerstedt [2010] a wider class of properties for dynamic multiagent networks is investigated by focusing on graph theoretic methods. A parallel research line investigates a slightly different property called structural observability Sundaram and Hadjicostis [2008]. Here, the objective is to choose the nonzero entries of the consensus matrix (i.e. the state matrix of the resulting network system) in order to obtain observability from a given set of nodes. However, in many contexts the structure of the system matrix is given (e.g. the Laplacian for average consensus). Thus, we believe that the problem studied in the paper is of interest. It is also worth noting that in Tanner [2004] simulations were provided showing that it is “unlikely” for a Laplacian based consensus network to be completely controllable. This suggests that a careful study for the analysis of such properties is useful. Observability and reachability for dynamic multiagent networks have recently received further attention as necessary structural properties for some interesting applications. In Ji et al. [2007] reachability is assumed in order to study optimal control based state transfer from a leader agent. Observability of the network is a necessary property to study the intrusion detection problem for multi-agent dynamic systems, see, e.g., Pasqualetti et al. [2011, 2010], Sundaram and Hadjicostis [2008].

The contribution of the paper is twofold. First, we provide necessary and sufficient conditions based on simple algebraic relations from number theory that completely characterize the observability (reachability) of grid and torus graphs under suitable conditions on the eigenvalue
We denote \( m \) the property that we let \( e \) with \( d \) of nodes \( i \) and set the multiplicities. We call these grid and torus graphs “simple”.

More in detail, on the basis of the node labels and the total number of nodes in the graph, we are able to (i) identify all and only the observable (reachable) nodes of the graph, (ii) say if the graph is observable (reachable) from a given set of nodes and (iii) construct a set of observation (leader) nodes from which the graph is observable (reachable).

Second, we provide a closed form expression for the unobservable (unreachable) eigenvalues and eigenvectors for any unobservable (unreachable) set of nodes. Though the starting point of this paper relies on the results in Parlangeli and Notarstefano [2010c], see also Parlangeli and Notarstefano [2010b], the methodologies are in fact novel and are based on mathematical tools from cartesian product of graphs and Kronecker product of matrices.

The paper is organized as follows. In Section 2 we introduce some preliminary definitions and properties of undirected graphs, describe the network model used in the paper and set up the observability and reachability problems. In Section 3 we recall results on the observability (reachability) of path and cycle graph that are at the basis of the new results on grid and torus graphs. In Section 4 we provide necessary and sufficient conditions for the observability (reachability) of a suitable class of grid and torus graphs obtained by the cartesian product of path and cycle graphs. For space constraints all proofs are omitted in this paper and will be provided in a forthcoming document.

**Notation** We let \( \mathbb{N}, \mathbb{N}_0 \), the \( \mathbb{R}_{>0} \) and \( \mathbb{R}_{\geq 0} \) denote the natural numbers, the non-negative integer numbers, positive real numbers and the non-negative real numbers, respectively. We denote \( d_0, d, d_1, d_2 \in \mathbb{N} \), the vector of dimension \( d \) with zero components and \( d_0 \times d_2 \), \( d_1 \), \( d_2 \in \mathbb{N} \), the matrix with \( d_1 \) rows and \( d_2 \) columns with zero entries. For \( i \in \mathbb{N} \) we let \( e_i \) be the \( i \)-th element of the canonical basis, e.g., \( e_1 = [1 \ 0 \ ... \ 0]^T \). For a matrix \( A \in \mathbb{R}^{d_1 \times d_2} \) we denote \([A]_{i,j}\) the \((i,j)\)th element and \([A]\) the \( i \)th column of \( A \). For a vector \( v \in \mathbb{R}^d \) we denote \( [v]_i \) the \( i \)th component of \( v \) so that \( v = ([v]_1, \ldots, [v]_d)^T \). Adopting the usual terminology of number theory, we will say that \( k \) divides a nonzero integer \( m \) (written \( k | m \)) if there is an integer \( q \) with the property that \( m = kq \). When this relation holds, \( k \) is said a factor or divisor of \( m \). If two integers \( b \) and \( c \) satisfy for a given \( m \) the relation \( m|(b - c) \) then we say that \( b \) is congruent to \( c \) modulo \( m \) (written \( b \equiv c \mod m \) or equivalently \( b \mod m = c \)).

2. PRELIMINARIES AND PROBLEM SET-UP

In this section we present some preliminary terminology on graph theory, introduce the network model, set up the observability problem and provide some standard results on observability of linear systems that will be useful to prove the main results of the paper.

2.1 Preliminaries on graph theory

Let \( G = (I, E) \) be a static undirected graph with set of nodes \( I = \{1, \ldots, n\} \) and set of edges \( E \subset I \times I \). We denote \( N_i \) the set of neighbors of agent \( i \), that is, \( N_i = \{j \in I \mid (i, j) \in E\} \), and \( d_i = \sum_{j \in N_i} 1 \) the degree of node \( i \). The maximum degree of the graph is defined as \( \Delta = \max_{i \in I} d_i \). The degree matrix \( D \) of the graph \( G \) is the diagonal matrix defined as \( [D]_{ii} = d_i \). The adjacency matrix \( A \in \mathbb{R}^{n \times n} \) associated to the graph \( G \) is defined as

\[
[A]_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}
\]

The Laplacian \( L \) of \( G \) is defined as \( L = D - A \). The Laplacian is a symmetric positive semidefinite matrix with \( k \) eigenvalues in 0, where \( k \) is the number of connected components of \( G \). If the graph is connected the eigenvector associated to the eigenvalue 0 is the vector \( 1 = [1 \ldots 1]^T \).

Next, we introduce the notion of cartesian product of graphs. Let \( G = (I, E) \) and \( G' = (I', E') \) be two undirected graphs. The cartesian product \( G \square G' \) is a graph with vertex set \( I \times I' \) (i.e. the cartesian product of the two vertex sets) and edge set defined as follows. Nodes \( [i, i'] \in I \times I' \) and \( [k, k'] \in I \times I' \) are adjacent in \( G \square G' \) if either \( i = k \) and \( (i', k') \in E' \) or \( i' = k' \) and \( (i, k) \in E \). The cartesian product is commutative and associative. Thus, a \( d \in \mathbb{N} \) dimensional product graph, \( \prod_{i=1}^d G_i \), is constructed by combining the above definition with the associative property.

We introduce some special graphs that will be of interest in the rest of the paper. A path graph is a graph in which there are only nodes of degree two except for two nodes of degree one. The nodes of degree one are called external nodes, while the other are called internal nodes. From now on, without loss of generality, we will label the external nodes with 1 and \( n \), and the internal nodes so that the edge set is \( E = \{(i, i+1) \mid i \in \{1, \ldots, n-1\}\} \). A cycle graph is a graph in which all the nodes have degree two. From now on, without loss of generality, we will label the nodes so that the edge set is \( E = \{(i, i \mod (n+1)) \mid i \in \{1, \ldots, n\}\} \). A d-dimensional grid graph is the cartesian product of \( d \) paths (of possibly different length). In a grid graphs the nodes have degree from \( d \) up to \( 2d \). We call the nodes with degree \( d \) corner nodes. Corner nodes are obtained from the product of external nodes in the paths. A d-dimensional torus graph is the cartesian product of \( d \) cycle graphs.

2.2 Network of agents running average consensus

We consider a collection of agents labeled by a set of identifiers \( I = \{1, \ldots, n\} \), where \( n \in \mathbb{N} \) is the number of agents. We assume that the agents communicate according to a time-invariant undirected communication graph \( G = (I, E) \), where \( E = \{(i, j) \in I \times I \mid i \text{ and } j \text{ communicate}\} \). That is, we assume that the communication between any two agents is bi-directional. The agents run a consensus algorithm based on a Laplacian control law (see e.g. Olfati-Saber et al. [2007] for a survey). The dynamics of the agents evolve in continuous time \( t \in \mathbb{R}_{\geq 0} \) and are given by

\[
\dot{x}_i(t) = - \sum_{j \in N_i} (x_i(t) - x_j(t)), \ i \in \{1, \ldots, n\}.
\]

Using a compact notation the dynamics may be written as

\[
\dot{x}(t) = -Lx(t), \ t \in \mathbb{R}_{\geq 0},
\]

where \( x = ([x_1] \ldots [x_n])^T = [x_1 \ldots x_n]^T \) is the vector of the agents’ states and \( L \) is the graph Laplacian.
Remark 2.1. (Discrete time system). In discrete time, we can consider the following dynamics
\[ x_i(t+1) = x_i(t) - \epsilon \sum_{j \in N_i} (x_i(t) - x_j(t)), \quad i \in \{1, \ldots, n\}, \]
where \( \epsilon \in \mathbb{R} \) is a given parameter. A compact expression for the dynamics is
\[ x(t+1) = (I - \epsilon L)x(t), \quad i \in \mathbb{N}_0. \]
For \( \epsilon \in (0, 1/\Delta) \) (\( \Delta \) is the maximum degree of the graph), \( P = (I - \epsilon L) \) is a nonnegative, doubly stochastic, stable matrix.
It can be easily shown that the continuous and discrete time systems have the same unobservable properties (eigenvalues and subspace). Therefore, the results shown in the paper also hold in this discrete time set-up. \( \square \)

2.3 Network observability and reachability

In this section we describe the mathematical framework that we will use to study the observability of a network system. We start by describing the scenario that motivates our work. We imagine that an external processor (not running the consensus algorithm) collects information from some nodes in the network. We call these nodes observation nodes. In particular, we assume that the external processor can read the state of each observation node. Formally, we consider the following model. For each observation node \( i \in I \), we have the following output
\[ y_i(t) = x_i(t). \]
Therefore the output matrix is
\[ C_i = [e_i^T] \]
If the set of observation nodes \( I_o \) in the network has cardinality greater than one, say \( I_o = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\} \), then the output is \( y_{i_o}(t) = [x_{i_1}(t) \ x_{i_2}(t) \ \ldots \ x_{i_p}(t)]^T \). Therefore, the output matrix is
\[ C_{i_o} = \begin{bmatrix} e_{i_1}^T & \cdots & e_{i_p}^T \end{bmatrix} \]
It is a well known result in linear systems theory that the observability properties of the pair \( (L, C_{i_o}) \) correspond to the controllability properties of the pair \( (L^T, C_{i_o}^T) = (L_C, C_{i_o}^T) \). The associated dual network system is
\[ \dot{x}(t) = -Lx(t) + C_{i_o}^Tu(t), \quad (1) \]
where \( u \in \mathbb{R}^{p_o} \) is the input vector. It follows easily that each component \( u_{i_o} \) fully controls the dynamics of the \( i_o \)-th node, so that this turns to be the model of a leader-follower network. Thus, our results apply also to the controllability problem in a leader-follower network, where the observation nodes correspond to the leader nodes. For the sake of space, from now on we will concentrate on the observability.

Remark 2.2. Straightforward results from linear system theory can be also used to prove that the controllability problem studied in Rahmani et al. [2009] and Martini et al. [2010] and the dual observability problem studied in Ji and Egerstedt [2007] can be equivalently formulated in our set up. \( \square \)

2.4 Standard results on observability of linear systems

The observability problem consists of looking for nonzero values of \( x(0) \) that produce an identically zero output \( y(t) \).
Here, we recall an interesting result on the observability of time-invariant linear systems known as Popov-Belevich-Hautus (PBH) lemma.

Lemma 2.3. (PBH lemma). Let \( A \in \mathbb{R}^{n \times n} \) and \( C \in \mathbb{R}^{p \times n} \), \( n, p \in \mathbb{N} \), be the state and output matrices of a linear time-invariant system. The pair \( (A, C) \) is observable if and only if
\[ \text{rank} \left[ \begin{array}{c} C \\ A - \lambda I \end{array} \right] = n \]
for all \( \lambda \in \mathbb{C} \). \( \square \)
Combining the PBH lemma with the fact that the state matrix is symmetric (and therefore diagonalizable) the following corollary may be proven.

Corollary 2.4. Let \( X_{un} \) be the unobservable subspace associated to the pair \( (L, C) \), where \( L \) is a symmetric matrix. Then \( X_{un} \) is spanned by vectors \( v_l \) satisfying
\[ Cv_l = 0_p, \quad Lv_l = \lambda v_l, \]
for \( \lambda \in \mathbb{R} \). \( \square \)
In the rest of the paper we will call unobservable eigenvalues and eigenvectors the eigenvalues and eigenvectors for which (2.4) is satisfied.

3. PREVIOUS RESULTS ON THE OBSERVABILITY OF PATH AND CYCLE GRAPHS

In this section we briefly recall the results in Parlangeli and Notarstefano [2010c], see also Parlangeli and Notarstefano [2010b], on the observability of path and cycle graphs. The characterization of the observability for grid and torus graphs relies on these results.

We start analyzing the observability of the path by using the PBH lemma in the form expressed in Corollary 2.4. First, it is known, Parlangeli and Notarstefano [2010a], completely characterize the observability of a path by means of simple rules from number theory.

Theorem 3.1. (Path observability). Given a path graph of length \( n \), let \( n = 2^{n_0} \prod_{j=1}^{k} p_j \) be a prime number factorization for some \( k \in \mathbb{N} \) and (odd) prime numbers \( p_1, \ldots, p_k \). The following statements hold:

1. the path is not completely observable from a node \( i \in \{2, \ldots, n-1\} \) if and only if
\[ (n - i) \text{ mod } p = (i - 1) \]
for some odd prime \( p \) dividing \( n \);
2. for each odd prime factor \( p \) of \( n \), all nodes with index \( ip \text{ mod } p \) \( i \in \{1, \ldots, \frac{p}{2}\} \) are unobservable and share the following unobservable eigenvalues
\[ \lambda_p^\nu = 2 - 2 \cos \left( \frac{(2\nu - 1)}{p} \pi \right), \quad \nu \in \{1, \ldots, \frac{p-1}{2}\}; \]

}\]
and eigenvectors of the unobservable subspace
\[ V_\nu^p = [v_\nu^p, 0 - (Iv_\nu^p)T] -v_\nu^p T \ldots (1) \tilde{F} (Iv_\nu^p)T]^T, \]
where \((v_\nu^p)_a = \sin \left( \frac{(p-1)/2 + \alpha(2\nu - 1)\pi}{p} \right), \alpha \in \{1, \ldots, (p-1)/2\}, \nu \in \{1, \ldots, (p-1)/2\}; \) and
(iii) if node \(i\) satisfies \((n-i) \mod n = (i-1)\) for \(l \leq k\) distinct prime factors \(p_j\), then the eigenvalues in (2) and their corresponding eigenvectors in (2) are unobservable from \(i\).
\[ \square \]

The following corollary follows straight from Theorem 3.1 and characterizes all and only the path graphs that are observable from any node.

**Corollary 3.2.** Given a path graph of length \(n = 2^k\) for some \(k \in \mathbb{N}\), then the path is observable from any node.

Next, we characterize the observability of a cycle graph. We start with a negative result, namely that a cycle graph is not observable from a single node. First, we need a well known result in linear systems theory, see, e.g., Antsaklis and Michel [1997].

**Lemma 3.3.** If a state matrix \(A \in \mathbb{R}^{n \times n}, n \in \mathbb{N}\), has an eigenvalue with geometric multiplicity \(\mu > p\), then for any \(C \in \mathbb{R}^{n \times n}\) the pair \((A, C)\) is unobservable. \[ \square \]

All but at most two eigenvalues of the Laplacian of the cycle have geometric multiplicity two. Thus, applying the previous lemma next proposition follows.

**Proposition 3.4.** A cycle graph is not observable from a single node for any choice of the observation node with \(\left\lfloor \frac{n+1}{2} \right\rfloor\) unobservable eigenvalues. \[ \square \]

It is worth noting that, due to the symmetry of the cycle, the observability properties are determined by the relative distance between each pair of consecutive observation nodes. The following theorem (Theorem 4.9 in Parlangeli and Notarstefano [2010b]) parallels Theorem 3.1.

**Theorem 3.5.** (Cycle observability). Given a cycle graph of length \(n\), let \(n = \prod_{\nu=1}^{k} p_\nu\) be a prime number factorization for some \(k \in \mathbb{N}\) and prime numbers \(p_1, \ldots, p_k\) (including the integer 2). The following statements hold:
(i) the cycle graph is observable from the set of observation nodes \(L_0 = \{i_1, \ldots, i_p\}\) if and only if \((i_2 - i_1), (i_3 - i_2), \ldots, ((i_1 - i_p) \mod(n))\) are coprime; and
(ii) for each prime factor \(p\) of \(n\) and for each fixed \(i \in \{1, \ldots, p\}\), the set of nodes \(\{i + kp\}, k \in \{0, \ldots, \frac{n}{p}-1\}\), is unobservable with the following unobservable eigenvalues
\[ \lambda_\nu^p = 2 - 2 \cos \left( \frac{\pi}{p} \right) \nu \in \{1, \ldots, p-1\} \]
and eigenvectors of the unobservable subspace, for \(i=1,\)
\[ V_\nu^p = [0 v_\nu^p T \ldots 0 v_\nu^p T]^T, \]
where \((v_\nu^p)_a = \sin \left( \frac{\alpha \pi}{p} \right), \alpha \in \{1, \ldots, p-1\}, \nu \in \{1, \ldots, p-1\}\).
\[ \square \]

Next corollaries provide respectively an easy way to choose two observation nodes to get observability (for any cycle length) and the class of cycle graphs (lengths) for which observability is guaranteed for any pair of observation nodes.

**Corollary 3.6.** Any cycle graph is observable from two adjacent nodes. \[ \square \]

**Corollary 3.7.** A cycle graph of length \(n\) is observable from any pair of nodes if and only if \(n\) is prime.

4. **OBSERVABILITY OF SIMPLE GRID AND TORUS GRAPHS**

In this section we give the main results of the paper on the observability (reachability) of grid and torus graphs.

4.1 **Laplacian eigenstructure of cartesian-product graphs**

An important property of graphs obtained as the cartesian product of other graphs is that the Laplacian can be obtained from the Laplacian of their constituent graphs by using the Kronecker product of two matrices, see Merris [1998]. Given two matrices \(A \in \mathbb{R}^{d \times d}\) and \(B \in \mathbb{R}^{l \times l}\) with \([A]_{ij} = \alpha_{ij}\), their Kronecker product \(A \otimes B \in \mathbb{R}^{dl \times dl}\) is defined as
\[ A \otimes B = [a_{11} B a_{12} B \ldots a_{1d} B \]
\[ a_{21} B a_{22} B \ldots a_{2d} B \ldots \]
\[ a_{l1} B a_{l2} B \ldots a_{ld} B], \]
and their Kronecker sum as
\[ A \oplus B = A \otimes I_l + I_d \otimes B. \]

Given the cartesian product of the graphs \(G_1, \ldots, G_d\) with Laplacian matrices \(L_1, \ldots, L_d\), the Laplacian \(L_\square\) of \(G_1 \square \ldots \square G_d\) is given by
\[ L_\square = L_1 \oplus \ldots \oplus L_d. \]

This structure on the Laplacian induces a structure also on its eigenvalues and eigenvectors. We state it in the next lemma, see Merris [1998].

**Lemma 4.1.** (Laplacian eigenstructure of cartesian product graphs) Let \(G_1, \ldots, G_d\) be \(d \in \mathbb{N}\) undirected graphs and \(G = G_1 \square \ldots \square G_d\) their cartesian product. Let \(\lambda_1^\kappa, \ldots, \lambda_k^\kappa\) be the Laplacian eigenvalues of the graphs \(G_\kappa\) and \(v_1^\kappa, \ldots, v_l^\kappa\) the corresponding eigenvectors for \(\kappa \in \{1, \ldots, d\}\). The Laplacian eigenvalues and their corresponding eigenvectors of \(G\) are
\[ \lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_d} \]
and
\[ v_{i_1} v_{i_2} \otimes \ldots \otimes v_{i_d}\]
for \(i_1 \in \{1, \ldots, n_1\}, \ldots, i_d \in \{1, \ldots, n_d\}. \]
\[ \square \]

We are now ready to define a simple cartesian product graph.

**Definition 4.2.** (Simple cartesian-product graphs). Let \(G\) and \(G'\) be two undirected graphs and let \(\{\lambda_1, \ldots, \lambda_k\}\) and \(\{\lambda'_1, \ldots, \lambda'_k\}\) be the sets of distinct eigenvalues among all the Laplacian eigenvalues of respectively \(G\) and \(G'\).

We say that the graph \(G \square G' = \text{simple}\) if the set \(\{\lambda_1 + \lambda'_i \mid i \in \{1, \ldots, k\}, \alpha \in \{1, \ldots, k\}\}\) contains only distinct eigenvalues.
\[ \square \]

Using the associative property of the cartesian product the definition easily generalizes to the product of more than two graphs.
4.2 Observability of simple grid graphs

Before stating our results we need some notation. Given a $d$-dimensional grid graph $G = P_1 \boxtimes \ldots \boxtimes P_d$, we denote $i = [(i_1), \ldots, (i_d)]$ a node of $G$, where the component $(i_k)$ identifies the position of the node on the $k$th path. Also, given a Laplacian eigenvector of the graph, $w$, $w \in \mathbb{R}^{n_1 \ldots n_d}$, we say “the component $[(i_1), \ldots, (i_d)]$ of $w$” meaning “the component $(i_1 \cdot n_1 \cdot \ldots \cdot n_d)^{(i_1)} \cdot (i_2 \cdot n_2 \cdot \ldots \cdot n_d)^{(i_2)} \ldots \cdot (i_d \cdot n_d)^{(i_d)}$ of $w$”.

We start with a lemma that relates the observability of a grid from a single node to the observability of its constitutive paths.

**Lemma 4.3.** Let $P_1, \ldots, P_d$, $d \in \mathbb{N}$, be path graphs of length respectively $n_1, \ldots, n_d$ and let $G = P_1 \boxtimes \ldots \boxtimes P_d$ be a simple grid. The grid graph $G$ is not observable from a node $[(i_1), \ldots, (i_d)]$, $i_1 \in \{1, \ldots, n_1\}$, $\ldots$, $i_d \in \{1, \ldots, n_d\}$, if and only if there exists $\ell \in \{1, \ldots, d\}$ such that $P_\ell$ is not observable from $(i_\ell)$.

**Corollary 4.4.** Any simple grid graph is observable from any corner node.

We are now ready to characterize the observability of simple grid graphs.

**Theorem 4.5.** (Grid observability). Let $G$ be a simple grid graph.

(i) The grid graph $G$ is not observable from a node $i = [(i_1), \ldots, (i_d)]$, $i_1 \in \{1, \ldots, n_1\}$, $\ldots$, $i_d \in \{1, \ldots, n_d\}$, if and only if there exists $\ell \in \{1, \ldots, d\}$ such that

\[
(n_\ell - (i_\ell) \mod p) = (i_\ell - 1),
\]

for some odd prime $p$ dividing $n_\ell$.

(ii) For any direction $\ell \in \{1, \ldots, d\}$ of $G$ the following holds. For each odd prime factor $p$ of $n_\ell$, all nodes $i = [(i_1), \ldots, (i_d)]$, such that $(i_\ell)p - (i_\ell) \in \{1, \ldots, \frac{n_\ell}{p}\}$, and $(i_1), \ldots, (i_{\ell-1})$, $(i_{\ell+1}), \ldots, (i_d)$ arbitrary, are unobservable and share the following unobservable eigenvalues

\[
\lambda_{\nu, \ell} = 2 - 2\cos\left(\frac{(2\nu - 1)\pi}{p}\right) + \lambda_1 + \ldots + \lambda_{\ell-1} + \lambda_{\ell+1} + \ldots + \lambda_d,
\]

and unobservable eigenvectors of the unobservable subspace

\[
w_{\nu, \ell} = u_1 \otimes u_2 \otimes \ldots \otimes u_{\ell-1} \otimes V_{\nu, \ell} \otimes u_{\ell+1} \otimes \ldots \otimes u_d
\]

with $\nu \in \{1, \ldots, (p - 1)/2\}$, and $\mu$, respectively $u_\mu$, $\mu \neq \ell$, any arbitrary eigenvector, respectively eigenvector, of $P_\ell$;

(iii) if a node $i = [(i_1), \ldots, (i_d)]$ satisfies (5) for $\ell \leq d$ distinct directions and, in each direction $\ell$, for $k_\ell \leq n_\ell$ distinct prime factors, then the set of eigenvalues (eigenvectors), defined as in (6) (respectively in (7)) for each direction $\ell$ and each prime factor are unobservable from $i$.

Next, we show, on the basis of the results in Theorem 4.5, how to check the observability of a simple grid from a given set of nodes or, equivalently, how to construct a set of observation nodes such that the grid is observable. For the sake of clarity we present the procedure for a two dimensional grid ($d = 2$), however the procedure can be easily generalized to higher dimensions.

First, we introduce some notation. Given two sets $X = \cup_{\nu=1}^{k_1} \{x_1, x_2, \ldots, x_{\nu}\}$ and $Y = \cup_{\mu=1}^{k_2} \{y_1, y_2, \ldots, y_{\mu}\}$, with $x_1, x_2, \ldots, x_{\nu} \in \mathbb{R} \times \mathbb{R}$ and $y_1, y_2, \ldots, y_{\mu} \in \mathbb{R} \times \mathbb{R}$, we say that $X \cap Y \neq \emptyset$ if and only if there exists $x_1, x_2, \ldots, x_{\nu} \in X$ and $y_1, y_2, \ldots, y_{\mu} \in Y$ such that $x_1 = y_1$, $x_2 = y_2$, i.e. $x_1, y_1 = y_2$ and $x_2, y_2 = y_2$.

Consider a two dimensional simple grid graph $G = P_1 \boxtimes P_2$ with $P_1$ and $P_2$ of dimension $n_1$ and $n_2$ respectively. Let, for each $\ell \in \{1, 2\}$, $n_\ell = 2^{\nu_{\ell}} \cdot \prod_{\ell=1}^{p_{\ell}} p_{\ell, \nu_{\ell}}$ be a prime number factorization for some $k_2 \in \mathbb{N}$ and odd prime numbers $p_{1, 1}, \ldots, p_{k_2}$.

Let $I_0 = \{i_1, \ldots, i_p\}$ be a set of observation nodes with $i_\alpha = [(i_{\alpha 1}, i_{\alpha 2})], \alpha \in \{1, \ldots, p\}$. Now, we construct $p$ sets $O_1, \ldots, O_p$ that will be used to define a simple rule for observability. For the sake of clarity we provide the rule to construct a set $O_\alpha$ for a specific case.

The general case can be easily deduced from the example. Suppose that $i_j$ satisfies condition (5) for $p_{1, 1}$ and $p_{1, 2}$ along direction 1 and for $p_{2, 3}$ along direction 2. Now, define the set $O_\alpha$ as follows.

\[O_\alpha = \{p_{1, 1}, p_{2, 1}\} \cup \ldots \cup \{p_{1, 1}, p_{2, k_2}\} \cup \{p_{1, 2}, p_{2, 1}\} \cup \ldots \cup \{p_{1, 2}, p_{2, k_2}\} \cup \{p_{1, 2}, p_{2, 3}\} \cup \{p_{1, 3}, p_{2, 1}\} \cup \ldots \cup \{p_{1, 3}, p_{2, k_2}\}.
\]

We call $O_1, \ldots, O_p$ an observability partition of the set $I_0$.

The following proposition follows from the main theorem. The proof is straightforward and thus omitted.

**Proposition 4.6.** Let $G = P_1 \boxtimes P_2$ be a simple grid graph and $I_0$ a set of observation nodes with observability partition $O_1, \ldots, O_p$. Then $G$ is observable from $I_0$ if and only if $O_1 \cap \ldots \cap O_p = \emptyset$.

The following examples can be easily explained by using the proposition above. If at least one of the observation nodes, say $i_1$, is observable in any direction, then the grid is observable. Indeed, the set $O_1$ will be empty. If all $n_\ell \in \{1, \ldots, d\}$, are prime, then the grid is observable if and only if one of the observation nodes is observable in any direction. Indeed, any $O_\alpha$, $\alpha \in \{1, \ldots, p\}$, can be either $O_\alpha = \{i_1, \ldots, i_p\}$ or $O_\alpha = \emptyset$.

Next, we show a graphical interpretation of the observability test based on the observability partition. We present it through an example. In Figure 1 we show a two dimensional grid of length $7 \times 15$. It can be easily tested that this grid is simple. In each direction $\ell \in \{1, 2\}$, for each prime factor of $n_\ell$ we associate a unique symbol to the rows (columns) of nodes that satisfy (5) for that prime number (in that direction). In particular, for the grid in Figure 1, we associate a cross to the columns satisfying (5) for the prime factor 5 of 15, a triangle the columns satisfying (5) for the prime factor 3 of 15, and a pentagon to the unique row satisfying (5) for the prime number 7.

Clearly all the nodes that are not crossed by any line are observable. Also, a subset of nodes from which the graph is observable can be easily constructed by suitably combining the different symbols. Equivalently, given a set of observation nodes, testing the associated symbols easily gives the observability property from the given set of nodes. We let the reader play with it and have fun.
4.3 Observability of simple torus graphs

To characterize the observability of a simple torus graph, we start with a negative result that gives the minimum number of observation nodes to get observability.

Proposition 4.7. A simple $d$-dimensional torus graph is not observable from any set of nodes with cardinality less than $2^d$. □

The previous result says that in an $d$-dimensional torus at least $2^d$ observation nodes are needed. In the next proposition we provide an important design result. Namely, we prove that $2^d$ nodes are in fact sufficient to obtain observability and provide a strategy to choose the observation nodes.

Proposition 4.8. Given a simple $d$-dimensional torus graph $T = C_1 \Box \ldots \Box C_d$, there exists a set of $2^d$ observation nodes such that the graph is observable. A set of observation nodes achieving the result is given by

$$I_o := \{(i_1)_1, \ldots, (i_1)_d, \ldots, (i_p)_1, \ldots, (i_p)_d\}$$

$$= \{(i_1^1, i_1^2, \ldots, i_1^p), \ldots, (i_d^1, i_d^2, \ldots, i_d^p)\}$$

i.e., the set of all possible combinations of nodes such that the $\ell$-th element is either $i_1^\ell$ or $i_d^\ell$, where the nodes $i_1^\ell$ and $i_d^\ell$ guarantee observability on the cycle $C_\ell$. □

Next lemma provides necessary conditions for observability of a torus from a set of observation nodes based on the observability of its constitutive cycles.

Proposition 4.9. Let $C_1, \ldots, C_d, d \in \mathbb{N}$, be cycle graphs of length respectively $n_1, \ldots, n_d$ and let $T = C_1 \Box \ldots \Box C_d$ be a simple torus. The torus graph $T$ is not observable from a set of nodes $I_o = \{i_1, \ldots, i_p\}$, with $i_\alpha = ([i_{\alpha1}], \ldots, [i_{\alpha d}])$, $\alpha \in \{1, \ldots, p\}$, if $p < 2^d$ or there exists $\ell \in \{1, \ldots, d\}$ such that $C_\ell$ is not observable from $I_o^\ell = \{(i_{\ell1}), \ldots, (i_{p\ell})\}$. □

5. CONCLUSIONS

In this paper we have characterized the observability (by duality the reachability) of a suitable class of grid and torus graphs in terms of simple algebraic rules from number theory. In particular, we have shown what are all and only the unobservable set of nodes and provided simple routines to choose a set of observation nodes that guarantee observability.