Studying topological characteristics of digital images is a fundamental issue in image analysis and understanding. In the present paper we first propose a linear time constant-working space algorithm for determining the genus of a connected digital image. The computation is based on a combinatorial relation for digital images that may be of independent interest as well. We also propose definitions of dimension for planar digital images. These definitions serve as an alternative to the one proposed by Mylopoulos and Pavlidis\cite{1}, and make up some of its shortcomings. We study various properties of the so-defined image dimension, in particular, characterization of dimension in terms of Euler characteristic. We also show that image dimension can be found within linear time and constant memory.

Keywords: Digital topology, digital image, hole, connected component, genus, dimension, constant working-space algorithm

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1. Introduction

Studying different characteristics and properties of images is a fundamental issue in image analysis and understanding. Equally important are various geometric features of a given image characterizing its shape (such as convexity, presence of concavities, points of inflection, etc.), metric characteristics (such as curvature, curve length or region area), as well as the topological characteristics of the image. Defining and extracting such features is performed for continuous images, as well as for their digital counterparts. These can be digital images obtained, for example, by digitization of real-world images, or synthetic digital images obtained on the basis of certain definition through some computational process. In the present paper we study some topological properties of images of the last type. Specifically, we are concerned with the genus of a given digital image and its dimension.

The genus of an image is a basic topological invariant providing important information about the image topology (e.g., the degree of its connectedness). Therefore, important applications are expected in image analysis, computer vision, computer graphics, as well as the design of build-in software. Investigations on related problems (for example, about Euler’s number computation) have been already carried out (see, e.g., \cite{2}). To our knowledge of the available literature, however, the present article provides for the first time rigorous mathematical analysis of the space and time complexity of the computation.

We propose a constant-working space algorithm for determining the genus of a digital image. More precisely, given an $m \times n$ binary array representing the image, we show how one can count the number of holes of the array with an optimal number of $O(mn)$ integer arithmetic operations and optimal $O(1)$ working space. Our considerations cover the two basic possibilities for image and hole types determined by the adjacency relation adopted for the image and for the background, i.e.: (a) 0-connected image with 1-connected holes; (b) 1-connected image with 0-connected holes.

Dimension is another fundamental concept in topology. It is a topological invariant \cite{4} and plays an important role in defining and studying properties of basic geometric images, such as curves and surfaces \cite{5}.

In digital topology the notion of dimension has attracted comparatively little attention, unlike some other topological notions (such as connectivity, tunnels, gaps, cavities, and others, see, e.g., \cite{6,3,7,8}). In fact, still in 1971 Mylopoulos and Pavlidis \cite{1} provided a definition of dimension for subsets of discrete spaces and, to our knowledge, it is the only one available in the literature. Recently Brimkov and Klette \cite{9} applied that definition for defining digital curves and hypersurfaces. Overall, however, it has not been used very often in digital geometry.

Here we expose some shortcomings of the Mylopoulos-Pavlidis definition, for

\footnote{0/1-connectedness in the grid-cell model correspond to 8/4-connectedness in the grid-point model, see \cite{3}.}
instance the fact that there may be 3-dimensional images in a 2-dimensional digital space. Then we propose an alternative definition for the case of planar digital images. Our definition makes up the “defects” of the one from 1 and implies dimensionality properties analogous to those familiar from classical topology. In particular, it makes possible to define a digital curve as a one-dimensional “digital” continuum (see Section 4.5), that parallels the classical definition of a curve proposed by Urysohn 10 and Menger 11. We also provide characterization of dimension in terms of Euler characteristic, which is another basic topological invariant. Moreover, we show how dimension of a digital image can be found in linear time and constant memory. Finally, we discuss on possible applications to the theory of digital curves.

The paper is organized as follows. In the next section we introduce some basic notions and results to be used in the sequel. In Section 3 we consider the problem of computing the genus of a digital image. The computation is based on some technical results about gaps in digital images, which may be of independent interest. In Section 4 we define and study properties of dimension of digital images. In particular, characterization of dimension in terms of Euler characteristic is provided, and efficient computation of dimension of a digital image is described. We conclude with some remarks in Section 5.

2. Preliminaries

In this sections we introduce some basic notions of digital geometry to be used in the sequel. We also introduce and briefly discuss on the notion of space-efficient algorithm.

2.1. Basic Definitions of Digital Geometry

We conform to the terminology used in 3 (see also 12,13,6,14).

All considerations take place in the grid cell model that consists of the grid cells of $\mathbb{Z}^2$, together with the related topology. In the grid cell model we represent pixels as squares, called 2-cells. Their edges and vertices are called 1-cells and 0-cells, respectively. For every $i = 0, 1, 2$ the set of all $i$-cells is denoted by $C_2^{(i)}$. Further, we define the space $C_2 = \bigcup_{i=0}^{2} C_2^{(i)}$. We say that two 2-cells are 0-adjacent (1-adjacent) if $e \cap e' \in C_2^{(0)}$ (resp., $e \cap e' \in C_2^{(1)}$). The relation of 0-adjacency (resp., 1-adjacency) is denoted by $A_0$ (resp., $A_1$). Given a 2-cell $p$, by $A_0(p)$ and $A_1(p)$ we denote the $A_0$ and $A_1$ adjacency neighborhood of $p$, respectively, that are the sets of all 2-cells that are 0-adjacent (resp. 1-adjacent) to $p$. We also denote by $A_0(p) = A_0^*(p) \cup \{p\}$, $A_1(p) = A_1^*(p) \cup \{p\}$, $\alpha = 0 \text{ or } 1$, the incidence neighborhood of $p$. We denote by $A_0$ and $A_1$ the relations of 0-incidence and 1-incidence, respectively, that can directly be defined by the corresponding incidence neighborhoods, as follows: $eA_\alpha e'$ if $e \in A_\alpha(e')$, where $\alpha \in \{0, 1\}$.

The grid cell model can also be viewed as an abstract cell complex $(C_2, <, \dim)$ (see 15). Here $<$ is a bounding relation, that is antisymmetric, irreflexive, and transi-
tive, and such that for every \( e, e' \in C_2 \), \( e < e' \) if and only if \( e\hat{e}e' \), where \( I = A_0 \cup A_1 \), and \( \text{dim}(e) < \text{dim}(e') \). Hence \( < \) is a strict partial order on \( C_2 \) and the corresponding order topology \( \tau(<) \) is called the grid cell topology. In this topology the basic open sets (i.e., the open sets of the base) are precisely the sets \( U \subseteq C_2 \), such that, for every \( u \in U \) and every \( v \in C_2 \) with \( u < v \), we have \( v \in U \).

A digital image (or digital picture, binary picture) \( D \) is any finite set of pixels in \( C_2^{(2)} \).

Next we recall some other notions of digital geometry. Let \( \alpha = 0 \) or \( 1 \). An \( \alpha \)-path in a digital image \( D \) is a sequence of pixels from \( D \) such that every two consecutive pixels are \( \alpha \)-adjacent. Two pixels are \( \alpha \)-connected if there is an \( \alpha \)-path between them. A digital image \( D \) is \( \alpha \)-connected if there is an \( \alpha \)-path joining any two pixels of \( D \). Otherwise \( D \) is \( \alpha \)-disconnected. Clearly, an image may be \( 0 \)-connected but not \( 1 \)-connected.

Let \( M \) be a subset of a binary picture \( S \). If \( S \setminus M \) is not \( k \)-connected, then the set \( M \) is said to be \( k \)-separating in \( S \). Now let \( M \) be a finite set of pixels that is \( k \)-separating in \( C_2 \) (note: \( k = 0 \) or \( k = 1 \)).

The maximal (by inclusion) \( k \)-connected subsets of a digital image \( S \) are called \( k \)-(connected) components of \( S \). Components are nonempty, and distinct \( k \)-components are disjoint.

### 2.2. Space-Efficient Algorithms

In recent years, developing specialized algorithms and software for intelligent peripherals is becoming increasingly important. This is crucial for peripherals used in image analysis and processing—such as digital cameras, scanners, or printers—where the size of the problem input may be huge. At the same time, such kinds of devices are equipped with considerably lower working space than the usual computers. Therefore, a hot topic of research is the design of space-efficient algorithms, that is, ones working within a limited amount of memory.

Of special interest are algorithms whose working space size is limited by a constant (preferably, not too large). Some general terminology and theoretical foundations have already been introduced in relation to the work on several specific problems (see, for example, \( 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28 \), among others). A number of diverse computation models have been considered. For a short discussion on these and related matters the reader is referred to \( 16, 17 \). For example, in the so-called in-place algorithms, the input data are given by a constant number of arrays that can be used (under some restrictions) as working space for the algorithm. Another (more restrictive) model assumes that the input data are given as a read-only array whose values cannot be changed during the algorithm’s execution, and the algorithm can use working space of constant size, i.e., that does not depend on the input size\(^b\). In this paper we conform to this last model of actual constant-working space.

\(^b\)Strictly speaking, in complexity theory algorithms with constant-working space correspond to
space algorithms.

3. Holes and Genus

3.1. Problem Statement

Let $S$ be a finite digital image in $\mathbb{C}_2$ and $\bar{S}$ its complement to the whole space $\mathbb{C}_2$. It is clear (and well-known) that $\bar{S}$ has exactly one infinite connected component with respect to an adjacency relation $A_k$ ($k = 0$ or $1$) and, possibly, a number of finite components. The latter are called $k$-holes of $S$, or also connectivity of $S$. See Figure 1.

The number of holes of an image $A \subseteq \mathbb{C}_2$ (or $A \subseteq \mathbb{R}^2$) is equivalent to the genus of $A$, which is the minimal number of “cuts” that makes the set simply connected (i.e., homeomorphic to the unit disc). Holes and genus are also defined for any set $A \subseteq \mathbb{R}^n$ as well as for digital images in any dimension. For detailed accounting of the matter the reader is referred to 3.

Note that different adjacencies may be used in defining the connectedness of an image and its background (e.g., $A_0$ for $S$ and $A_1$ for $\bar{S}$ or vice-versa). In fact, this is often the preferred approach, since 0- and 1-adjacencies form a good pair of adjacency relations. Basically, good pairs characterize separation of the image holes from the infinite background. For details and discussion about the usefulness of this notion we refer to 9, 30, 3, 6. Here we mention it only to justify the consideration of the following two meaningful cases:

(A) Given a 0-connected digital image $S$, one looks for the number of all 1-holes of $S$ (i.e., the number of the finite 1-connected components of $\bar{S}$).

(B) Given a 1-connected digital image $S$, one looks for the number of all 0-holes of $S$ (i.e., the number of the finite 0-connected components of $\bar{S}$).

See for illustration Figure 1.

In Section 3.3 we show how one can solve the above problems in linear time and constant-working space. Before that, in the next section we prove some subsidiary technical results.

3.2. Subsidiary Technical Results

Let $S$ be a digital image consisting of $p$ pixels (0-cells). Denote by $P_S$ the rectilinear polygon obtained as a union of all 2-cells of $S$. Let $\partial(P_S)$ be the boundary of $P_S$.

Following 31, a vertex (0-cell) or an edge (1-cell) of $S$ is called free iff it belongs to $\partial(P_S)$. Otherwise it is called non-free. (See Figure 2, left.) Let $v$ and $e$ be the
number of vertices and edges of \( S \), and let \( v = v^* + v', e = e^* + e' \), where \( v^*, v' \), \( e^* \), and \( e' \) denote the number of the free vertices, non-free vertices, free edges, and non-free edges of \( S \).

Denote by \( B \) and \( b \) the number of \( 2 \times 2 \) and \( 2 \times 1 \) blocks in \( S \), respectively (see Figure 2, middle). It is easy to see that we have the following equalities:

\[
v' = B, \quad e' = b
\]  

We say that \( S \) has a gap located at a vertex (0-cell) \( x \in S \cap \mathbb{Z}_2^0 \) if there are exactly two strictly 0-adjacent pixels \( p_1, p_2 \in S \) with \( p_1 \cap p_2 = x \) (see Figure 2, right).

Denote by \( g \) the number of gaps in \( S \). The following fact was proved in \(^{31}\).

**Proposition 3.1.** Let \( S \) be a digital image. Then

\[
g = e^* - v^*,
\]

where \( e^* \) and \( v^* \) denote the number of free edges and free vertices of \( S \), respectively.
In the above theorem is proved by induction. For this, all possible \(3 \times 3\) different configurations of pixels have been examined, which made the proof too long. Here we provide a much shorter proof based on graph-theoretic approach. Note that both proofs apply to images that have arbitrarily many components.

**Proof of Proposition 3.1**

Every boundary vertex of \(S\) is incident to either two or four boundary edges, the latter being the case if the vertex locates a gap; otherwise, the former case holds.

Since each boundary vertex is incident to an even number of edges, there must be an Eulerian cycle consisting of free edges belonging to \(\partial(P_S)\), as each such edge is used in the cycle exactly once (Figure 3). Along that cycle every edge and every vertex that does not expose a gap are counted once, while free vertices that expose gaps are counted twice. Since edges and vertices alternate on the cycle, we obtain \(e^* = v^* + g\), which completes the proof.

Proposition 3.1 implies the following corollary.

**Corollary 3.1.** Let \(S\) be a digital image. Then the following combinatorial relation holds:

\[
h - c + p - b + B - g = 0,
\]

where \(h\) is the number of holes, \(c\) is the connectivity, \(p\) is the number of pixels, \(b\) is the number of \(2 \times 1\) blocks, \(B\) is the number of \(2 \times 2\) blocks, and \(g\) is the number of gaps of \(S\).

**Proof:** Consider the planar graph \(G_S(V, E)\) defined as follows: (i) The elements of the set \(V\) of vertices of \(G_S\) are labeled by the vertices (0-cells) of \(S\), and (ii) The edges of \(G_S\) are the edges (1-cells) of \(S\).
Applying to $G_S$ the well-known Euler’s formula, we obtain

$$v - e + f = 1 + c,$$

where $f$ is the number of faces of $G_S$.

Now we observe that, in fact, $f$ counts the pixels of $S$ and its holes, plus one more unit for the infinite background of $S$, i.e., we have

$$v - e + p + h + 1 = 1 + c.$$

Since $e = e^* + e'$ and $v = v^* + v'$, we obtain

$$v^* + v' - e^* - e' + p + h = c.$$

Then equality (3) follows by substitution from (1) and (2).

We will use the above corollary in the computation of the number of holes in a digital image.

### 3.3. Counting Holes

Combinatorial relation (3) suggests that in order to compute the number of holes of a connected digital image $S$, it suffices to know the number of pixels in $S$ and to compute the number $b$ of $2 \times 1$ blocks, the number $B$ of $2 \times 2$ blocks, and the number $g$ of gaps. This can be done as follows.

Given an input binary array $A[m,n]$, one scans it row by row and counts the parameters of interest. This can be achieved, e.g., by the following procedure.

```plaintext
procedure CountHole();
    p := 0; b := 0; B := 0; g := 0;
    for j := 0 to n + 1 do A[m + 1, j] := 0;
    for i := 1 to m do A[i, n + 1] := 0;
    for i := 1 to m do A[i, 0] := 0;
    for i := 1 to m do
        for j := 1 to n do
            if A[i, j] = 1 then
                p := p + 1; c := 0;
                if A[i + 1, j] = 1 then b := b + 1; c := c + 1;
                if A[i, j + 1] = 1 then b := b + 1; c := c + 1;
                if A[i + 1, j + 1] = 1 then
                    if c = 2 then B := B + 1;
                    elseif c = 0 then g := g + 1;
                    if A[i + 1, j - 1] = 1 and A[i + 1, j] = 0 and A[i, j - 1] = 0 then g := g + 1;
                end
            end
        end
    end
end
```
Let us first consider version (A) of the problem where $S$ is 0-connected. We are looking for the number $h$ of 1-holes of $S$. Having $p$, $b$, $B$, and $g$ computed, we find the number of holes by a direct application of formula (3) with $c = 1$:

$$h = 1 - p + b - B + g$$  \hspace{1cm} (4)

Let us now consider version (B) of the problem where $S$ is 1-connected. We are looking for the number $h$ of 0-holes of $S$. With the help of Figure 4, we can easily realize that:

- If $q$ is a 0-connected hole of $S$ and $q$ features $m$ gaps, then it consists of $m + 1$ one-connected components (see Figure 4, left);
- If $q$ is a 0-connected set of cells of $\bar{S}$ with $m$ gaps, that is 0-connected to the infinite background, then clearly $q$ is not a 0-hole of $S$ (see Figure 4, right).

Since 1-connected images have no 0-gaps, by (4) the number 0-holes of $S$ is

$$h = 1 - p + b - B.$$  \hspace{1cm} (5)

The procedure described above is linear in the number of pixels of $S$. The working space consists of the fields in which one stores the current values of the few counters used in formulas (4) and (5).

As a last remark we would like to mention a recent work by Asano and Buzer, that provides an $O(n^2 \log n)$ time constant-working space algorithm for computing the 1-connected components of a digital image of size $n \times n$. The algorithm works
under the same model adopted in the present paper. Note: that algorithm may require $\Omega(n^2 \log n)$ operations even in the case of a single component. Combining the above-mentioned result from \cite{19} and those of the present paper, one obtains an $O(n^2 \log n)$ time constant-working space algorithm for computing the number of holes of a not necessarily connected digital image.

4. Dimension in Digital Spaces

In this section we propose a new definition of dimension in digital spaces and study various related properties. In particular, relations with the Euler number of a digital image are considered. We also provide a time and space efficient algorithm for computing the dimension of a digital set.

4.1. Review of Mylopoulos-Pavlidis Theory of Dimension

Mylopoulos and Pavlidis \cite{1} proposed definition of dimension of a (finite or infinite) set of $n$-cells $D \subseteq \mathbb{C}_n$ with respect to an adjacency relation $A_n$ (see \cite{32} for more details; for its use see also \cite{3}).

Let $\overline{A}_n(c)$ be the union of $A_n(c)$ with all $n$-cells $c'$ for which there exist $c_1, c_2 \in A_n^*(c)$ such that a shortest $\alpha$-path from $c_1$ to $c_2$ not passing through $c$ passes through $c'$. Note that for $n = 2$ we have $\overline{A}_1(c) = \overline{A}_0(c) = A_0(c)$.

We also denote $\overline{A}_n(c) = \overline{A}_n(c) \setminus \{c\}$.

A nonempty set $D \subseteq \mathbb{C}_n$ is called totally $\alpha$-disconnected if $A_n^*(x) \cap D = \emptyset$ for each $x \in D$.

$D \subseteq \mathbb{C}_n$ is called linearly $\alpha$-connected whenever $|A_n^*(x) \cap D| \leq 2$ for all $x \in D$ and $|A_n^*(x) \cap D| > 0$ for at least one $x \in D$.

**Definition 4.1.** Let $D$ be a digital image and $A_n$ an adjacency relation on $\mathbb{C}_n$. The *dimension* $\dim_n(D)$ is defined as follows:

1. $\dim_n(D) = -1$ if and only if $D = \emptyset$,
2. $\dim_n(D) = 0$ if $D$ is a totally $\alpha$-disconnected nonempty set (i.e., there is no pair of cells $c, c' \in D$ such that $c \neq c'$ and $\{c, c'\}$ is $\alpha$-connected),
3. $\dim_n(D) = 1$ if $D$ is linearly $\alpha$-connected,
4. $\dim_n(D) = \max_{c \in D} \dim_n(\overline{A}_n(c) \cap D) + 1$ otherwise.

We will call a 2-*block* any $2 \times 2$ square of pixels, a 1-*block* any $2 \times 1$ rectangle of pixels, and an L-*block* any 2-block with exactly one pixel missing (see Figure 5a,b).

The following characterization of 2-dimensionality in $\mathbb{C}_2$ was given in \cite{3}:

**Proposition 4.1.** $M \subseteq \mathbb{C}_2$ is two dimensional with respect to adjacency relation $A_n$ if and only if:

\[\text{For higher dimensions these sets may not coincide; e.g., for } n = 3, \text{ we have } \overline{A}_2(c) = \overline{A}_1(c) = A_1(c) \text{ and } \overline{A}_0(c) = A_0(c) \neq A_1(c), \text{ i.e., } \overline{A}_2(c) \neq \overline{A}_0(c).\]
For $\alpha = 0$, $M$ contains an $L$-block as a proper subset;
For $\alpha = 1$, $M$ contains a 2-block as a proper subset.

The above proposition suggests that a 2-dimensionality of a digital image is equivalent to existence of $L$- (resp. 2-) blocks in a digital image. Note however that, according to Definition 4.1, an $L$-block (resp. 2-block) itself, is one-dimensional with respect to $A_0$ (resp. $A_1$). To us, this is a shortcoming of this definition.

Another “defect” of the definition, which seems to be even more serious to us, is that a digital image in the two-dimensional digital space $C_2$ may have dimension three! This can be easily seen if we apply Definition 4.1 to an image that contains a $(3 \times 3)$-block (see Figure 5c (right)).

With the above examples in mind, in the next section we provide definitions of dimension, through which the above problems are resolved.

4.2. Properties of Dimension of Digital Sets

Definition 4.2. Let $D$ be a digital image and let the space $C_2$ be equipped with an adjacency relation $A_\alpha$, $\alpha \in \{0, 1\}$. The dimension of $D$ relative to $A_\alpha$ adjacency is denoted by $\dim_\alpha(D)$ and defined as follows:

1. $\dim_\alpha(D) = -1$ if $D = \emptyset$;
2. $\dim_\alpha(D) = 0$ if $D$ is totally $\alpha$-disconnected;
3. $\dim_\alpha(D) = 1$ if: $\alpha = 0$ and $D$ is not totally $\alpha$-disconnected and does not contain any $L$-block; or $\alpha = 1$ and $D$ is not totally $\alpha$-disconnected and does not contain any 2-block;
4. $\dim_\alpha(D) = 2$ otherwise (more precisely, if $\alpha = 0$ and $D$ contains at least one $L$-block or $\alpha = 1$ and $D$ contains at least one 2-block).

Remark 4.1. Note that in Definition 4.2, points (3) and (4) can be reformulated by using mathematical morphology. (Remember that in these cases $D$ is not totally $\alpha$-disconnected.) More precisely, we can define $\dim(D) = 1$ iff $\alpha = 1$ and
\[ \epsilon_B(D) = D \ominus B = \emptyset \] where the structuring element \( B \) is a 2-block, or \( \alpha = 0 \) and 
\[ \bigcup_{i=1}^{4} \epsilon_{L_i}(D) = \bigcup_{i=1}^{4} D \ominus L_i = \emptyset \] where \( L_i, i = 1, \ldots, 4, \) represents all possible L-blocks (see Figure 5b displaying the four possible L-blocks). Further, \( \dim(D) = 2 \) iff \( \alpha = 1 \) and \( \epsilon_B(D) \neq \emptyset \), or \( \alpha = 0 \) and \( \bigcup_{i=1}^{4} \epsilon_{L_i}(D) \neq \emptyset \).

In order to give a sort of “local” characterization of dimension, we now define dimension of a point of a digital image \( D \).

**Definition 4.3.** Let \( D \) be a nonempty digital image and \( p \in D \). The local dimension of \( p \) within \( D \) with respect to \( A_0 \) is denoted \( \dim_0(p, D) \) and defined as follows:

1. \( \dim_0(p, D) = 0 \) if \( A_0(p) \cap D = \emptyset \);
2. \( \dim_0(p, D) = 1 \) if \( A_0(p) \cap D \) is totally 0-disconnected;
3. \( \dim_0(p, D) = 2 \) otherwise (i.e., if \( A_0(p) \cap D \) is not totally 0-disconnected).

**Lemma 4.1.** Let \( D \) be a nonempty digital image and \( p \in D \). Then \( \dim_0(p, D) = 2 \) iff \( p \) belongs to an L-block in \( D \).

**Proof** Let \( \dim_0(\tilde{p}, D) = 2 \), i.e., \( A_0(p) \cap D \) is not totally 0-disconnected. Then there are at least two pixels \( p_1, p_2 \in A_0(p) \cap D \) such that \( (p_1, p_2) \in A_0 \). Up to symmetries with respect to \( \tilde{p} \), there are only two possible cases: \( (p_1, p_2) \in A_1 \) or \( (p_1, p_2) \in A_0 \setminus A_1 \). In both, the pixels \( \tilde{p}, p_1, p_2 \) form an L-block (see Figure 6a). Conversely, if \( p \) belongs to an L-block, then \( A_0(p) \cap D \) is not totally 0-disconnected and, by Definition 4.3, \( \dim_0(p, D) = 2 \).

**Remark 4.2.** As a consequence of Lemma 4.1, we have that if in a digital image there is a pixel of dimension 2, then the same image must contain at least two other distinct pixels of the same dimension.

The next proposition characterizes 0-dimensionality in terms of local 0-dimensionality.

**Proposition 4.2.** Let \( D \) be a nonempty digital image. Then \( \dim_0(D) = \max\{\dim_0(p, D) : p \in D\} \).
Proof First let us suppose that \( \dim_0(D) = 0 \), i.e., \( D \) is totally 0-disconnected. This is equivalent to saying that for every \( p \in D \), \( A_0(p) \cap D = \emptyset \), that is, \( \dim_0(p, D) = 0 \). Hence \( \max\{\dim_0(p, D) : p \in D\} = 0 \).

Now suppose that \( \dim_0(D) = 1 \), i.e., there is no \( L \)-block in \( D \). By Lemma 4.1, for every \( p \in D \), \( \dim_0(p, D) \leq 1 \) and at least one pixel has dimension one. Thus \( \max\{\dim_0(p, D) : p \in D\} = 1 \). Conversely, suppose that \( \max\{\dim_0(p, D) : p \in D\} = 1 \), i.e., for every \( p \in D \), \( \dim_0(p, D) \leq 1 \) and there exists some \( \tilde{p} \in D \) such that \( \dim_0(\tilde{p}, D) = 1 \). By Lemma 4.1 it follows that no pixel of \( D \) can belong to an \( L \)-block. So, by Definition 4.2, \( \dim_0(D) \leq 1 \). Since \( \dim_0(\tilde{p}, D) = 1 \), it follows that \( \dim_0(D) \neq 0 \). Hence, \( \dim_0(D) = 1 \).

Finally, let \( \dim_0(D) = 2 \). Then \( D \) contains at least one \( L \)-block \( L \). By Lemma 4.1, every pixel \( p \in L \) is such that \( \dim_0(p, D) = 2 \). Thus \( \max\{\dim_0(p, D) : p \in D\} = 2 \). Conversely, suppose that \( \max\{\dim_0(p, D) : p \in D\} = 2 \). Then there exists at least one pixel \( \tilde{p} \in D \) such that \( \dim_0(\tilde{p}, D) = 2 \) and, by Lemma 4.1, \( \tilde{p} \) belongs to some \( L \)-block \( L \). This implies that \( \dim_0(D) = 2 \).

We also have the following property.

**Proposition 4.3.** Let \( D \) be a nonempty digital image and let \( p \in E \subseteq D \). Then \( \dim_0(p, E) \leq \dim_0(p, D) \).

**Proof** Suppose by contradiction that \( \dim_0(p, E) > \dim_0(p, D) \). We have the following cases:

1. \( \dim_0(p, E) = 2 \) and \( \dim_0(p, D) \leq 1 \).
2. \( \dim_0(p, E) = 1 \) and \( \dim_0(p, D) = 0 \).

Consider first case 1. Since \( \dim_0(p, E) = 2 \), we have that \( A_0(p) \cap E \) is not totally 0-disconnected with respect to \( A_0 \). We have however that \( A_0(p) \cap E \subseteq A_0(p) \cap D \). This means that \( A_0 \cap (D) \) is not totally 0-disconnected with respect to \( A_0 \), too. Then \( \dim_0(p, D) = 2 \), which is a contradiction.

Now consider case 2. Since \( \dim_0(p, E) = 1 \), we have that \( A_0(p) \cap E \) is a nonempty totally 0-disconnected set. However, \( A_0(p) \cap E \subseteq A_0(p) \cap D \). So, \( A_0(p) \cap D \) cannot be the empty set. This implies that \( \dim_0(p, D) \neq 0 \), which is a contradiction. \( \square \)

Propositions 4.2 and 4.3 immediately imply the following.

**Corollary 4.1.** Let \( D \) be a nonempty digital image and let \( E \subseteq D \). Then \( \dim_0(E) \leq \dim_0(D) \).

Definition 4.3 and the related results allow us to give the following definition of local dimension with respect to \( A_1 \) adjacency.

**Definition 4.4.** Let \( D \) be a nonempty digital image and let \( p \in D \). The **local dimension** of \( p \) within \( D \) with respect to \( A_1 \) is the nonnegative integer \( \dim_1(p, D) = \dim_0(A_0(p) \cap D) \).
Lemma 4.2.

Let $D$ be a nonempty digital image and $p \in D$. Then $\dim_1(p, D) = 2$ if and only if $p$ belongs to a 2-block in $D$.

Proof Let $\dim_1(p, D) = 2$. Then by Definition 4.4 we have that $\dim_0(p, (A_0(p) \cap D)) = 2$. Therefore, by Definition 4.2, $A_0(p) \cap D$ contains at least one L-block $L$. This implies that $B = \{p\} \cup L$ is a 2-block of $D$ and $p$ a pixel of its. Now let $p$ belong to a 2-block. Then we have that $A_0(p) \cap D$ contains an L-block. Therefore $\dim_0(A_0(p) \cap D) = 2$. That is, by Definition 4.4, $\dim_1(p, D) = 2$. \qed

We also have the following characterization of dimension with respect to $A_1$ adjacency.

Proposition 4.4. Let $D$ be a nonempty digital image. Then $\dim_1(D) = \max\{\dim_1(p, D) : p \in D\}$.

Proof Suppose first that $\dim_1(D) = 0$. Then $D$ is totally 1-disconnected. This is equivalent to saying that for every $p \in D$, $A_1(p) \cap D = \emptyset$. Then, for every $p \in D$, we have that $A_0(p) \cap D$ is totally 0-disconnected, i.e., $\dim_0(A_0(p) \cap D) = 0$. In fact if, by contradiction, $A_0(p) \cap D$ was not totally 0-disconnected, then there should exist at least two 0-joined pixels $p_1, p_2 \in A_0 \cap D$. Wlog, consider pixel $p_1$. Clearly, it must belong to $A_0(p) \setminus A_1(p)$, since otherwise $A_1(p) \cap D \neq \emptyset$. However, $p_2$ is 0-adjacent to $p_1$. So, it must be in one of the positions $a$ or $b$ depicted in Figure 6b. Hence, $p_2 \in A_1(p) \cup D$, which is clearly a contradiction. All this implies that $\max\{\dim_0(A_0(p) \cap D) = \dim_1(p, D) : p \in D\} = 0$.

Now, let us suppose that $\dim_1(D) = 1$, i.e., that there is no 2-block contained in $D$. By Lemma 4.2, for every $p \in D$, $\dim_1(p, D) \leq 1$ and there is at least one pixel $\widetilde{p}$ such that $\dim_1(\widetilde{p}, D) = 1$. Thus $\max\{\dim_1(p, D) : p \in D\} = 1$. Conversely, suppose that $\max\{\dim_1(p, D) : p \in D\} = 1$. Then for every $p \in D$, $\dim_1(p, D) \leq 1$ and there exists some $\widetilde{p} \in D$ such that $\dim_1(\widetilde{p}, D) = 1$. By Lemma 4.2, it follows that no pixel of $D$ can belong to a 2-block. Therefore, by Definition 4.2, $\dim_1(D) \leq 1$. Since $\dim_1(\widetilde{p}, D) = 1$, $\dim_1(D) \neq 0$. So, $\dim_1(D) = 1$.

Finally, suppose that $\dim_1(D) = 2$. Then $D$ contains at least one 2-block $B$. By Lemma 4.2, every pixel $p \in B$ has dimension 2. Thus $\max\{\dim_1(p, D) : p \in D\} = 2$. Conversely, suppose that $\max\{\dim_1(p, D) : p \in D\} = 2$. Then there exists at least one pixel $\widetilde{p} \in D$ such that $\dim_1(\widetilde{p}, D) = 2$ and, by Lemma 4.2, $\widetilde{p}$ belongs to some 2-block $B$. This implies that $\dim_1(D) = 2$. \qed

Corollary 4.2. Let $D$ be a digital image. Then $\dim_1(D) \leq \dim_0(D)$.

Proof Let $p \in D$. Then $\dim_0(p, (A_0(p) \cap D)) \leq \dim_0(p, D) \leq \max\{\dim_0(p, D) : p \in D\} = \dim_0(D)$. Hence $\dim_0(D)$ is greater than each element of the set $\{\dim_0(p, A_0(p) \cap D) : p \in D\}$. So, $\dim_1(D) = \max\{\dim_1(p, D) : p \in D\} = \max\{\dim_0(p, (A_0(p) \cap D)) : p \in D\} \leq \dim_0(D)$. \qed

In a similar way one can prove the following properties that parallel well-known properties of dimension theory in $\mathbb{R}^n$. 
Proposition 4.5. Let $D$ be a nonempty digital image and $p \in E \subseteq D$. Then $\dim_1(p, E) \leq \dim_1(p, D)$.

Proposition 4.6. Let $D$ be a nonempty digital image and $E \subseteq D$. Then $\dim_1(E) \leq \dim_1(D)$.

Proposition 4.7. Let $D_1$ and $D_2$ be two mutually disjoint digital images. Then $\dim_\alpha(D_1 \cup D_2) = \max(\dim_\alpha(D_1), \dim_\alpha(D_2))$, where $\alpha \in \{0, 1\}$.

4.3. Dimension and Euler Characteristic

In this section we establish relations between dimension of digital images and their Euler characteristic. In combinatorial topology, Euler characteristic is a fundamental theoretic concept and basic topologic invariant. Recall that, given a subset $D$ of the abstract cell complex $(\mathbb{C}_2, \prec, \dim)$, its Euler characteristic is the number

$$\chi(D) = c_0 - c_1 + c_2, \quad (6)$$

where $c_i$ is the number of the $i$-dimensional cells of $D$, $i = 0, 1, 2$. In describing our results, we will also use the notion of a skeleton of a digital image, which we introduce next.

Definition 4.5. Let $D$ be a non-empty digital image and let the space $\mathbb{C}_2$ be equipped with an adjacency relation $A_\alpha$, $\alpha \in \{0, 1\}$. We call a skeleton of $D$ the graph $S_\alpha(D) = (V, E)$ (for short), whose set of vertices $V$ are labeled by the elements of $D$ (i.e., we may think that $V = D$), and, given two vertices $p$ and $q$, $(p, q) \in E \iff p$ and $q$ are $\alpha$-adjacent.

In what follows, we will characterize dimensionality in $\mathbb{C}_2$ with respect to $A_1$ adjacency, the characterization with respect to $A_0$ adjacency being similar. Because of Proposition 4.4, it is enough to consider the case of connected digital images. We have the following theorem.

Theorem 4.1. Let $D$ be a 1-connected digital image with a skeleton $S(D) = (V, E)$. In terms of the denotations in equality 6, $|V| = |D| = c_2$. Let $|E| = m$. Then the following holds:

1. $\dim_1(D) = -1$, if $c_2 = 0$
2. $\dim_1(D) = 0$, if $c_2 \neq 0$ and $m = 0$
3. $\dim_1(D) = 1$, if $c_2 > m > 0$
4. If $m = c_2 > 0$, then
   a. $\dim_1(D) = 1$ if $\chi(D) = 0$
   b. $\dim_1(D) = 2$ if $\chi(D) > 0$
5. If $m > c_2 > 0$,
   a. $\dim_1(D) = 1$ if $\chi(D) < 0$
   b. $\dim_1(D) = 2$ if $\chi(D) \geq 0$
Before proving the above theorem, we recall some well-known elementary properties from graph theory (see, e.g., [35]).

**Proposition 4.8.** Let $G$ be a connected graph with $n$ vertices and $m$ edges. We have that:

1. $n \leq m + 1$,
2. $G$ is a tree iff $n = m + 1$, or, equivalently, if $G$ has no cycle,
3. $G$ has a unique cycle $C_i$ of length $i \geq 3$ iff $m = n$.

We will also use the following simple lemma.

**Lemma 4.3.** Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then $G$ has at least two distinct cycles iff $m > n$.

**Proof** Suppose that $G$ has at least two cycles. Then, by Part 2 of Proposition 4.8, $G$ is not a tree, and $n < m + 1$. Since $G$ has at least two cycles, by Part 3 of Proposition 4.8, we have that $n \neq m$. Hence, $m > n$.

Conversely, if $m > n$, by Part 2 of Proposition 4.8 we have that $G$ is not a tree. Since $G$ is connected, it has at least one cycle. If we assume by contradiction that there are no other cycles, by Part 3 of Proposition 4.8 we will have $m = n$—a contradiction.

Let us also list the following fact.

**Lemma 4.4.** Let $D \subset \mathbb{C}^2$. Then

\[ c_0 - c_1 = c - h - c_2, \tag{7} \]

where $c$ and $h$ are the number of the (0-)connected components and (1-)holes of $D$, respectively.

Indeed, the following Euler-Poincaré result is well-known in combinatorial topology:

\[ \chi(D) = c_0 - c_1 + c_2 = \beta_0 - \beta_1 + \beta_2, \]

where $\beta_0, \beta_1$, and $\beta_2$ are the Betti numbers (see, e.g., [3]). These count respectively the number of connected components, tunnels, and cavities of a cell-complex. Since a plane digital image $D$ is homotopic to a one-dimensional CW-complex [36], we clearly have $\beta_2 = 0$, from where we get the result stated.

We are now ready to prove Theorem 4.1.

**Proof of Theorem 4.1** For Cases 1 and 2 the proof is trivial. We prove for the other three cases.

**Case 3.** Let $c_0 > m$. By Proposition 4.8 (Part 2), $S(D)$ is a tree. Suppose, by contradiction, that $\dim_1 D \neq 1$. Since $n > m > 0$, it necessarily follows that $\dim_1(D) = 2$, i.e., $D$ admits at least one 2-block. Hence, $S(D)$ has at least one cycle $C_4$, which contradicts the fact that $S(D)$ is a tree.
Case 4. Let \( c_0 = n \) and \( \chi(D) > 0 \). Suppose, by contradiction, that \( \dim_1(D) = 1 \). Then \( D \) contains no 2-blocks or, equivalently, \( S(D) \) contains no cycle \( C_4 \) as a subgraph. Then by Proposition 4.8 (Part 3), \( S(D) \) must have at least one cycle \( C_4 \), where \( i \) is an even number greater than or equal to 8.

Since \( D \) is connected, \( c = 1 \) and the equality of Lemma 4.4 becomes \( c_1 - c_0 = c_2 + h - 1 \) \((c_0, c_1, c, h \neq 0)\). Since \( c_1 - c_0 < c_2 \), we have that \( c_2 + h - 1 < c_2 \). It follows that \( h < 1 \), that is, \( h = 0 \), a contradiction.

Now let \( c_2 = m \) and \( \chi(D) = 0 \). Suppose, by contradiction, that \( \dim_1(D) = 2 \). So, \( D \) contains at least a 2-block and, consequently its skeleton has a \( C_4 \) subgraph.

As before, by Lemma 4.4, \( c_1 - c_0 = c_2 + h - 1 \). Since \( \chi(D) = 0 \), we obtain \( c_2 + h - 1 = c_2 \), i.e., \( h = 1 \). By Lemma 4.3, this implies that \( m > c_2 \), a contradiction.

Case 5. Let \( m > c_2 > 0 \) and \( c_0 - c_1 > 0 \). Suppose by contradiction that \( \dim_1(D) = 1 \). Then \( D \) has no 2-block and, consequently, \( S(D) \) has no cycle \( C_4 \) as a subgraph. Since \( m > c_2 \), we have by Lemma 4.3 that \( S(D) \) has at least two cycles different than \( C_4 \). Hence, \( h > 1 \). Since \( \chi(D) = c_0 - c_1 + c_2 = 0 \) and from Lemma 4.4, it follows that \( h \leq 1 \), a contradiction. \( \square \)

Remark 4.3. It is easy to see that Theorem 4.1 covers all possibilities. Thus a digital image \( D \) has dimension \( k \), \(-1 \leq k \leq 2\) only if \( D \) satisfies a corresponding condition of the theorem. Formally, the case “\( c_2 = m \) and \( \chi(D) < 0 \)” is not considered. However, it is easily verified to be non-admissible. In fact, by Proposition 4.8 (Part 3) we have that \( S(D) \) has a unique cycle. Here we distinguish two subcases.

Case (a): The unique cycle is \( C_4 \). We have \( c = 1 \) and \( h = 0 \). Case (b): The unique cycle is \( C_{2n} \), \( n > 3 \). In this case \( c = 1 \) and \( h = 1 \). So, in both cases \( 1 = c \geq h \), i.e., \( \chi(D) \geq 0 \). However, \( \chi(D) = c - h = 1 - h < 0 \), which implies \( 1 < h \), a contradiction.

Lemma 4.4 and Theorem 4.1 imply the following corollary.

Corollary 4.3. Let \( D \) be a 1-connected digital image whose skeleton \( S(D) \) has \( c_2 \) vertices and \( m \) edges. Then the following implications hold:

1. If \( c_2 > m \) then \( \dim_1(D) = 1 \).
2. If \( c_2 = m \) and \( h < 1 \) then \( \dim_1(D) = 2 \).
3. If \( c_2 = m \) and \( h = 1 \) then \( \dim_1(D) = 1 \).
4. If \( m > c_2 \) and \( h \leq 1 \) then \( \dim_1(D) = 2 \).
5. If \( m > c_2 \) and \( h > 1 \) then \( \dim_1(D) = 1 \).

Characterization of dimension under 0-adjacency is similar to one under 1-adjacency. The points from 1 through 4 of Theorem 4.1 can be reformulated also for 0-adjacency. Note that, since there may be 1-dimensional images with \( m > c_2 > 0 \) and \( \chi(D) < 0 \), the last item of the theorem changes as follows:

5'. If \( m > c_2 > 0 \) and \( \chi(D) \geq 0 \), then \( \dim_0(D) = 2 \).
4.4. **Time and Space Efficient Computation of Dimension**

It is not hard to see that, similar to counting holes, digital dimension admits time and space-efficient computation.

To test for 0-dimensionality, it suffices to check if a given (nonempty) digital image $S$ contains or does not contain a pixel that has a (0 or 1) neighbor. In the latter case, $S$ is 0-dimensional, while in the former it is either 1- or 2-dimensional.

With respect to 1-adjacency, the test for 2-dimensionality is provided by the procedure ContHoles, as it identifies and counts the $(2 \times 2)$-blocks (if any) in $S$. If $S$ contains $(2 \times 2)$-blocks, then it is two-dimensional. Otherwise, if $S$ is not 0-dimensional, then it is 1-dimensional.

With respect to 0-adjacency, one needs to test $S$ for existence of $L$-blocks, which can be done similar to counting gaps.

All these tests are linear in time and clearly required only constant amount of memory.

Obviously, Lemmas 4.1 and 4.2 allow us to compute the local dimension at a point with constant time and space.

4.5. **Application to Digital Curves and Surfaces**

Various definitions of digital curves and surfaces are available in the literature (see, e.g., 37). A short survey of these is found in 9, 30. This last paper provided the first definitions of digital curves and hypersurfaces involving the notion of dimension as introduced in 1. Recall that, since Urysohn 10 and Menger 11, a curve $\gamma \subset \mathbb{R}^2$ is
known to be a one-dimensional continuum\textsuperscript{e}. By analogy, in \textsuperscript{38} digital continuum has been defined as a nonempty, finite, and (α-)connected set of cells in a digital space. On that basis, a digital curve (with respect to a certain adjacency relation) has been defined in \textsuperscript{38} as a one-dimensional continuum. In other words, the classical Urysohn-Menger’s definition applies to the case of digital curves, as well. Figure 7 represents digital curves with respect to one- and two-adjacencies.

The modified definition of digital dimension presented in this paper in turn refines the above notion of a digital curve. We believe that this could be one more step towards developing a unified topological theory for both continuous and discrete spaces.

5. Concluding Remarks

In the present paper we have first proposed a linear time constant-working space algorithm for determining the genus of a connected digital image. The computation is based on a combinatorial relation for digital pictures that may also be of independent interest. The algorithm is also applicable to the case of digital images with more than one connected component, provided that the number of components is known in advance. A challenging task is seen in constructing an equally time and space efficient algorithm for the case of images with unknown connectivity. Extension to higher dimensions is another important direction of future research.

We have also proposed definitions of dimension for planar digital images. These definitions serve as an alternative to the one proposed by Mylopoulos and Pavlidis \textsuperscript{1}, and make up some of its shortcomings. We believe that the notion of image dimension in digital spaces will play an increasing role in theoretical research, helping to make notions and results of digital topology compatible with those from classical topology. Ongoing research is focused on extending the presented definitions and results to higher dimensions.

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References


\textsuperscript{e}Continuum in $\mathbb{R}^2$ is a nonempty subset of a topological space that is compact (closed and bounded) and topologically connected.
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