A weakening of Blum's Axioms for abstract computational complexity is introduced in order to take into a better account measures that can be finite even when the computations diverge. How the new axioms affect the theory and how they can be used to get an insight in the theory of computations using a finite amount of resource, is shown.

**INTRODUCTION**

The theory of abstract computational complexity is based on the axiomatization of the notion of amount of resource (time, tape, ink, etc.) required to perform a given computation on an abstract computing device [1].

Let \( \{\phi_i\}^\infty_{i=0} \) be an acceptable Gödel numbering [2] of partial recursive functions of one variable.

Let \( \Phi = \{\Phi_i\}^\infty_{i=0} \) be an infinite set of partial recursive functions of one variable. \( \Phi \) is said to be an acceptable measure of complexity if it satisfies the following axioms (usually referred to as "Blum's axioms"):

1. \((\forall i)(\forall x)[\Phi_i(x) \text{ is defined if and only if } \phi_i(x) \text{ is defined}].^1 \)
2. The relation \( \Phi_i(x) = n \) is recursive in \( i, x, n \).^2

Blum's axioms only under a convention fit some measures like the number of squares or the number of reversals required by a Turing machine (TM) computation. In fact,

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1 As synonymous we will use the notations:
   \( f \text{ defined}, f \text{ def}, f \text{ convergent}, f \text{ conv} \) and
   \( f \text{ undefined}, f \text{ undef}, f \text{ divergent}, f \text{ div} \).
2 As a convention we assume that if \( \Phi_i(x) \) is undefined, then \((\forall n)[\Phi_i(x) \nrightarrow n] \).
in these cases, the amount of resource required by the machine may be finite even if
the machine itself does not halt. The way Blum’s axioms are usually made to be
valid also for these measures is the following: Instead of defining

\[ \Phi_i(x) = \text{number of squares (reversals) required by the computation of } \varphi_i(x), \]

we define

\[ \Phi_i(x) = \begin{cases} 
\text{number of squares (reversals) required by the computation of } \varphi_i(x) & \text{if } \varphi_i(x) \text{ is defined; } \\
\text{divergent otherwise.} & \text{otherwise.}
\end{cases} \]

This is but one example of the situations where Blum’s axioms seem not to be fully
adequate to represent all possible measures of complexity (another example is provided
by the running time of programs with error condition). In other words, while Blum’s
axioms are adequate to study the complexity of properly halting programs, they don’t
allow the study of the behavior of simple but interesting programs like the programs
cycling on a finite amount of storage or halting under an error message.

The basic feature of this kind of situations is that exactly in these cases we are in
front of a subset of programs for which the halting problem is decidable. For example,
suppose we know that a TM with 1 tape, c characters (including blank), and s states
requires only n squares, then we start the machine and we let it run for at most
\( c^n \cdot s \cdot n \) steps \(^3\): If it doesn’t halt by that time we are able to assert that it will not halt
at all.

**Weaker Axioms for Abstract Computational Complexity**

In order to have a better representation of the aforesaid measures Blum’s axioms
can be modified as follows:

(A1a) \((\forall i)(\forall x)\) if \( \varphi_i(x) \) is defined, then \( \Phi_i(x) \) is defined; 

(A1b) there is a partial recursive function \( N \) of two variables such that

\[ (\forall i)(\forall x)\] if \( \Phi_i(x) \) is defined, then \( N(i, x) \) is defined

and \( N(i, x) = 1 \), if \( \varphi_i(x) \) is defined,

\[ = 0, \text{otherwise}; \]

(A2) the relation \( \Phi_i(x) = n \) is recursive in \( i, x, \) and \( n \).

\(^3\) This value comes from a very large evaluation, taking into account all possible configurations
on \( n \) tape squares, and can be strongly reduced.
As it can be easily seen Blum's axiom (2) is left unchanged while axiom (1) is split into two parts: The first part asserts that the definedness of a function implies the finiteness of the amount of resource required to perform its computation; the second part asserts that the finiteness of the amount of resource required to compute a function implies that we can effectively decide whether the function is defined or not.

**FACT 1.** *The axioms are independent.*

*Proof.* \( \Phi_i(x) = \varphi_i(x) \) satisfies (A1a) and satisfies (A1b) with
\[
N(i, x) = 1, \text{if } \varphi_i(x) \text{ is defined},
\]
\[
= \text{undefined otherwise},
\]
but does not satisfy (A2): 
\[
\Phi_i(x) = 0 \text{ satisfies (A1a) and (A2) but not (A1b),}
\]
\[
\Phi_i(x) \text{ always divergent satisfies (A1b) and (A2) but not (A1a).} \quad \text{Q.E.D.}
\]

**FACT 2.** *Axioms (A1a), (A1b), and (A2) are weaker than Blum's axioms.*

*Proof.* Immediate. \quad \text{Q.E.D.}

Any measure satisfying the weaker axioms will be called an acceptable weak measure of complexity, while any measure satisfying the old axioms will be called a strong measure.

The first question we ask about the weaker axioms is whether or not they preserve the main results of the theory, that is whether the most important machine independent results about properties of complexity measures are still valid or they become dependent on a class of measures satisfying not only the weaker axioms but also Blum's axioms.

The answer is substantially positive.

First of all, many relevant theorems have been proved under even weaker axioms: For example, Borodin's gap theorem [4] and Meyer–McCreight's union theorem are valid for the so-called measured sets, that is, infinite sets of partial recursive functions satisfying (A2). For this reason, the only interesting case is when the proofs of the theorems make use of Blum's axiom (1) in the form of the implication "\( \varphi_i(x) \) is defined if \( \Phi_i(x) \) is defined". Then two situations are the most interesting.

In the first one the statement of the theorem remains valid and we only need a slight modification of the argument to prove the result under the weaker axioms. This is so, for example, with Blum's speed-up and compression theorems [1] or in Theorem 3.1 in Ref. [6], where the aforesaid implication occurs all along the proof. When this happens we only need to change the implication in the following way,

\[
\quad \text{Q.E.D.}
\]

\* Of course machine dependent results, like the one given in Ref. [3], preserve their validity.
“ϕ_i(x) is defined if Φ_i(x) is defined and N(i, x) = 1” and the proof is carried on as before.

The second interesting situation is when the statement of the theorem has to be modified; then what really matters is that the “meaning” of the theorem is preserved. For example, this happens with the theorem asserting that two measures are recursively related: The former statement of the theorem was:

“Given any two measures Φ and Φ_i there is a function g ∈ R_2 such that (∀i) (∀x) [g(x, Φ_i(x)) ≤ Φ_i(x) and g(x, Φ_i(x)) ≤ Φ_i(x)]”; the new statement will be “Given any two weak measures Φ and Φ_i there is a function g ∈ R_2 such that (∀i) (∀x) [ϕ_i(x) def → g(x, Φ_i(x)) ≤ Φ_i(x) and g(x, Φ_i(x)) ≤ Φ_i(x)]” (The proof is straightforward). It is clear that the interesting part of the statement is preserved (that is: If a function is defined the amounts of resources that we need for its computation, as functions of the argument have the same behavior, modulo a recursive function) while the case in which the theorem is now unvalid is when the function is not defined (because under the new axioms one resource can be finite while the other one can be infinite).

**Structure of the Set of Programs of Finite Measure**

Let us consider the set of all functions whose computation requires a finite amount of resource.

**Definition 1.** $S_0 = \{(x, y) | \Phi_2(y) \text{ is defined}\}$.

**Definition 2.** $S = \{x | \Phi_2(x) \text{ is defined}\}$.

**Fact 3.** There is no acceptable weak measure Φ such that one of the following is true:

(i) $S_0$ or $S$ is recursive (in particular for no measure can be $S = N$);

(ii) $h$ is the index of a universal function (that is if (∀x)(∀y)[ϕ_h(⟨x, y⟩) = ϕ_2(y)]) and $\{x \mid \Phi_h(x) \text{ is defined}\}$ is recursive;

(iii) given any $i$ we can find a measure $Φ_i$ such that $Φ_i(x)$ is defined for all $x$’s.

**Proof.** Immediate. Q.E.D.

Fact 4 points out that no resource whose amount is an acceptable weak measure of complexity can be finite for any computation. Actually, we can prove more: There are programs diverging everywhere that for no value of the argument use a finite amount of resource.

**Fact 4.** $\exists i \in \mathbb{N} \text{ such that } (∀x)[ϕ_2(x) \text{ is divergent and } Φ_i(x) \text{ is divergent}]$. 
Proof. Let us define $s$ such that

$$\varphi_{s(i)}(x) = 0, \text{ if } N(i, x) = 0,$$

$$= \text{divergent otherwise.}$$

By recursion theorem there is an integer $e$ such that

$$\varphi_e(x) = \varphi_{s(i)}(x) = 0, \text{ if } N(e, x) = 0,$$

$$= \text{divergent otherwise.}$$

Then $\varphi_e(x) \text{ conv } N(e, x) = 0$, but also $\varphi_e(x) \text{ conv } \Phi_e(x) \text{ conv } N(e, x) = 1$ that is a contradiction. Hence $\varphi_e(x)$ must always be divergent. On the other side, $\varphi_e(x) \text{ div } \Phi_e(x) \text{ conv } N(e, x) = 0 \rightarrow \varphi_e(x) \text{ conv.}$ Hence also $\Phi_e(x)$ must always be divergent. Q.E.D.

Thus the structure of $S$ is not trivial and we are going to devote this section to its characterization.

**Theorem 1.** $S$ is a nonrecursive r.e. set.

Proof. We have already proved that $S$ cannot be recursive. In order to prove that it is r.e. let us define $f$ in the following way:

**Stage 0:** If $\Phi_0(0) = 0$ set $F = \{0\}$;

otherwise $F = \emptyset$.

**Stage $n$:** For every $i \leqslant n$ and $i \notin F$;

if $\Phi_i(i) = n - i$, set $F = F \cup \{i\}$,

otherwise $F = F$.

Then: $f(n) = (n + 1)$-th element of the list $F$.

Clearly, $f$ enumerates $S$. Q.E.D.

**Definition 3.** $K = \{x \mid \varphi_x(x) \text{ is defined}\}$.

**Theorem 2.** $S - K$ is r.e.

Proof. By slight modification of the proof of Theorem 1. Q.E.D.

**Remark.** This is intuitively clear in case of tape complexity for TM's: We can enumerate all programs cycling on $1, 2, ..., n, ...$ squares by “dovetailing”[2].

**Definition 4.** $A \equiv_T B$ if $A$ is $T$-reducible to $B$ and $B$ is $T$-reducible to $A$ ($A \triangleleft_T B$ and $B \triangleleft_T A$).
Theorem 3. $S \equiv_T K$.

Proof. $K \leq_T S$, in fact

$$e_k(x) = 1, \text{ if } e_s(x) = 1 \text{ and } N(x, x) = 1,$$

$$= 0, \text{ otherwise},$$

where $c_k$ is the characteristic function of $K$ if $c_s$ is the characteristic function of $S$. Since $K$ is $T$-complete (i.e., is a maximal degree) also $S$ is $T$-complete; hence $S$ and $K$ are equivalent. Q.E.D.

Remark. Theorem 3 does not give any deep insight in the structure of $S$ because under Turing reducibility, even $K$ and $\overline{K}$ are equivalent. The first question we ask is, therefore, is $S$ creative?

Definition 5. An r.e. set $A$ is creative if there is a partial recursive function $\psi$ such that

$$(\forall i)[Wi \subseteq A \rightarrow \psi(i) \in \overline{Wi} \cup A].$$

Theorem 4. $S$ is creative.

Outline of the proof. By implicit use of s-m-n theorem and recursion theorem we can define $\psi$ in the following way: For every $i$, $\psi(i)$ is equal to the index of the following program: “With input $Z$, compute the index of the program itself. Call it $j$. If $j \in W_i$, then converge; if $j \in S - K$, then converge; if $j \in K$, then diverge; in any other case diverge.”

Hence for every $i$ such that $W_i \subseteq S$ the only possibility that doesn’t bring to a contradiction is that $\psi(i) \in \overline{S} \cup W_i$ Q.E.D.

Corollary 5. $S$ is recursively isomorphic to $K$.

Proof. $S$ creative $\rightarrow S 1$-complete $\rightarrow S \equiv_1 K \rightarrow S \equiv K$. Q.E.D.

Remark. This result refines the information about the unsolvability degree of $S$ and tells us that we have effective ways to go from halting programs to programs using a finite amount of resource and viceversa.

Axiomatic Definition of TapeLike Measures

As we have seen, the weaker axioms, while providing a more adequate representation of measures for computational complexity preserve the validity of all the main results of the theory and, besides, allow us to extend our research to nonhalting programs with particularly interesting features.
As an example of the kind of results that we can achieve under the new axioms (though they were not even statable in the old axiomatic system) we can prove the following theorems in which well-known machine dependent facts are given an abstract formulation.

**Definition 6.** Two r.e. sets $A$ and $B$ are **recursively inseparable** if there is no recursive set $R$ such that $A \subseteq R$ and $B \subseteq R$, that is such that $A \subset R \subset B$.

**Fact 5.** Let $P = \{Z \mid \text{the TM } M_z \text{ cycles on a finite amount of tape when given its own index as input} \}$, $P$ and $K$ are recursively inseparable.

**Proof.** Suppose there is a recursive set $R$ such that $P \subseteq R \subseteq K$, then we could define

$$
\varphi_{\tau(i)}(x) = \begin{cases} 
\text{convergent, if } i \in R, \\
\text{cycling, if } i \notin R.
\end{cases}
$$

By recursion theorem there is an integer $j$ such that

$$
\varphi_j = \varphi_{\tau(i)}.
$$

Hence,

$$
\begin{align*}
&j \in R \rightarrow \varphi_j(j) \text{ convergent } \rightarrow j \in K \rightarrow j \notin R, \\
&j \notin R \rightarrow \varphi_j(j) \text{ cycling } \rightarrow j \in P \rightarrow j \in R.
\end{align*}
$$

Since both alternatives lead to a contradiction there is no such recursive set $R$. Q.E.D.

**Theorem 6.** Given any strong measure $\Phi$, there is a weak measure $\tilde{\Phi}$ such that $\tilde{\Phi} \subseteq K$ and $K$ are recursively inseparable.

**Proof.** Let $\Phi$ be a measure satisfying Blum’s axiom (1), i.e., $(\forall i)(\forall x)[\Phi_i(x) \text{ is defined iff } \varphi_i(x) \text{ is defined}]$.

Let $\sigma$ be such that

$$
\varphi_{\sigma(i)}(x) = \begin{cases} 
\text{convergent, if } \varphi_{\sigma(i)}(x) \neq 0, \\
\text{divergent, if } \varphi_{\sigma(i)}(x) = 0, \\
\text{divergent, if } \varphi_{\sigma(i)}(x) \text{ is divergent}
\end{cases}
$$

and $\sigma(j)$ is an increasing function.

Let us define

$$
\tilde{\Phi}_i(x) = \begin{cases} 
\Phi_i(x) \text{ if } i \neq \sigma(j), \\
\Phi_{\sigma(i)}(x) \text{ if } i = \sigma(j);
\end{cases}
$$

and

$$
\tilde{N}(i, x) = \begin{cases} 
N(i, x), \text{ if } i \neq \sigma(j), \\
0, \text{ if } \varphi_{\sigma(i)}(x) = 0, \\
1, \text{ if } \varphi_{\sigma(i)}(x) \neq 0, \\
\text{divergent, if } \varphi_{\sigma(i)}(x) \text{ is divergent}
\end{cases}
$$

if $i = \sigma(j)$. Q.E.D.
The new measure is acceptable because

if $i \neq \sigma(j)$:

\[
q_i(x) \overset{\text{def}}{\leftrightarrow} \Phi_i(x) \overset{\text{def}}{\leftrightarrow} \check{\Phi}_i(x),
\]

\[
\check{\Phi}_i(x) \overset{\text{def}}{\rightarrow} \Phi_i(x) \overset{\text{def}}{\rightarrow} N(i, x) \overset{\text{def}}{=} \check{N}(i, x) \text{ and } \check{N}(i, x) = N(i, x),
\]

\[
\Phi_i(x) = m \overset{\text{def}}{\leftrightarrow} \check{\Phi}_i(x) = m;
\]

if $i = \sigma(j)$:

\[
q_i(x) \overset{\text{def}}{\rightarrow} \check{q}_i(x) \overset{\text{def}}{\leftrightarrow} \Phi_i(x) \overset{\text{def}}{\leftrightarrow} \check{\Phi}_i(x) = m \overset{\text{def}}{\leftrightarrow} \check{\Phi}_i(x) = m,
\]

\[
\check{\Phi}_i(x) \overset{\text{def}}{\rightarrow} \Phi_i(x) \overset{\text{def}}{\rightarrow} \check{q}_i(x) \overset{\text{def}}{\rightarrow} \check{N}(i, x) \overset{\text{def}}{=} \check{N}(i, x),
\]

and $\check{N}(i, x) = 1$ if $q_i(x) \neq 0$, i.e., $q_i(x)$ def,

$= 0$ if $q_i(x) = 0$, i.e., $q_i(x) \div$;

\[
\check{q}_i(x) = m \overset{\text{def}}{\rightarrow} \check{q}_i(x) = m.
\]

Let $R$ be a recursive set; then $\exists e \in N$ such that

\[
q_e(x) = 1, \text{ if } x \in R,
\]

\[
= 0, \text{ if } x \notin R;
\]

hence,

\[
q_{\sigma(e)}(x) = \text{ convergent, if } x \in R,
\]

\[
= \text{ divergent, if } x \notin R.
\]

Suppose $R$ is such that

\[
\hat{S} - K \subseteq R \subseteq \hat{K}
\]

(where, as we know, $\hat{S} - K = \{x \mid \check{\Phi}_x(x) \text{ is defined and } \check{N}(x, x) = 0\}$),

then $\sigma(e) \in R \rightarrow q_x(\sigma(e)) \overset{\text{def}}{\rightarrow} q_{\sigma(e)}(x)$ divergent but at the same time

$\sigma(e) \in R \rightarrow q_{\sigma(e)}(x)$ convergent by definition of $\sigma$: contradiction.

On the other side $\sigma(e) \notin R \rightarrow q_{\sigma(e)}(\sigma(e))$ divergent by definition of $\sigma$ and besides

$\sigma(e) \notin R \rightarrow q_{\sigma(e)}(\sigma(e)) = 0 \rightarrow \check{N}(\sigma(e), \sigma(e)) = 0,$

but at the same time

\[
\check{\Phi}_{\sigma(e)}(\sigma(e))(=\Phi_e(\sigma(e))) \text{ is defined};
\]

since $\check{\Phi}_{\sigma(e)}(\sigma(e))$ is defined and $\check{N}(\sigma(e), \sigma(e)) = 0$, we have $\sigma(e) \in \hat{S} - K \rightarrow \sigma(e) \in R$: contradiction. Since in both cases we get a contradiction it follows that either $\hat{S} - K \subseteq R$ or $R \subseteq \hat{K}$.

Q.E.D.
Another characteristic of the set of programs cycling on a finite amount of tape is to be isomorphic to the set of halting programs. Also of this fact we can give an axiomatic formulation and show that, for example, the same measure \( \Phi \) defined in Theorem 6 also has this property.

**Definition 7.** Two r.e. sets \( A \) and \( B \) are **effectively inseparable** if there exists a partial recursive function \( \psi \) of two variables such that for any \( u \) and \( v \), \( A \subseteq Wu \) and \( B \subseteq Wv \) and \( Wu \cap Wv = \varnothing \rightarrow \psi(u, v) \) convergent and \( \psi(u, v) \in \overline{Wu} \cup \overline{Wv} \).

**Theorem 7.** Let \( \Phi \) be defined as in theorem 6: \( K \) and \( \bar{S} - K \) are effectively inseparable.

**Proof.** Let us define

\[
\varphi_{s(u,v)}(x) = \begin{cases} 
0, & \text{if } x \text{ appears first in the enumeration of } Wu, \\
1, & \text{if } x \text{ appears first in the enumeration of } Wv, \\
\text{divergent}, & \text{otherwise.}
\end{cases}
\]

Let \( \sigma \) be defined as in Theorem 6; then \( \psi = \sigma \circ s \) is a recursive function satisfying Definition 7 that is, given \( u \) and \( v \) such that \( K \subseteq Wu \), \( \bar{S} - K \subseteq Wv \) and \( Wu \cap Wv = \varnothing \), \( \sigma(s(u, v)) \) is convergent and \( \sigma(s(u, v)) \in \overline{Wu} \cup \overline{Wv} \).

In fact, \( \sigma \) and \( s \) are total functions (by s-m-n theorem) and besides

\[
\sigma(s(u, v)) \in Wv \rightarrow \sigma(s(u, v)) \in K \rightarrow \varphi_{\sigma(s(u,v))}(\sigma(s(u, v))) \text{ div,}
\]

but at the same time

\[
\sigma(s(u, v)) \in Wv \rightarrow \varphi_{s(u,v)}(\sigma(s(u, v))) = 1 \rightarrow \varphi_{\sigma(s(u,v))}(\sigma(s(u, v))) \text{ conv}
\]

(by definition of \( \sigma \)) that is a contradiction.

On the other side

\[
\sigma(s(u, v)) \in Wu \rightarrow \varphi_{s(u,v)}(\sigma(s(u, v))) = 0
\]

and

\[
\sigma(s(u, v)) \in K \quad \text{or} \quad \sigma(s(u, v)) \in \bar{S}.
\]

(a) \( \sigma(s(u, v)) \in K \rightarrow \varphi_{\sigma(s(u,v))}(\sigma(s(u, v))) \text{ conv, but at the same time} \)

\[
\varphi_{s(u,v)}(\sigma(s(u, v))) = 0 \rightarrow \varphi_{\sigma(s(u,v))}(\sigma(s(u, v))) \text{ div}
\]

that is a contradiction.
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(b) \( \sigma(s(u, v)) \in \tilde{S} \rightarrow \tilde{\Phi}_{\sigma(s(u, v))}(\sigma(s(u, v))) \overset{\text{div}}{\rightarrow} \Phi_{s(u, v)}(\sigma(s(u, v))) \overset{\text{div}}{\rightarrow} \varphi_{s(u, v)}(\sigma(s(u, v))) \) div

that contradicts

\[ \varphi_{s(u, v)}(\sigma(s(u, v))) = 0. \]

Since in all cases we get a contradiction it follows that \( \sigma(s(u, v)) \in \bar{W}u \cup \bar{W}v \) Q.E.D.

**Corollary 8.** Let \( \tilde{\Phi} \) be defined as in Theorem 6: \( \tilde{S} - K \) is recursively isomorphic to \( K \).

**Proof.** If \( K \) and \( \tilde{S} - K \) are effectively inseparable they are both creative, hence they are recursively isomorphic. Q.E.D.

**Definition 8.** A weak measure \( \tilde{\Phi} \) is said to be *tapelike* if it is such that \( \tilde{S} - K \) and \( K \) are recursively inseparable and recursively isomorphic, that is the set of halting programs and the set of cycling programs are in the relation given by Theorems 6 and 7.

**Conclusions**

Beside providing an example of how to use the new axioms the abstract proof of the existence of tapelike measures, can be considered also under another point of view.

By weakening the axioms we contribute to increase the number of pathological measures that we already can find among strong measures and that people was trying to exclude by strengthening the axioms [5, 7–9]. On the other side, any natural measure we can think of (number of reversals of the head, amount of ink, etc.) can be machine dependently proved to be tapelike.

For this reason the way \( \tilde{\Phi} \) is derived from \( \Phi \) in the proof of Theorem 6 can indicate how we can force a weak measure to be tape like, by starting from an adequate strong measure.

Moving along an independent approach toward the same goal Ivan M. Havel [10] is quite recently arrived to the conclusion that a reasonable way of characterizing tape-like weak measures is to admit (by axiom) that we have the ability of forcing a program to loop.

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